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On a new application of summation of Jacobi Series by B-Method

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ABSTRACT:
In this research work, a general theorem on a new application of summation of Jacobi series by B-method or Borel method has been established. The main theorem includes as special cases, a number of well known results. Several new results can also be deduced from it. The result is an extension of classical results of [1].

Keywords: Summability, Fourier series, Borel methods
MSC: 42C10, 35A25, 40G10

1. Introduction:

Summability transformation help us to generalize the concept of limit of a sequence or series and thus provide us a method to assign limits even to sequences which are divergent. Thus transformations or methods can be classified into two ways:
(i) sequence to sequence transformations.
(ii) sequence to function transformations (sec [3]).
Knopp [4] has defined the generalized Borel summability. Stone [5] was the first mathematician to apply Borel summability to Fourier series. Sinvhal [6] has applied this method to the conjugate series of the derived Fourier series. Recently Sahney [1] considered the Borel summability of the series.

Let $\upsilon(z)\varepsilon L(-1,1)$ and the integral

$\int_{-1}^{1}(1-z)^{\lambda}\ (1+z)^{\varepsilon}\ \upsilon(z)dz,\ \lambda>-1,\ \in>-1$ exists.

The Jacobi series associated with the function $\upsilon(z)$ is

$$\upsilon(z) \sim \sum_{l=0}^{\infty}\alpha_l Q_l^{(\lambda,\varepsilon)}(z) \qquad \ldots \qquad (1)$$

where $\alpha_l = \dfrac{2l+\lambda+\varepsilon+1}{2^{\lambda+\varepsilon+1}}\cdot\dfrac{\Gamma(l+1)}{\Gamma(l+\lambda+1)}\cdot\dfrac{\Gamma(l+\lambda+\varepsilon+1)}{\Gamma(l+\varepsilon+1)}.$

$$\int_{-1}^{1}(1-i)^{\lambda}(1+u)^{\varepsilon}Q_l^{(\lambda,\varepsilon)}(u)\upsilon(u)du \qquad \ldots \qquad (2)$$

where $Q_l^{(\lambda,\,\varepsilon)}(z)$ are Jacobi polynomials of degree l and order $(\lambda,\ \varepsilon)$.

2. Main Result:

The object of this research paper is apply the Borel's method of summability to Jacobi series at an interior point of the interval (–1, 1).
The following theorem is an extension of Sahney's result [1] for Fourier series.

Theorem : If $\upsilon(z)\varepsilon L(-1,1)$,

$$\chi(v)=\int_0^v|\eta(x)|\ dx = O\left(\frac{v}{\log\frac{1}{v}}\right),\ v\to+0. \qquad \ldots \qquad (3)$$

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where $\eta(x)=\chi(z\pm x)-\chi(z)$ … (4)

then the Jacobi series (1) is summable (B) to the sum $\chi(z)$ at a point z interior to the open interval (–1, 1).

For the proof of the theorem, the following lemma is needed.

Lemma: Under the condition (3), we have

$$\chi_1(v)\equiv\int_0^v\left|\eta_1\left(u^1\right)\right|du^1=0\left(\frac{v}{\log\frac{1}{v}}\right), v\to+0 \quad \ldots \quad (5)$$

Where $\eta_1\left(u^1\right)=\chi\left\{\cos\left(\theta-u^1\right)\right\}-\chi(\cos\theta)$ … (6)

and $z=\cos\theta$

$z\pm x=\cos w$,

$\theta-w=u^1$,

$\lambda,\ \varepsilon\geq\frac{1}{2}$.

Proof of the theorem :

