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Common Fixed Point Results of Weakly Reciprocal and Commutativity Continuity in complete $\mathcal{D}^*$ −metric Spaces

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Abstract. The major goal of a present this manuscript is to an introduce and establish some of new common fixed point theorems in complete $\mathcal{D}^* -$ metric sp. utilizing the ideas of compatibility, variants of $\mathbb{R} -$ weakly commutativity and faintly reciprocal continuity. Various fundamental properties related to commuting mappings in complete $\mathcal{D}^* -$ metric spaces have been discussed. A results presented in this manuscript have been improved and sharpened many related results present in the existing literature. Moreover, various suitable examples in support of these conclusions have been provided.

Mathematics Subject Classification: primary(54H25); secondary (47H10).

Key-words: Complete $\mathcal{D}^*$ −metric;$\mathbb{R}$ −weakly commutativity; common fixed point; weakly reciprocal continuity.

1. Introduction

The common fixed point theorems in complete $\mathcal{D}^*$ −metric-sp play significant role in developing mathematical approaches for solving issues in pure and practical mathematics and other sciences. The Banach Fix. P. Theo. which was first presented by Banach in 1922 is significant result in Fix. P. Theo, and he proved a common fixed point theorem (briefly. comm. Fix. P. Theo.), which ensures under appropriate conditions. By Sessa [1] as sharper tool for obtaining common map Fix. P. As a result, all Fix. P. Theo. for commuting maps could be simply translated into the new notion of weak commutativity of maps. It lends a new push to the research of comm. Fix. Point. of maps meeting some contractive type constraints. G. Jungck[2] proved a comm. Fix. P. Theo. for commuting maps, which generalizes the Banachs Fix. P. Theo,. A major breakthrough was done by [3] when he proclaimed the new notion he called compatibility of map and its usefulness for obtaining (Comm. Fix. Point. Theo.)of maps was shown by him. Thereafter a flood of (Comm. Fix. P. Theo.) was produced by various researchers using the improved notion of compatibility of maps. The notion of $\mathbb{R} -$ weakly commutativity for single-valued maps was defined by Pant[4] to generalize the concept of commuting and weakly commuting maps ((briefly. W. Com. Maps)) [1]. Thereafter, Shahzad *and* Kamran[5] extended this concept to the setting of single and multi-valued maps, and studied the structure of comm. Fix. P. Some references dealing with$\mathbb{R} -$ (W.Com. Maps) are [6– 8]. After that, commutativity conditions have also been used to find coupled coincidence point[9, 10] . Pant in[11] introduced new reciprocal continuity and studied a comm. Fix. Point. Theo. utilizing compatibility in metric-sp. Furthermore, the concept of point wise $\mathbb{R} -$ (W.Com. Maps) broadened the scope of research of (Comm. Fix. P. Theo.) from the notion of compatible to point wise$\mathbb{R}$ −(W.Com. Maps). Following, multiple (Comm. Fix. P. Theo.) Were established by merging the concepts of $\mathbb{R} -$ (W.Com. Maps) and reciprocal continuity of maps in various settings. Also, in 2007, S.Shaban, et.al. [12] Have been established the meaning$\mathcal{D}^* -$ metric $-$ sp and proved several essential properties in $\mathcal{D}^*$ −metric-sp. Recently, AL. Jumaili in [13] used $\mathcal{D}^* -$ metric-sp *and* presented some coincidence (Fix. P. Theo.) for anon-decreasing$\varphi -$ maps impartially ordered complete generalized$\mathcal{D}^*$ −metric-sp Very recently, in (Comm. Fix. P. Theo.) in metric-sp several versions of weak commutativity have been considered via K. Das [14], maps which are not compatible have also been discussed in Comm. Fix. P. problems. In addition, the authors [15] extension the conception of $\mathcal{D}^* -$ metric-sp by changing $\mathbb{R}$ via an ordered Banach-sp in$\mathcal{D}^* -$ metric-sp *and* established some (Fix. Point. Theo.)in complete partially ordered G-cone-metric-sp. In this manuscript, numerous typical of (Comm. Fix. P. Theo.) for variants$\mathbb{R}$ −(W. Com. Maps) in complete$\mathcal{D}^*$ −metric-sp have been proved. This paper has been divided into various sections. First section provides a brief historical overview. The notion of $\mathbb{R} -$ (W.Com. Maps) and its analogues in $\mathcal{D}^*$ −metric-sp were presented in the second section. A (Comm. Fix. P. Theo.) has been proved utilizing $\mathbb{R}$ −(W.Com. Maps) in third section. The fourth section uses weak reciprocal continuity to prove some new (Comm. Fix. P. Theo.) in complete $\mathcal{D}^*$ −metric-sp for some forms of$\mathbb{R} -$ (W.Com. Maps) As a result of the findings reported in this manuscript,

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numerous well-known (Comm. Fix. P. Theo.)in the literature have been refined and sharpened. Also, some examples in support of these findings have been given.

2. Preliminaries

In this section, definitions and the core ideas that are important to our work have been introduced.

Definition2.1: [12] Let $\mathcal{X} \neq \emptyset$. A $\mathcal{D}^* -$ metric is $\mathcal{D}^*: \mathcal{X}^3 \to [0,\infty)$, satisfies the next conditions $\forall\, x, y, z, c \in \mathcal{X}$

$(\boldsymbol{\mathcal{D}_1^*})$ $\mathcal{D}^*(x,y,z) \geq 0, \forall\, x,y,z \in \mathcal{X}$;

$(\boldsymbol{\mathcal{D}_2^*})$ $\mathcal{D}^*(x,y,z) = 0 \iff x = y = z$;

$(\boldsymbol{\mathcal{D}_3^*})$ $\mathcal{D}^*(x,y,z) = \mathcal{D}^*(\mathcal{P}\{x,y,z\})$, *(Symmetry) (s. t) $\mathcal{P}$permutation function*,

$(\boldsymbol{\mathcal{D}_4^*})$ $\mathcal{D}^*(x,y,z) \leq \mathcal{D}^*(x,y,c) + \mathcal{D}^*(c,z,z)$

The function $\mathcal{D}^*$ is a $\mathcal{D}^*$ −metric and the pair $(\mathcal{X},\mathcal{D}^*)$ is called a $\mathcal{D}^* -$ metric-sp.

Definition 2.2: [12] A $\mathcal{D}^*$ −metric-sp is symmetric$\mathcal{D}^*$ −metric-sp if it satisfies $\forall x, y \in \mathcal{X}$,

$\mathcal{D}^*(x,x,y) = \mathcal{D}^*(x,y,y)$.

Definition2.3: [12] If $(\mathcal{X},\mathcal{D}^*)$ is $\mathcal{D}^* -$ metric-sp,:

$(\boldsymbol{i})$ A sequence$\{x_n\}$ converges to $x \in \mathcal{X}$ *iff* $\mathcal{D}^*(x_n,x_n,x) = \mathcal{D}^*(x,x,x_n) \longrightarrow 0$ as $n \longrightarrow \infty$. (i. e)

$\forall\, \varepsilon > 0, \exists$ a positive integer r (s. t),

$\forall\, n \geq r \Longrightarrow \mathcal{D}^*(x,x,x_n) < \varepsilon$, This is equivalent, $\forall\, \varepsilon > 0\ \exists$ positive integer r (s. t), $\forall\, n, m \geq r, \mathcal{D}^*(x,x_n,x_m) < \varepsilon$.