Let $\sigma_l(z)$ denote the l^{th} partial sum of the Jacobi series (1) is given by

$$\sigma_l(z)=\frac{2^{-\lambda-\varepsilon}}{2l+\lambda+\varepsilon+2}\ \frac{\Gamma(l+2)\ \Gamma(l+\lambda+\varepsilon+2)}{\Gamma(l+\lambda+1)\ \Gamma(l+\varepsilon+1)}\int_{-1}^{1}(1-u)^{\lambda}(1+u)^{\varepsilon}F_l^{(\lambda,\varepsilon)}(u,\ z)\upsilon(u)du$$

where

$$F_l^{(\lambda,\varepsilon)}(u,z)=\frac{Q_{l+1}^{(\lambda,\varepsilon)}(u)Q_l^{(\lambda,\varepsilon)}(z)-Q_l^{(\lambda,\varepsilon)}(u)Q_{l+1}^{(\lambda,\varepsilon)}(z)}{u-z}$$

and

$$G_l=\frac{2^{-\lambda-\varepsilon}}{2l+\lambda+\varepsilon+2}\ \frac{\Gamma(l+2)\Gamma(l+\lambda+\varepsilon+2)}{\Gamma(l+\lambda+1)\Gamma(l+\varepsilon+1)}\ \square\ 2^{-\propto-\beta-1}\{l+O(1)\}.$$

Taking $\chi(u)=1$ then we have

$$1=G_l\int_{-1}^{1}(1-u)^{\lambda}(1+u)^{\varepsilon}F_l^{(\lambda,\varepsilon)}(u,\ z)\ du$$

therefore

$$\sigma_l(z)-\zeta(z)=\int_{-1}^{1}(1-u)^{\lambda}(1+u)^{\varepsilon}F_l^{(\lambda,\varepsilon)}(u,\ z)K(u,\ z)du$$

where $K(u,z)=\zeta(u)-\zeta(z)$.

In view of the definition Borel method,
we have,

$$e^{-r}\sum_{l=0}^{\infty}\ \frac{r^l}{l!}\{\sigma_l(z)-\zeta(z)\}$$

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$$= e^{-r} \sum_{l=0}^{\infty} \frac{r^l}{l!} G_e \int_{-1}^{1} (1-u)^{\lambda} (1+u)^{\varepsilon} K(u, z) du$$

put $z = \cos\theta$ and $u = \cos w$, we obtain

$$e^{-r} \sum_{l=0}^{\infty} \frac{r^l}{l!} \{\sigma_l(\cos\theta) - \zeta(\cos\theta)\}$$

$$= e^{-r} \sum_{l=0}^{\infty} \frac{r^l}{l!} G_l \int_0^{\pi} (1-\cos w)^{\lambda} (1+\cos w)^{\varepsilon} K(\cos w, \cos\theta) F_l^{(\lambda,\varepsilon)}(\cos w, \cos\theta) \sin w \, dw$$

$$= e^{-\gamma} \sum_{l=0}^{\infty} \frac{\gamma^l}{l!} G_l \left(\int_0^{\gamma^{-1}} + \int_{\gamma^{-1}}^{\theta-\gamma^{-r}} + \int_{\theta-\gamma^{-1}}^{\theta+\gamma^{-\Delta}} + \int_{\theta+\gamma^{-\Delta}}^{\pi-\gamma^{-1}} + \int_{\pi-\gamma^{-1}}^{\pi} \right) (1-\cos w)^{\lambda} (1+\cos w)^{\varepsilon}$$

$$K(\cos w, \cos\theta) F_l^{(\lambda, \varepsilon)}(\cos w, \cos\theta) \sin w \, dw$$

$$= M_1 + M_2 + M_3 + M_4 + M_5 \qquad \ldots \qquad (7)$$

where $0 < \Delta < \frac{1}{2}$.

Let us consider the integral M_1 first
we have

$$M_1 = e^{-\gamma} \sum_{l=0}^{\infty} \frac{\gamma^l}{l!} G_l \int_0^{\gamma^{-1}} (1-\cos w)^{\lambda} (1+\cos w)^{\varepsilon} K(\cos w, \cos\theta) F_l^{(\lambda,\varepsilon)}(\cos w, \cos\theta) \sin w \, dw$$

$$= e^{-\gamma} \sum_{l=0}^{\infty} \frac{\gamma^l}{l!} G_l \int_0^{\gamma^{-1}} 2^{\lambda+\varepsilon} \left(\sin \frac{w}{2}\right)^{2\lambda} \left(\cos \frac{w}{2}\right)^{2\varepsilon} K(\cos w, \cos\theta) F_l^{(\lambda, \varepsilon)}(\cos w, \cos\theta) \sin w \, dw$$