$(\boldsymbol{ii})$ A seq $\{x_n\}$ in $\mathcal{X}$ is called$\mathcal{D}^* -$ Cauchy seq. if $\forall\, \mathcal{E} > 0, \exists$ a positive integer r (s.t), $\forall n, m \geq r, \mathcal{D}^*(x_n,x_n,x_m) < \varepsilon$.

$(\boldsymbol{iii})$ $(\mathcal{X},\mathcal{D}^*)$Complete $\mathcal{D}^*$ −metric, if $\forall\ \mathcal{D}^* - \text{Cauchy} - \text{seq}$ in $(\mathcal{X},\mathcal{D}^*)$convergent in $\mathcal{X}$.

Lemma2.4: [16] Let$(\mathcal{X},\mathcal{D}^*)$ be $\mathcal{D}^* -$ metric-sp *and* $\{x_n\}$ be a sequence in $\mathcal{X}$ and $x \in \mathcal{X}$. The following properties equivalent:

$(\boldsymbol{i})$ $\{x_n\}$ is $\mathcal{D}^*$ −convergent to $x \in \mathcal{X}$;

$(\boldsymbol{ii})\mathcal{D}^*(x_n,x_n,x) \longrightarrow 0, as(n \longrightarrow \infty)$;

$(\boldsymbol{iii})\mathcal{D}^*(x_n,x,x,) \longrightarrow 0, as(n \longrightarrow \infty)$.

Lemma2.5: [12] In D*, the following characteristics are equivalent: metric-sp:

$(\boldsymbol{i})$ A seq $\{x_n\}$ is $\mathcal{D}^*$ −Cauchy;

$(\boldsymbol{ii})$ $\forall\, \varepsilon > 0, \exists$ positive integer$r$ (s.t), $\mathcal{D}^*(x_n,x_n,x_m) < \varepsilon\ \forall\, n,m \geq r$.

Lemma 2.6: [12] suppose$(\mathcal{X},\mathcal{D}^*)$ is $\mathcal{D}^* -$ metric-sp so $\mathcal{D}^*(x,x,y) \leq 2\mathcal{D}^*(y,y,x) \forall x, y \in \mathcal{X}$.

Definition2.7: [17] Let $\mathcal{F},\mathcal{G}: (\mathcal{X},\mathcal{D}^*) \to (\mathcal{X},\mathcal{D}^*)$ be self-maps on set$\mathcal{X}$. *If* $\mathcal{F}(x) = \mathcal{G}(x) = r$ for some$x$ in $\mathcal{X}$, so x coincidence point of $\mathcal{F}$, $\mathcal{G}$, r apoint of coincidence of $\mathcal{F}$, $\mathcal{G}$.

Definition2.8: [18] Let$\mathcal{F},\mathcal{G}: (\mathcal{X},\mathcal{D}^*) \to: (\mathcal{X},\mathcal{D}^*)$ self-map on $\mathcal{X}$. If $\mathcal{F}(x) = \mathcal{G}(x) = x$ for some x in $\mathcal{X}$, then x (Comm. Fix. P.) of $\mathcal{F}$, $\mathcal{G}$.

*The following definitions from the [19]**:***

*Definition 2.9**:** A $(\mathcal{F},\mathcal{G})$ of self-maps of $(\mathcal{X},\mathcal{D}^*)$ called compatible map if* $\lim_{n\to\infty} \mathcal{D}^*\{\mathcal{F}(\mathcal{G}(x_n)), \mathcal{F}(\mathcal{G}(x_n)), \mathcal{G}(\mathcal{F}(x_n))\} = 0$ *or*

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$\lim_{n\to\infty} \mathcal{D}^*\{\mathcal{G}(\mathcal{F}(x_n)), \mathcal{G}(\mathcal{F}(x_n)), \mathcal{F}(\mathcal{G}(x_n))\} = 0,$

when $\{x_n\}$ in $\mathcal{X}$ (s. t) $\lim_{n\to\infty}\mathcal{F}(x_n) = \lim_{n\to\infty}\mathcal{G}(x_n) = \wp$ forsome $\wp \in \mathcal{X}$.

Definition2.10: Apair $(\mathcal{F}, \mathcal{G})$ of self-maps in $(\mathcal{X}, \mathcal{D}^*)$ called non-compatible maps *if* $\exists$ at least one seq. $\{x_n\}$ in$\mathcal{X}$ (s. t): $\lim_{n\to\infty} \mathcal{F}(x_n) = \lim_{n\to\infty} \mathcal{G}(x_n) = \wp \in \mathcal{X}$, *but either*

$\lim_{n\to\infty}\mathcal{D}^*\{\mathcal{F}(\mathcal{G}(x_n)), \mathcal{F}(\mathcal{G}(x_n)), \mathcal{G}(\mathcal{F}(x_n))\} \neq 0$ *and*

$\lim_{n\to\infty} \mathcal{D}^*\{\mathcal{G}(\mathcal{F}(x_n)), \mathcal{G}(\mathcal{F}(x_n)), \mathcal{F}(\mathcal{G}(x_n))\} \neq 0$, or non-existent.

Definition2.11: [20] If $\mathcal{F}, \mathcal{G}: (\mathcal{X}, \mathcal{D}^*) \to (\mathcal{X}, \mathcal{D}^*)$self-maps of $(\mathcal{X}, \mathcal{D}^*)$, $\mathcal{F}$ *and* $\mathcal{G}$ are reciprocally-cont. *if* $\lim_{n\to\infty} \mathcal{F}(\mathcal{G}(x_n)) = \mathcal{F}(r)$ *and* $\lim_{n\to\infty} \mathcal{G}(\mathcal{F}(x_n)) = \mathcal{G}(r)$. Consider the sequences $\{x_n\}$ in$\mathcal{X}$ (s. t): $\lim_{n\to\infty} \mathcal{F}(x_n) = \lim_{n\to\infty} \mathcal{G}(x_n) = r$ forsome $r \in \mathcal{X}$.

Definition 2.12: [21] Let $\mathcal{F}, \mathcal{G}: (\mathcal{X}, \mathcal{D}^*) \to (\mathcal{X}, \mathcal{D}^*)$ self-maps of $(\mathcal{X}, \mathcal{D}^*)$ then, $\mathcal{F}$, $\mathcal{G}$ are weakly reciprocally-cont. if:

$\lim_{n\to\infty} \mathcal{F}(\mathcal{G}(x_n)) = \mathcal{F}(r)$or $\lim_{n\to\infty} \mathcal{G}(\mathcal{F}(x_n)) = \mathcal{G}(r)$. Whenever the sequences $\{x_n\}$ in $\mathcal{X}$ (s. t): $\lim_{n\to\infty} \mathcal{F}(x_n) = \lim_{n\to\infty} \mathcal{G}(x_n) = r$, for some $r \in \mathcal{X}$.

3. A (Comm. Fix. Point) Theorem for$\mathbb{R}$ −Weakly Commuting Maps in Complete $\mathcal{D}^*$ − metric -spaces

This section is devoted to introduce comm. Fix. P. Theo. by $\mathbb{R}$ −weakly commuting maps in complete $\mathcal{D}^*$ − metric-sp, in addition, suitable example that supports our main results has been provided.

Definition 3.1: *Let* $\mathcal{F}, \mathcal{G}: (\mathcal{X}, \mathcal{D}^*) \to (\mathcal{X}, \mathcal{D}^*)$*self-maps of* $(\mathcal{X},\mathcal{D}^*)$. *Then,* $\mathcal{F}$, $\mathcal{G}$ *are:*

(i) Commuting if $\mathcal{F}(\mathcal{G}(x)) = \mathcal{G}(\mathcal{F}(x)), \forall\, x \in \mathcal{X}$.