$\leq |M_{11}| + |M_{12}|$ 'say'

where

$$M_{11} = e^{-\gamma} \sum_{l=0}^{\infty} \frac{\gamma^l 2^{\lambda+\varepsilon} G_l}{l!} \int_0^{\gamma^{-1}} \left(\sin \frac{w}{2}\right)^{2\lambda} \times$$

$$\left(\cos \frac{w}{2}\right)^{2\varepsilon} K(\cos w, \cos\theta) F_l^{(\lambda, \varepsilon)}(\cos w, \cos\theta) Q_{l+1}^{(\lambda, \varepsilon)}(\cos w) Q_l^{(\lambda, \varepsilon)}(\cos\theta) \sin w \, dw$$

and M_{12} is α similar expression with $Q_l^{(\lambda, \varepsilon)}(\cos w) Q_{l+1}^{(\lambda, \varepsilon)}(\cos\theta)$ in place of $Q_{l+1}^{(\lambda, \in)}(\cos w) Q_l^{(\lambda, \in)}(\cos\theta)$.

$$|M_{11}| = e^{-\gamma} \sum_{l=0}^{\infty} O(l) \int_0^{\gamma^{-1}} w^{2\lambda} |\zeta(\cos w) - \zeta(\cos\theta)| \sin w \, dw$$

$= e^{-\gamma} \sum_{l=0}^{\infty} O(l)\, O(l^{-1})$, using Lebesgue integral

$= O(1),\ l \to \infty,\ \gamma \to \infty$

similarly, $|M_{12}| = O(1)$.

Thus, $\quad M_1 = 0(1) \qquad \ldots \qquad (8)$

proceeding on the same lines as M_1,
we have

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$$M_5 = O(1) \qquad \ldots \qquad (9)$$

now, we consider M_2,

$$M_2 = e^{-\gamma} \sum_{l=0}^{\infty} \frac{\gamma^l}{l!} G_l \int_{\gamma^{-1}}^{\theta-\gamma^{-1}} (1-\cos w)^{\lambda} (1+\cos w)^{\varepsilon}$$

$$\left\{Q_{l+1}^{(\lambda,\varepsilon)}(\cos w)\, Q_l^{(\lambda,\varepsilon)}(\cos\theta) - Q_l^{(\lambda,\varepsilon)}(\cos w)\, Q_{l+1}^{(\lambda,\varepsilon)}(\cos\theta)\right\}$$

$$\frac{K(\cos w, \cos\theta)}{\cos w - \cos\theta} \sin w \; dw$$

$$= M_{21} - M_{22} \quad \text{(say)} \qquad \ldots \qquad (10)$$

$$M_{21} = \frac{e^{-\gamma}}{2^{\lambda+\varepsilon+1}} \int_{\gamma^{-1}}^{\theta-\gamma^{-1}} (1-\cos w)^{\lambda} (1+\cos w)^{\epsilon}$$

$$\frac{K(\cos w, \cos\theta)}{\cos w - \cos\theta} \sum_{l=0}^{\infty} \left[\frac{\{l+O(1)\}\gamma^l}{l!}\right] \left\{ l^{\frac{-1}{2}} L(\theta) \cos(N\theta+\alpha) + O\left(l^{\frac{-3}{2}}\right)\right\} \times$$

$$(l+1)^{\frac{-1}{2}} L(w) \left[\cos\{(N+1)w+2\} + \frac{O(1)}{(l+1)\sin w}\right] \sin w \; dw$$

$$= \frac{e^{-r}}{e^{\lambda+\varepsilon+1}} \sum_{l=0}^{\infty} \frac{\{l+O(1)\} r^l}{l!} \int_{r^{-1}}^{\theta-r^{-1}} (l-\cos w)^{\lambda} (1+\cos w)^{\varepsilon} \times$$

$$\frac{K(\cos w, \cos\theta)}{\cos w - \cos\theta} l^{\frac{-1}{2}} (l+1)^{\frac{-1}{2}} L(\theta) L(w) \cos(N\theta+\alpha) \left[\cos\{(N+1)w+\alpha\} + \frac{O(1)}{(l+1)\sin w}\right] \times$$