(ii) Weakly commuting if $\mathcal{D}^*\{\mathcal{F}(\mathcal{G}(x)), \mathcal{F}(\mathcal{G}(x)), \mathcal{G}(\mathcal{F}(x))\} \leq \mathcal{D}^*\{\mathcal{F}(x), \mathcal{F}(x), \mathcal{G}\,(x)\}$,

$(iii)\mathbb{R}$ − Weakly commuting at$x \in \mathcal{X}$ if $\exists$ some real number$k > 0$ (s. t):

$\mathcal{D}^*\{\mathcal{F}(\mathcal{G}(x)), \mathcal{F}(\mathcal{G}(x)), \mathcal{G}(\mathcal{F}(x))\} \leq k\mathcal{D}^*\{\mathcal{F}(x), \mathcal{F}(x), \mathcal{G}(x)\}\ \forall x \in \mathcal{X}.$

Theorem3.2: Let$(\mathcal{X}, \mathcal{D}^*)$ be complete$\mathcal{D}^*$ − metric-sp *and* $\mathcal{F}$, $\mathcal{G}$ $\mathbb{R}$ −weakly commuting self-maps of $(\mathcal{X}, \mathcal{D}^*)$satisfying the following properties:

$(i)\mathcal{F}(\mathcal{X}) \subseteq \mathcal{G}(\mathcal{X})$;

$(ii)\mathcal{F}$or $\mathcal{G}$ cont. map;

(iii) $\mathcal{D}^*(\mathcal{F}x, \mathcal{F}y, \mathcal{F}z) \leq k\mathcal{D}^*(\mathcal{G}x, \mathcal{G}y, \mathcal{G}z)$, $\forall\, x, y, z \in \mathcal{X}$ and $0 \leq k < 1$. So, $\mathcal{F}$, $\mathcal{G}$ have unique comm. Fix. P. in$\mathcal{X}$

Proof: Suppose x_0 is arbitrary point in$\mathcal{X}$, via(i), can select apoint $x_1 \in \mathcal{X}$ (s. t) $\mathcal{F}x_0 = \mathcal{G}x_1$. In general, choosex_{n+1} (s.t) $y_n = \mathcal{F}x_n = \mathcal{G}x_{n+1}$

Now, explain $\{y_n\}$ is $\mathcal{D}^*$ − Cauchy-seq. in $\mathcal{X}$. for proving,

Putting$x = x_n$, $y = x_n$, $z = x_{n+1}$, in part (iii), we get:

$\mathcal{D}^*(\mathcal{F}x_n, \mathcal{F}x_n\,, \mathcal{F}x_{n+1}) \leq k\mathcal{D}^*(\mathcal{G}x_n, \mathcal{G}x_n, \mathcal{G}x_{n+1}) = k\mathcal{D}^*(\mathcal{F}x_{n-1}, \mathcal{F}x_{n-1}\,, \mathcal{F}x_n)$

Consequently in the same method, we obtain

$\mathcal{D}^*(\mathcal{F}x_n, \mathcal{F}x_n\,, \mathcal{F}x_{n+1}) \leq k^n\, \mathcal{D}^*(\mathcal{F}x_0, \mathcal{F}x_0\,, \mathcal{F}x_1) \Longrightarrow \mathcal{D}^*(y_n, y_n\,, y_{n+1}) \leq k^n\mathcal{D}^*(y_0, y_0\,, y_1).$

Therefore, $\forall\, n, m \in N, n < m$, we obtain, $\mathcal{D}^*(y_n, y_n, y_{m)} \leq$

$\mathcal{D}^*(y_n, y_n, y_{n+1}) + \mathcal{D}^*(y_{n+1}, y_{n+1}, y_{n+2}) + \cdots + \mathcal{D}^*(y_{m-1},, y_{m-1}\,, y_m)$

$\leq (k^n + k^{n+1} + \cdots + k^{m-1})\, \mathcal{D}^*(y_0, y_0, y_1) \leq (k^n + k^{n+1} + \cdots)\, \mathcal{D}^*(y_0, y_0, y_1)$

$= \frac{k^n}{(1-k)} \mathcal{D}^*(y_0, y_0, y_1) \to 0$ as $n \to \infty$.

Hence $\{y_n\}$ is $\mathcal{D}^*-$Cauchy-seq. in $\mathcal{X}$. Since$(\mathcal{X}, \mathcal{D}^*)$ is complete$\mathcal{D}^*-$ metric-sp , consequently, $\exists\, z \in \mathcal{X}$ (s. t) $\lim_{n\to\infty} y_n = \lim_{n\to\infty} \mathcal{G}x_n = \lim_{n\to\infty} \mathcal{F}x_n = z$.

Assume the map $\mathcal{F}$ is cont. Consequently:

$\lim_{n\to\infty} \mathcal{F}\mathcal{G}x_n = \lim_{n\to\infty} \mathcal{F}\mathcal{F}\, x_n = \mathcal{F}z$. Since, $\mathcal{F}$, $\mathcal{G}$ $\mathbb{R}-$weakly commuting,

$\mathcal{D}^*(\mathcal{F}\mathcal{G}x_n, \mathcal{F}\mathcal{G}x_n, \mathcal{G}\mathcal{F}x_n) \leq \mathbb{R}\, \mathcal{D}^*(\mathcal{F}x_n, \mathcal{F}x_n, \mathcal{G}x_n), (s.t)\ \mathbb{R} > 0$. When $n \to \infty$, we obtain $\lim_{n\to\infty} \mathcal{G}\mathcal{F}x_n = \lim_{n\to\infty} \mathcal{F}\mathcal{G}x_n = \mathcal{F}\, z$.

Now, prove $\mathcal{F}z = z$. Assume $\mathcal{F}z \neq z$, then $\mathcal{D}^*(z, z, \mathcal{F}z) > 0$. On putting $x = x_n, y = x_n, z = \mathcal{F}x_n$ in part $(\boldsymbol{iii})$, get:

$\mathcal{D}^*(\mathcal{F}x_n, \mathcal{F}x_n, \mathcal{F}\mathcal{F}x_n) \leq k\, \mathcal{D}^*(\mathcal{G}\, x_n, \mathcal{G}\, x_n, \mathcal{G}\mathcal{F}x_n)$. When limit as $n \to \infty$, obtain:

$\mathcal{D}^*(z, z, \mathcal{F}z) \leq k\ \mathcal{D}^*(z, z, \mathcal{F}z) < \mathcal{D}^*(z, z, \mathcal{F}z)$, This contradiction. Therefore, $\mathcal{F}z = z$.

Since $\mathcal{F}(\mathcal{X}) \subseteq \mathcal{G}(\mathcal{X})$, we discover $z_1 \in \mathcal{X}$ (s. t):

$z = \mathcal{F}z = \mathcal{G}z_1$. Now put $x = y = \mathcal{F}x_n, z = z_1$ in part(iii), get:

$\mathcal{D}^*(\mathcal{F}\mathcal{F}x_n, \mathcal{F}\mathcal{F}x_n, \mathcal{F}z_1) \leq k\, \mathcal{D}^*(\mathcal{G}\mathcal{F}x_n, \mathcal{G}\mathcal{F}x_n, \mathcal{G}z_1)$.

Taking *limit as* $n \to \infty$, obtain:

$\mathcal{D}^*(\mathcal{F}z, \mathcal{F}z, \mathcal{F}z_1) \leq k\mathcal{D}^*(\mathcal{F}z, \mathcal{F}z, \mathcal{G}z_1) = k\, \mathcal{D}^*(\mathcal{F}z, \mathcal{F}z, \mathcal{F}z) = 0$, Which implies $\mathcal{F}z = \mathcal{F}z_1$,

i.e., $z = \mathcal{F}z = \mathcal{F}z_1 = \mathcal{G}z_1$. In addition, utilizing definition of $\mathbb{R}-$ weakly commutativity, we have

$\mathcal{D}^*(\mathcal{F}z, \mathcal{F}z, \mathcal{G}z) = \mathcal{D}^*(\mathcal{F}\mathcal{G}z_1, \mathcal{F}\mathcal{G}z_1, \mathcal{G}\mathcal{F}z_1) \leq \mathbb{R}\, \mathcal{D}^*(\mathcal{F}z_1, \mathcal{F}z_1, \mathcal{G}z_1) = 0$,

Implies $\mathcal{F}z = \mathcal{G}z = z$. Consequently, z is Comm. Fix. of a maps $\mathcal{F}, \mathcal{G}$.

Uniqueness: Suppose$p(\neq z)$be a anther Comm. Fix. P. of $\mathcal{F}$, $\mathcal{G}$. Then,$\mathcal{D}^*(z, z, p) > 0$ *and*

$\mathcal{D}^*(z, z, p) = \mathcal{D}^*(\mathcal{F}z, \mathcal{F}z, \mathcal{F}p) \leq k\mathcal{D}^*(\mathcal{G}z, \mathcal{G}z, \mathcal{G}p) = k\, \mathcal{D}^*(z, z, p) < \mathcal{D}^*(z, z, p)$,

this contradiction, so $z = p$, thus uniqueness fellows.

Next, we present the following example to illustrate the validity of theorem (3.2).

*Example3.2***:** Let$\mathcal{X} = [-1,1]$ *and* $\mathcal{D}^*: \mathcal{X}^3 \to [0, \infty)$, be the $\mathcal{D}^*-$metric defined by $\mathcal{D}^*(x, y, z) = (|x - y| + |y - z| + |z - x|)$, $\forall\, x, y, z \in \mathcal{X}$ *.Then* $(\mathcal{X}, \mathcal{D}^*)$*is complete* $\mathcal{D}^*$ *−metric-sp. Def. a self-maps* $\mathcal{F}, \mathcal{G}$*:* $\mathcal{F}: (\mathcal{X}, \mathcal{D}^*) \to (\mathcal{X}, \mathcal{D}^*)$ *(s. t)*

$\mathcal{F}(x) = x$ *and* $\mathcal{G}: (\mathcal{X}, \mathcal{D}^*) \to (\mathcal{X}, \mathcal{D}^*)$(s. t) $\mathcal{G}(x) = 2x - 1, \forall\, x \in \mathcal{X}$:

$(\boldsymbol{i})\mathcal{F}(\mathcal{X}) \subseteq \mathcal{G}(\mathcal{X})$;

$(\boldsymbol{ii})\mathcal{F}$ *Cont. On* $\mathcal{X}$;

$(\boldsymbol{iii})$ $\mathcal{D}^*(\mathcal{F}x, \mathcal{F}y, \mathcal{F}z) \leq k\mathcal{D}^*(\mathcal{G}x, \mathcal{G}y, \mathcal{G}z)$, holds$\forall x, y, z \in \mathcal{X}$, *and* $\frac{1}{2} \leq k < 1$.

A maps $\mathcal{F}$, $\mathcal{G}$ $\mathbb{R}-$W. Com. are thus all the conditions of Theo **(3.2)** are satisfied and $x = 1$ is the unique Comm.. Fix. of $\mathcal{F}$, $\mathcal{G}$.

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4. Common Fix. P. theorems for variants$\mathbb{R}-$ weakly commuting maps utilizing faintly reciprocal continuity

In this section, new Comm. Fix. P. Theo. in complete $\mathcal{D}^*-$metric$-$sp forsome forms of $\mathbb{R}-$(W.Com. Map) have been established by the notion of weak reciprocal continuity.

*Definition 4.1***:** Let $\mathcal{F},\mathcal{G}:(\mathcal{X},\mathcal{D}^*)\rightarrow(\mathcal{X},\mathcal{D}^*)$ self-maps of $(\mathcal{X},\mathcal{D}^*)$,then, $\mathcal{F},G$:

$(\boldsymbol{i})$ $\mathbb{R}-$(W.Com.Map) of type$\left(\mathcal{A}_{\mathcal{G}}\right)$ if $\exists$ some $k>0$ (s. t):

$\mathcal{D}^*\{\mathcal{F}(\mathcal{F}(x)),\mathcal{F}(\mathcal{F}(x)),\mathcal{G}(\mathcal{F}(x))\}\leq k\,\mathcal{D}^*\{\mathcal{F}(x),\mathcal{F}(x),\mathcal{G}(x)\}\forall\, x\in\mathcal{X}.$

$(\boldsymbol{ii})$ $\mathbb{R}-$(W.Com.Map) of type $(\mathcal{A}_{\mathcal{F}})$ if $\exists$ some $k>0$ (s. t):

$\mathcal{D}^*\{\mathcal{F}(\mathcal{G}(x)),\mathcal{F}(\mathcal{G}(x)),\mathcal{G}(\mathcal{G}(x))\}\leq k\mathcal{D}^*\{\mathcal{F}(x),\mathcal{F}(x),\mathcal{G}(x)\}\forall x\in\mathcal{X}$.

$(\boldsymbol{iii})$ $\mathbb{R}-$(W.Com.Map) of type$(\mathcal{P})$ if $\exists$ some$k>0$ (s. t):

$\mathcal{D}^*\{\mathcal{F}(\mathcal{F}(x)),\mathcal{F}(\mathcal{F}(x)),\mathcal{G}(\mathcal{G}(x))\}\leq k\mathcal{D}^*\{\mathcal{F}(x),\mathcal{F}(x),\mathcal{G}(x)\}\forall\, x\in\mathcal{X}.$

Theorem4.2: Let$\mathcal{F}$, G weakly reciprocally cont. self-maps of complete$(\mathcal{X},\mathcal{D}^*)$ satisfying the following properties:

$(\boldsymbol{i})\mathcal{F}(\mathcal{X})\subseteq\mathcal{G}(\mathcal{X})$;

$(\boldsymbol{ii})$ $\mathcal{D}^*(\mathcal{F}x,\mathcal{F}y,\mathcal{F}z)\leq k\mathcal{D}^*(\mathcal{G}x,\mathcal{G}y,\mathcal{G}z)$, $\forall\, x,y,z\in\mathcal{X}$; $0\leq k<1$.

If $\mathcal{F}$, G are either compatible or$\mathbb{R}-$(W.Com.Map) type($\mathcal{A}_{\mathcal{G}}$) or $\mathbb{R}-$(W.Com.Map) type($\mathcal{A}_{\mathcal{F}}$) or $\mathbb{R}-$(W.Com.Map) type$(\mathcal{P})$, so $\mathcal{F}$, $\mathcal{G}$ have unique Com. Fix. P.

Proof: Presume x_0 is an arbitrary in$\mathcal{X}$. As $\mathcal{F}(\mathcal{X})\subseteq\mathcal{G}(\mathcal{X})$, consequently $\exists\,\{x_n\},(s.t)\,\mathcal{F}x_n=\mathcal{G}x_{n+1}$.

Def a seq. $\{y_n\}$ in $\mathcal{X}$ via $y_n=\mathcal{F}x_n=\mathcal{G}x_{n+1}$. *……………………….. (4.2.1)*

A Seq$\{y_n\}$ is $\mathcal{D}^*-$ Cauchy seq in $\mathcal{X}$ (proof of this fact similar to that of Theo-(3.2)).