$$\sin w \, dw + O(1)$$

$$= \frac{e^{-r}}{2^{\lambda+\varepsilon+1}} \left(\sum_{l=0}^{\infty} \frac{l\, r^l}{l! \sqrt{l(l+1)}} \left[\int_{r^{-1}}^{\theta-r^{-l}} (1-\cos w)^{\lambda} (1+\cos w)^{\lambda} \frac{K(\cos w, \cos\theta)}{\cos w - \cos\theta} L(\theta) L(w) \times \right.\right.$$

$$\cos(N\theta+\alpha) \cos\{(N+1)w+\alpha\} \sin w \, dw$$

$$\left. + \int_{r^{-1}}^{\theta-r^{-l}} (1-\cos w)^{\lambda} (1+\cos w)^{r} \frac{K(\cos w, \cos\theta)}{\cos w - \cos\theta} L(\theta) L(w) \cos(N\theta+2) \frac{O(1)}{l+1} dw \right] +$$

$$\left. \sum_{l=0}^{\infty} \frac{0(1).\gamma'}{l.l!} \int_{\gamma^{-1}}^{\theta-\gamma^{-1}} \frac{|\zeta(\cos w) - \zeta(\cos\theta)|}{|\cos w - \cos\theta|} dw + \sum_{l=0}^{\infty} \frac{O(1)\gamma'}{l(l+1)l!} \int_{\gamma^{-1}}^{\theta-\gamma^{-1}} \frac{|\zeta(\cos w) - \zeta(\cos\theta)|}{|\cos w - \cos\theta|} dw \right)$$

$= J_1 + J_2 + J_3 + J_4$, say.

Now, we observe that

$J_2 = O(1),\ J_3 = O(1)$ and $J_4 = O(1)$

we have

$$J_1 = \frac{e^{-\gamma}}{2^{\lambda+\varepsilon+1}} \sum_{l=0}^{\infty} \frac{\gamma'}{l!} \int_{\gamma^{-1}}^{\theta-\gamma^{-1}} (1-\cos w)^{\lambda} (1+\cos w)^{\varepsilon} \frac{\zeta(\cos w) - \zeta(\cos\theta)}{\cos w - \cos\theta} L(\theta)\, L(w) \times$$

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$$\left[\cos\left\{N(\theta+w)+w+2\alpha\right\}+\cos\left\{N(\theta-w)+w\right\}\right]\sin w\, dw$$

$= J_{11} + J_{12}$ say,

where

$$J_{11}=\frac{e^{-\gamma}}{2^{\lambda+\varepsilon+1}}\sum_{l=0}^{\infty}\frac{\gamma'}{l!}\int_{\gamma^{-1}}^{\theta-\gamma^{-1}}(1-\cos w)^{\lambda}\,(1+\cos w)^{\varepsilon}\times$$

$$\frac{\zeta(\cos w)-\zeta(\cos\theta)}{\cos w-\cos\theta}\,L(\theta)\,L(w)\,\cos\left\{N(\theta+w)+w+2\alpha\right\}\sin w\, dw$$

and

$$J_{12}=\frac{e^{-\gamma}}{2^{\lambda+\varepsilon+1}}\int_{\gamma^{-1}}^{\theta-\gamma^{-1}}(1-\cos w)^{\lambda}\,(1+\cos w)^{\varepsilon}\times$$

$$\frac{\zeta(\cos w)-\zeta(\cos\theta)}{\cos w-\cos\theta}\sum_{l=0}^{\infty}\gamma':\frac{\cos\left\{N(w-\theta)+w\right\}}{l\,!}\sin w\, dw$$

substituting the value of N *i.e.*

$l+\dfrac{\lambda+\varepsilon+1}{2}$ and simplifying, we obtain

$$J_{12}=\frac{e^{-\gamma}}{2^{\lambda+\varepsilon+1}}\int_{\gamma^{-1}}^{\theta-\gamma^{-1}}(1-\cos w)^{\lambda}\,(1+\cos w)^{\varepsilon}\times$$

$$\frac{\zeta(\cos w)-\zeta(\cos\theta)}{\cos w-\cos\theta}\sin w\left[\sum_{l=0}^{\infty}\frac{\gamma'\cos\left\{(\theta-w)l\right\}}{l\,!}\right]\times$$

$$\cos\left\{w-\frac{\lambda+\varepsilon+1}{2}(\theta-w)\right\}+\sum_{l=0}^{\infty}\frac{\gamma'\sin\left\{l(\theta-w)\right\}}{l!}\sin\left\{w-\frac{\lambda+\varepsilon+1}{2}(\theta-w)\right\}$$