Now, since$(\mathcal{X},\mathcal{D}^*)$ complete$\mathcal{D}^*-$metric-sp, therefore $\exists$ $z\in\mathcal{X}$ *(s. t)* $\lim_{n\to\infty}y_n=z$. *Hence,* $\lim_{n\to\infty}y_n=\lim_{n\to\infty}\mathcal{F}x_n=\lim_{n\to\infty}\mathcal{G}x_n=z$.

Assume $\mathcal{F}$, $\mathcal{G}$ are compatible-maps, therefore a faint reciprocal cont. of $\mathcal{F}$and $\mathcal{G}\Longrightarrow\lim_{n\to\infty}\mathcal{F}\mathcal{G}x_n=\mathcal{F}z$ *or* $\lim_{n\to\infty}\mathcal{G}\mathcal{F}x_n=\mathcal{G}z$. Suppose that $\lim_{n\to\infty}\mathcal{G}\,\mathcal{F}x_n=\mathcal{G}z$,so compatibility of $\mathcal{F},\mathcal{G}$ gives, $\lim_{n\to\infty}\mathcal{D}^*(\mathcal{F}\mathcal{G}x_n,\mathcal{F}\mathcal{G}\,x_n,\mathcal{G}\mathcal{F}x_n)=0$, i.e, $\mathcal{D}^*\left(\lim_{n\to\infty}\mathcal{F}\mathcal{G}x_n\,,\mathcal{F}\mathcal{G}x_n,\mathcal{G}z\right)=0$. Thus, $\lim_{n\to\infty}\mathcal{F}\mathcal{G}x_n=\mathcal{G}z$. Utilizing (4.2.1), we obtain $\lim_{n\to\infty}\mathcal{F}\mathcal{G}x_{n+1}=\lim_{n\to\infty}\mathcal{F}\mathcal{F}x_n=\mathcal{G}z$. Therefore, utilizing part(ii), get $\mathcal{D}^*(\mathcal{F}z,\mathcal{F}z,\mathcal{F}\mathcal{F}x_n)\leq k\,\mathcal{D}^*(\mathcal{G}z,\mathcal{G}z,\mathcal{G}\mathcal{F}x_n)$.

On both sides, when $n\to\infty$, obtain $\mathcal{D}^*(\mathcal{F}z,\mathcal{F}z,\mathcal{G}z)\leq k\,\mathcal{D}^*(\mathcal{G}z,\mathcal{G}z,\mathcal{G}z)=0$.

This gives, $\mathcal{F}z=\mathcal{G}z$. Once more compatibility of $\mathcal{F}$, $\mathcal{G}$ implies commutability at a coincidence point, therefore$\mathcal{G}\mathcal{F}z=\mathcal{F}\mathcal{G}z=\mathcal{F}\mathcal{F}z=\mathcal{G}\mathcal{G}z$.

Now, prove $\mathcal{F}z=\mathcal{F}\mathcal{F}z$. Assume $\mathcal{F}z\neq$, than utilizing part (ii), get,

$\mathcal{D}^*(\mathcal{F}z,\mathcal{F}z,\mathcal{F}\mathcal{F}z)\leq k\mathcal{D}^*(\mathcal{G}z,\mathcal{G}z,\mathcal{G}\mathcal{F}z)$ *and* $\mathcal{D}^*(\mathcal{F}z,\mathcal{F}z,\mathcal{F}\mathcal{F}z)\leq k\mathcal{D}^*(\mathcal{F}z,\mathcal{F}z,\mathcal{F}\mathcal{F}z)$, which

is contraction since $k\in[0,1)$. Hence, $\mathcal{F}z=\mathcal{F}\mathcal{F}z=\mathcal{G}\mathcal{F}z$ *and* $\mathcal{F}z$ is Com. Fix. P. of $\mathcal{F},\mathcal{G}$.

Now, presume $\lim_{n\to\infty}\mathcal{F}\mathcal{G}x_n=\mathcal{F}z$. Then, $\mathcal{F}(\mathcal{X})\subseteq\mathcal{G}(\mathcal{X})\Longrightarrow\mathcal{F}z=\mathcal{G}p$ for some$p\in\mathcal{X}$ so, $\lim_{n\to\infty}\mathcal{F}\mathcal{G}x_n=\mathcal{G}p$. Also, a compatibility of $\mathcal{F}$, $\mathcal{G}$ implies to, $\lim_{n\to\infty}\mathcal{G}\mathcal{F}x_n=\mathcal{G}p$. By hypothesis of theorem (4.2), we get $\lim_{n\to\infty}\mathcal{F}\mathcal{G}x_{n+1}=\lim_{n\to\infty}\mathcal{F}\mathcal{F}x_n=\mathcal{G}p$.

Utilizing part (ii), obtain $\mathcal{D}^*(\mathcal{F}p,\mathcal{F}p,\mathcal{F}\mathcal{F}x_n)\leq k\,\mathcal{D}^*(\mathcal{G}p,\mathcal{G}p,\mathcal{G}\mathcal{F}x_n)$.

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On both sides, lettingn $\to \infty$, we have $\mathcal{D}^*(\mathcal{F}p,\mathcal{F}p,\mathcal{G}p) \le k\,\mathcal{D}^*(\mathcal{G}p,\mathcal{G}p,\mathcal{G}p) = 0$, this gives, $\mathcal{F}p = \mathcal{G}p$, as well compatibility of $\mathcal{F},\mathcal{G} \implies \mathcal{F}\mathcal{G}p = \mathcal{G}\mathcal{G}p = \mathcal{F}\mathcal{F}p = \mathcal{G}\mathcal{F}p$.

Eventually, prove $\mathcal{F}p = \mathcal{F}\mathcal{F}p$. Presume, $\mathcal{F}p \ne \mathcal{F}\mathcal{F}p$, then utilizing part (ii), we obtain

$\mathcal{D}^*(\mathcal{F}p,\mathcal{F}p,\mathcal{F}\mathcal{F}p) \le k\,\mathcal{D}^*(\mathcal{G}p,\mathcal{G}p,\mathcal{G}\mathcal{F}p)$ *and* $\mathcal{D}^*(\mathcal{F}p,\mathcal{F}p,\mathcal{F}\mathcal{F}p) \le k\,\mathcal{D}^*(\mathcal{F}p,\mathcal{F}p,\mathcal{F}\mathcal{F}p)$,

Once more which gives a contradiction, since $k \in [0,1)$. consequently $\mathcal{F}p = \mathcal{F}\mathcal{F}p = \mathcal{G}\mathcal{F}p$, thus, $\mathcal{F}p$ is Comm. Fix. of $\mathcal{F},\mathcal{G}$.

Now, assume $\mathcal{F},\mathcal{G}$ are $\mathbb{R}-$ weak commutatively of type$(\mathcal{A}_\mathcal{G})$. Weak reciprocal-cont. of $\mathcal{F},\mathcal{G} \implies \lim_{n\to\infty}\mathcal{F}\mathcal{G}x_n = \mathcal{F}z$ or $\lim_{n\to\infty}\mathcal{G}\mathcal{F}x_n = \mathcal{G}z$. Let us first suppose $\lim_{n\to\infty}\mathcal{G}\mathcal{F}x_n = \mathcal{G}z$. So, $\mathbb{R}-$ *weak* commutatively type$(\mathcal{A}_\mathcal{G})$ of $\mathcal{F},\mathcal{G} \implies$ to,

$\mathcal{D}^*(\mathcal{F}\mathcal{F}x_n,\mathcal{F}\mathcal{F}x_n,\mathcal{G}\mathcal{F}x_n) \le \mathbb{R}\ \mathcal{D}^*(\mathcal{F}x_n,\mathcal{F}x_n,\mathcal{G}x_n)$ where $\mathbb{R} > 0$. On both sides, when $n\to\infty$, we obtain $\mathcal{D}^*\left(\lim_{n\to\infty}\mathcal{F}\mathcal{F}x_n,\mathcal{F}\mathcal{F}x_n,\mathcal{G}z\right) \le \mathbb{R}\,\mathcal{D}^*(z,z,z) = 0$. This gives, $\lim_{n\to\infty}\mathcal{F}\mathcal{F}x_n = \mathcal{G}z$, utilizing part (ii), obtain: $\mathcal{D}^*(\mathcal{F}z,\mathcal{F}z,\mathcal{F}\mathcal{F}x_n) \le k\,\mathcal{D}^*(\mathcal{G}z,\mathcal{G}z,\mathcal{G}\mathcal{F}x_n)$

On both sides, when $n\to\infty$, $\mathcal{D}^*(\mathcal{F}z,\mathcal{F}z,\mathcal{G}z) \le k\mathcal{D}^*(\mathcal{G}z,\mathcal{G}z,\mathcal{G}z) = 0$.

Therefore, obtain $\mathcal{F}z = \mathcal{G}z$. Once more utilizing $\mathbb{R}-$ weak commutatively type$(\mathcal{A}_\mathcal{G})$,

$\mathcal{D}^*(\mathcal{F}\mathcal{F}z,\mathcal{F}\mathcal{F}z,\mathcal{G}\mathcal{F}z) \le \mathbb{R}\mathcal{D}^*(\mathcal{F}z,\mathcal{F}z,\mathcal{G}z) = 0$.

This implies $\mathcal{F}\mathcal{F}z = \mathcal{G}\mathcal{F}z$. consequently, $\mathcal{F}\mathcal{F}z = \mathcal{F}\mathcal{G}z = \mathcal{G}\mathcal{F}z = \mathcal{G}\mathcal{G}z$.

Now, establish $\mathcal{F}z = \mathcal{F}\mathcal{F}z$. Assume $\mathcal{F}z \ne \mathcal{F}\mathcal{F}z$, utilizing part$(ii)$, obtain:

$\mathcal{D}^*(\mathcal{F}z,\mathcal{F}z,\mathcal{F}\mathcal{F}z) \le k\mathcal{D}^*(\mathcal{G}z,\mathcal{G}z,\mathcal{G}\mathcal{F}z)$*and* $\mathcal{D}^*(\mathcal{F}z,\mathcal{F}z,\mathcal{F}\mathcal{F}z) \le k\mathcal{D}^*(\mathcal{F}z,\mathcal{F}z,\mathcal{F}\mathcal{F}z)$ *is contradiction.* Thus, $\mathcal{F}z = \mathcal{F}\mathcal{F}z = \mathcal{G}\mathcal{F}z$ and $\mathcal{F}z$ is Comm. Fix. of $\mathcal{F},\mathcal{G}$, further by same means, we can establish second case if, $\lim_{n\to\infty}\mathcal{F}\mathcal{G}x_n = \mathcal{F}z$.

In addition, if $\mathcal{F},\mathcal{G}$ $\mathbb{R}-$ (W.Com. Maps) of type$(\mathcal{A}_\mathcal{F})$, then via following the similar steps as introduced above, readily be verified $\mathcal{F}z$ is (Comm. Fix. P.) of $\mathcal{F},\mathcal{G}$.

Eventually, presume$\mathcal{F},\mathcal{G}$ $\mathbb{R}-$(W.Com.Map) type$(\mathcal{P})$ so weakreciprocal $\mathcal{F}$ and cont. of $\mathcal{G} \implies \lim_{n\to\infty}\mathcal{F}\mathcal{G}x_n = \mathcal{F}z$ *or* $\lim_{n\to\infty}\mathcal{G}\mathcal{F}x_n = \mathcal{G}z$.

Let us suppose $\lim_{n\to\infty}\mathcal{G}\mathcal{F}x_n = \mathcal{G}z$, since $(\mathcal{F},\mathcal{G})$ $\mathbb{R}-$ (W.Com. Maps) type(P), therefore $\mathcal{D}^*(\mathcal{F}\mathcal{F}x_n,\mathcal{F}\mathcal{F}x_n,\mathcal{G}\mathcal{G}x_n) \le \mathbb{R}\ \mathcal{D}^*(\mathcal{F}x_n,\mathcal{F}x_n,\mathcal{G}x_n)$ where $\mathbb{R} > 0$.

On both sides, when $n\to\infty$, we obtain $\mathcal{D}^*\left(\lim_{n\to\infty}\mathcal{F}\mathcal{F}x_n,\lim_{n\to\infty}\mathcal{F}\mathcal{F}x_n,\lim_{n\to\infty}\mathcal{G}\mathcal{G}x_n\right) \le \mathbb{R}\ \mathcal{D}^*(z,z,z) = 0$. This gives, $\mathcal{D}^*\left(\lim_{n\to\infty}\mathcal{F}\mathcal{F}x_n,\lim_{n\to\infty}\mathcal{F}\mathcal{F}x_n,\lim_{n\to\infty}\mathcal{G}\mathcal{G}x_n\right) = 0$.

Utilizing part(i), and (4.2.1), we have $\mathcal{G}\mathcal{F}x_{n-1} = \mathcal{G}\mathcal{G}x_n \to \mathcal{G}z$ as $n\to\infty$, this gives, $\mathcal{F}\mathcal{F}x_n \to \mathcal{G}z$ as $n\to\infty$. In addition, by part(ii), $\mathcal{D}^*(\mathcal{F}z,\mathcal{F}z,\mathcal{F}\mathcal{F}x_n) \le k\,\mathcal{D}^*(\mathcal{G}z,\mathcal{G}z,\mathcal{G}\mathcal{F}x_n)$.

On both sides, When $n\to\infty$, we obtain $\mathcal{D}^*(\mathcal{F}z,\mathcal{F}z,\mathcal{G}z) \le k\,\mathcal{D}^*(\mathcal{G}z,\mathcal{G}z,\mathcal{G}z) = 0$. This implies that $\mathcal{F}z = \mathcal{G}z$. once more, by $\mathbb{R}-$(W.Com.Map) type$(\mathcal{P})$ get

$\mathcal{D}^*(\mathcal{F}\mathcal{F}z,\mathcal{F}\mathcal{F}z,\mathcal{G}\mathcal{G}z) \le \mathbb{R}\,\mathcal{D}^*(\mathcal{F}z,\mathcal{F}z,\mathcal{G}z) = 0$, Where $\mathbb{R} > 0$.

This implies $\mathcal{F}\mathcal{F}z = \mathcal{G}\mathcal{G}z \implies \mathcal{F}\mathcal{F}z = \mathcal{F}\mathcal{G}z = \mathcal{G}\mathcal{F}z = \mathcal{G}\mathcal{G}z$.

Finally, prove $\mathcal{F}z = \mathcal{F}\mathcal{F}z$. Assume $\mathcal{F}z \ne \mathcal{F}\mathcal{F}z$ so utilizing part(ii),:

$\mathcal{D}^*(\mathcal{F}z,\mathcal{F}z,\mathcal{F}\mathcal{F}z) \le k\mathcal{D}^*(\mathcal{G}z,\mathcal{G}z,\mathcal{G}\mathcal{F}z)$ and $\mathcal{D}^*(\mathcal{F}z,\mathcal{F}z,\mathcal{F}\mathcal{F}z) \le k\mathcal{D}^*(\mathcal{F}z,\mathcal{F}z,\mathcal{F}\mathcal{F}z)$, which is a contradiction. Hence, $\mathcal{F}z = \mathcal{F}\mathcal{F}z$, therefore $\mathcal{F}z = \mathcal{F}\mathcal{F}z = \mathcal{G}\mathcal{F}z$ and $\mathcal{F}z$ Comm. Fix. of $\mathcal{F},\mathcal{G}$. This result holds good even if $\lim_{n\to\infty}\mathcal{F}\mathcal{G}x_n = \mathcal{F}z$ is considered instead of $\lim_{n\to\infty}\mathcal{G}\mathcal{F}x_n = \mathcal{G}z$.

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Uniqueness of the Comm. Fix. P. in each of the three types of maps can readily be obtained utilizing part(ii).

We present the following example to explain validity of Theo-(4.2).

Example4.3: Let$(\mathcal{X},\mathcal{D}^*)$ $\mathcal{D}^*$ −metric-sp (s. t) $\mathcal{X} = [2,20]$ $\forall x, y, z \in \mathcal{X}$,

$\mathcal{D}^*(x, y, z) = (|x - y| + |y - z| + |z - x|)$. Define $\mathcal{F},\mathcal{G}: (\mathcal{X},\mathcal{D}^*) \to (\mathcal{X},\mathcal{D}^*)$ as the following

$$\mathcal{F}(x) = \begin{cases} 2, & if\ x = 2 \text{ or } x > 5 \\ 6, & if\ 2 < x \leq 5 \end{cases} \quad and \quad \mathcal{G}(x) = \begin{cases} 2, & if \quad x = 2 \\ 12, & if \quad 2 < x \leq 5 \\ \frac{x+1}{3} & if \quad x > 5. \end{cases}$$

It can be readily established:

$(\boldsymbol{i})\mathcal{F}(\mathcal{X}) \subseteq \mathcal{G}(\mathcal{X})$;

$(\boldsymbol{ii})\mathcal{F},\mathcal{G}$satisfies condition$(ii)$ of Theo-(4.2);

$(\boldsymbol{iii})\mathcal{F},\mathcal{G}$ are$\mathbb{R}$ −(W.Com.Map) type$(\mathcal{A}_{\mathcal{G}})$:

$(\boldsymbol{iv})\mathcal{F},\mathcal{G}$ are weakly reciprocally cont. for $\{x_n\} = \{2\}$ or $\{x_n\} = \left\{\frac{5n+1}{n}\right\}$ $\forall\ n$ in$\mathcal{X}$. So, $\mathcal{F}$, $\mathcal{G}$ satisfy all the conditions of Theo-(4.2) and have unique Comm. Fix. P. at $x = 2$.

Verify following Comm. Fix. P. Theo. for non-compatible pair of self-maps in$\mathcal{D}^*$ − metric-sp:

Theorem4.4: Let$\mathcal{F},\mathcal{G}$ weaklyreciprocally cont. non-compatible self-map of $(\mathcal{X},\mathcal{D}^*)$ satisfying the following properties:

$(\boldsymbol{i})\mathcal{F}(\mathcal{X}) \subseteq \mathcal{G}(\mathcal{X})$;

$(\boldsymbol{ii})\mathcal{D}^*(\mathcal{F}\mathcal{X},\mathcal{F}y,\mathcal{F}z) \leq k\mathcal{D}^*(\mathcal{G}\mathcal{X},\mathcal{G}y,\mathcal{G}z), \forall\ k \geq 0, x, y, z \in \mathcal{X}$;

$(\boldsymbol{iii})\mathcal{D}^*(\mathcal{F}\mathcal{X},\mathcal{F}y,\mathcal{F}z) <$

$$max\left\{\begin{matrix} \mathcal{D}^*(\mathcal{G}\mathcal{X},\mathcal{G}\mathcal{X},\mathcal{G}\mathcal{F}\mathcal{X}), \mathcal{D}^*(\mathcal{F}\mathcal{X},\mathcal{F}\mathcal{X},\mathcal{G}\mathcal{X}), \mathcal{D}^*(\mathcal{F}\mathcal{F}\mathcal{X},\mathcal{F}\mathcal{F}\mathcal{X},\mathcal{G}\mathcal{F}\mathcal{X}) \\ \mathcal{D}^*(\mathcal{F}\mathcal{X},\mathcal{F}\mathcal{X},\mathcal{G}\mathcal{F}\mathcal{X}),\ \mathcal{D}^*(\mathcal{G}\mathcal{X},\mathcal{G}\mathcal{X},\mathcal{F}\mathcal{F}\mathcal{X}) \end{matrix}\right\}$$

$\forall\ x \in \mathcal{X}$, Supplied the right hand side is non-zero. If $\mathcal{F},\mathcal{G}$ are$\mathbb{R}$ −(W.Com.Map) type$(\mathcal{A}_{\mathcal{G}})$ or $\mathbb{R}$ −(W.Com.Map) type$(\mathcal{A}_{\mathcal{F}})$, then $\mathcal{F},\mathcal{G}$have Comm. Fix. P.

Proof: Given$\mathcal{F},\mathcal{G}$ noncompatible maps, consequently $\exists$ at least one seq. $\{x_n\}$ (s. t):

$\lim_{n\to\infty}\mathcal{F}x_n = \lim_{n\to\infty}\mathcal{G}x_n = z \in \mathcal{X}$, But either

$\lim_{n\to\infty} \mathcal{D}^*(\mathcal{F}\mathcal{G}x_n,\mathcal{F}\mathcal{G}x_n,\mathcal{G}\mathcal{F}x_n) \neq 0$, $\lim_{n\to\infty} \mathcal{D}^*(\mathcal{G}\mathcal{F}x_n,\mathcal{G}\mathcal{F}x_n,\mathcal{F}\mathcal{G}x_n) \neq 0$ or non-existent. By part

$(\boldsymbol{i})$ for each x_n, $\exists\ y_n \in \mathcal{X}$ (s. t) $\mathcal{F}x_n = \mathcal{G}y_n$. Thus, $\lim_{n\to\infty}\mathcal{F}x_n = \lim_{n\to\infty}\mathcal{G}x_n \lim_{n\to\infty}\mathcal{G}y_n = z$.

Utilizing part $(\boldsymbol{ii})$, obtain $\mathcal{D}^*(\mathcal{F}x_n,\mathcal{F}x_n,\mathcal{F}y_n) \leq k\mathcal{D}^*(\mathcal{G}x_n,\mathcal{G}y_n,\mathcal{G}y_n)$ and

$\mathcal{D}^*\left(z, \lim_{n\to\infty}\mathcal{F}y_n, \lim_{n\to\infty}\mathcal{F}y_n\right) \leq k\ \mathcal{D}^*(z,z,z) = 0$. This gives:

$\lim_{n\to\infty}\mathcal{F}y_n = z$. Therefore, $\lim_{n\to\infty}\mathcal{F}x_n = \lim_{n\to\infty}\mathcal{G}x_n = \lim_{n\to\infty}\mathcal{G}y_n = \lim_{n\to\infty}\mathcal{F}y_n = z$.

Assume $\mathcal{F}$, $\mathcal{G}$ are$\mathbb{R}$ −(W.Com.Map) type$(\mathcal{A}_{\mathcal{G}})$. So, via weak reciprocal continuity of $\mathcal{F}$, $\mathcal{G}$ we have $\lim_{n\to\infty}\mathcal{F}\mathcal{G}x_n = \mathcal{F}z$ or $\lim_{n\to\infty}\mathcal{G}\mathcal{F}x_n = \mathcal{G}z$. by same method, get $\lim_{n\to\infty}\mathcal{F}\mathcal{G}x_n = \mathcal{F}z$ or $\lim_{n\to\infty}\mathcal{G}\mathcal{F}y_n = \mathcal{G}z$.

Now, first suppose $\lim_{n\to\infty}\mathcal{G}\mathcal{F}y_n = \mathcal{G}z$. Then $\mathbb{R}$ −(W.Com.Map) type$(\mathcal{A}_{\mathcal{G}})$ of $\mathcal{F},\mathcal{G}$ $\Longrightarrow$ that $\mathcal{D}^*(\mathcal{F}\mathcal{F}y_n,\mathcal{F}\mathcal{F}y_n,\mathcal{G}\mathcal{F}y_n) \leq$ $\mathbb{R}$ $\mathcal{D}^*(\mathcal{F}y_n,\mathcal{F}y_n,\mathcal{G}y_n)$ and

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$\mathcal{D}^*\left(\lim_{n\to\infty}\mathcal{F}\mathcal{F}y_n, \lim_{n\to\infty}\mathcal{F}\mathcal{F}y_n, \mathcal{G}z\right) \leq \mathbb{R}\ \mathcal{D}^*(z,z,z) = 0$, where $\mathbb{R} > 0$. This gives, $\lim_{n\to\infty}\mathcal{F}\mathcal{F}y_n = \mathcal{G}z$, and via part (ii), obtain: $\mathcal{D}^*(\mathcal{F}\mathcal{F}y_n, \mathcal{F}\mathcal{F}y_n, \mathcal{F}z) \leq k\ \mathcal{D}^*(\mathcal{G}\mathcal{F}y_n, \mathcal{G}z, \mathcal{G}z)$.

On both sides, When $n \to \infty$, get$\mathcal{D}^*(\mathcal{G}z, \mathcal{G}z, \mathcal{F}z) \leq k\ \mathcal{D}^*(\mathcal{G}z, \mathcal{G}z, \mathcal{G}z) = 0$.

This $\Longrightarrow \mathcal{F}z = \mathcal{G}z$, once more by means of $\mathbb{R}$ −(W.Com.Map) of type $(\mathcal{A}_{\mathcal{G}})$,:

$\mathcal{D}^*(\mathcal{F}\mathcal{F}z, \mathcal{F}\mathcal{F}z, \mathcal{G}\mathcal{F}z) \leq \mathbb{R}\mathcal{D}^*(\mathcal{F}z, \mathcal{F}z, \mathcal{G}z) = 0$.

This $\Longrightarrow$ $\mathcal{F}\mathcal{F}z = \mathcal{G}\mathcal{F}z$ and $\mathcal{F}\mathcal{F}z = \mathcal{F}\mathcal{G}z = \mathcal{G}\mathcal{F}z = \mathcal{G}\mathcal{G}z$. If $\mathcal{F}z \neq \mathcal{F}\mathcal{F}z$, so via part$(\boldsymbol{ii})$ obtain,

$$\mathcal{D}^*(\mathcal{F}z, \mathcal{F}z, \mathcal{F}\mathcal{F}z) \leq max\begin{Bmatrix} \mathcal{D}^*(\mathcal{G}z, \mathcal{G}z, \mathcal{G}\mathcal{F}z),\ \mathcal{D}^*(\mathcal{F}z, \mathcal{F}z, \mathcal{G}z),\ \mathcal{D}^*(\mathcal{F}\mathcal{F}z, \mathcal{F}\mathcal{F}z, \mathcal{G}\mathcal{F}z), \\ \mathcal{D}^*(\mathcal{F}z, \mathcal{F}z, \mathcal{G}\mathcal{F}z),\ \mathcal{D}^*(\mathcal{G}z, \mathcal{G}z, \mathcal{F}\mathcal{F}z) \end{Bmatrix}$$

$$= max\begin{Bmatrix} \mathcal{D}^*(\mathcal{F}z, \mathcal{F}z, \mathcal{F}\mathcal{F}z),\ \mathcal{D}^*(\mathcal{F}z, \mathcal{F}z, \mathcal{F}z),\ \mathcal{D}^*(\mathcal{F}\mathcal{F}z, \mathcal{F}\mathcal{F}z, \mathcal{F}\mathcal{F}z), \\ \mathcal{D}^*(\mathcal{F}z, \mathcal{F}z, \mathcal{F}\mathcal{F}z),\ \mathcal{D}^*(\mathcal{F}z, \mathcal{F}z, \mathcal{F}\mathcal{F}z) \end{Bmatrix}\ and$$

$$\mathcal{D}^*(\mathcal{F}z, \mathcal{F}z, \mathcal{F}\mathcal{F}z) < \mathcal{D}^*(\mathcal{F}z, \mathcal{F}z, \mathcal{F}\mathcal{F}z),$$

This contraction. Consequently,$\mathcal{F}z = \mathcal{F}\mathcal{F}z = \mathcal{G}\mathcal{F}z$,$\mathcal{F}z$ Comm. Fix. P. of $\mathcal{F}$, $\mathcal{G}$. Result holds well even if $\lim_{n\to\infty}\mathcal{F}\mathcal{G}y_n = \mathcal{F}z$ is considered instead of $\lim_{n\to\infty}\mathcal{G}\mathcal{F}y_n = \mathcal{G}z$. If $\mathcal{F}, \mathcal{G}$ are$\mathbb{R}$ − weakly comuting type$(\mathcal{A}_{\mathcal{F}})$,then proof following similar lines.

Next, introduce following example to illustrate validity Theo-(4.4).

Example4.5: Let$(\mathcal{X}, \mathcal{D}^*)$ be $\mathcal{D}^*$ −metric-sp, (s. t) $\mathcal{X} = [2,20]$ $\forall x, y, z \in \mathcal{X}$,

$\mathcal{D}^*(x, y, z) = (|x - y| + |y - z| + |z - x|)$. Define $\mathcal{F}, \mathcal{G}: (\mathcal{X}, \mathcal{D}^*) \to (\mathcal{X}, \mathcal{D}^*)$ as:

$$\mathcal{F}x = \begin{cases} 2, & if\ x = 2 \text{ or } x > 5 \\ 6, & if\ 2 < x \leq 5 \end{cases} \quad and \quad \mathcal{G}x = \begin{cases} 2, & if\ x = 2 \\ 11, & if\ \ 2 < x \leq 5 \\ \frac{x+1}{3} & if\ \ x > 5. \end{cases}$$

Suppose $\{x_n\}$in $\mathcal{X}$ (s. t) either $\{x_n\} = \{2\}$ *or* $\{x_n\} = \left\{\frac{5n+1}{n}\right\}$ $\forall$ n. obviously, $\mathcal{F}, \mathcal{G}$ satisfy properties of Theo-(4.4) $x = 2$ is Comm. Fix. P.

CONCLUSION

The metric fixed point theory is very important and useful in Mathematics; it can be applied in various areas, for instant, variational inequalities, optimization, and approximation theory . Therefore, new (Comm. Fix. P. Theo) in complete$\mathcal{D}^*$ − metric-sp by ideas of compatibility, weakly commutativity, and weakly reciprocal continuity are all R variations have been introduced and studied. Several fundamental properties related to commuting mappings in complete $\mathcal{D}^*$ −metric-sp have been discussed. Furthermore, the results presented in this manuscript have been improved *and* sharpened many related results present in the existing literature.

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