$= J_{1,2,1} + J_{1,2,2}$ 'say'

since, we have

$$\sum_{l=0}^{\infty}\frac{\gamma'\sin u' l}{l}=e^{\gamma\cos u'}\sin(\gamma\sin u'),$$

$$e^{-\gamma}\sum_{l=0}^{\infty}\frac{\gamma'\sin lu'}{l!}=\frac{\sin(\gamma\sin u')}{e^{\{\gamma(1-\cos u')\}}}$$

and

$$e^{-\gamma}\sum_{l=0}^{\infty}\frac{\gamma'\cos lu'}{l\,!}=e^{\gamma\cos u'}\cos\left\{u'+\gamma\sin u'\right\}$$

therefore, we have

$$J_{1,2,2}=O(1)\int_{\gamma^{-1}}^{\theta-\gamma^{-1}}\left|\eta_1(u')\right|\sin(\gamma\sin u')\,du',\ \text{where}\ \gamma^{-1}<\delta<\theta-\gamma'.$$

$$=O(1)\left\{\frac{\gamma}{e^{\frac{1}{\gamma}}}\right\}\int_{\gamma^{-1}}^{\delta}\left|\eta_1(u')\right|\,du'$$

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$$=O(1)\,O\left(\frac{1}{\gamma}\right)O(1).$$

Thus, $J_{1,2,2}=O(1)$ as $\gamma\to\infty$ and $0<\gamma^{-1}<\delta$.

Similarly, we have

$$J_{1,2,1}=O(1)$$

$$\therefore\quad J_{12}=O(1).$$

In the same way, we obtain

$$J_{11}=O(1).$$

Thus, $J_1=O(1)$

consequently, $M_{21}=O(1)$,

$$M_{22}=O(1)$$

and $M_2=O(1)$.

Also, we have, as M_2

$$M_4=O(1) \qquad \ldots \qquad (11)$$

now,

$$M_3=e^{-\gamma}\sum_{l=0}^{\infty}\frac{\gamma' G_l}{l\,!}\int_{\theta-\gamma^{-1}}^{\theta+\gamma^{-1}}(1-\cos w)^{\lambda}(1+\cos w)^{\varepsilon}\times$$

$$\sin w\left\{\zeta(\cos w)-\zeta(\cos\theta)\right\}F_l^{(\lambda,\,\varepsilon)}(\cos w,\,\cos\theta)\,dw$$

using the result to [2], we have

$$M_3=\frac{e^{-\gamma}}{2^{\lambda+\varepsilon+2}}\sum_{l=0}^{\infty}\frac{\gamma'}{l\,!}\int_{\theta-\gamma^{-1}}^{\theta+\gamma^{-1}}(1-\cos w)^{\lambda}\,(1+\cos w)^{\varepsilon}\,\left\{\zeta(\cos w)-\zeta(\cos\theta)\right\}\times$$

$$L(\theta)\,L(w)\left[\frac{\sin\{\left(N+\frac{1}{2}\right)(\theta+w)2\alpha}{\sin\frac{\theta+w}{2}}+\frac{\sin\{\left(N+\frac{1}{2}\right)(\theta-w)}{\sin\frac{\theta-w}{2}}+O(1)\right]\sin w\,dw$$

$$=M_{31}+M_{32}+M_{33}\qquad \text{‘say’}$$

now, first we take up M_{32}, we have

$$M_{32}=\frac{L(\theta)e^{-\gamma}}{2^{\lambda+\varepsilon+2}}\int_{\theta-\gamma^{-1}}^{\theta+\gamma^{-1}}(1-\cos w)^{\lambda}(1+\cos w)^{\varepsilon}\left\{\zeta(\cos w)-\zeta(\cos\theta)\right\}\times$$

$$\left[\sum_{l=0}^{\infty}\frac{\gamma^{l}}{l\,!}\frac{\sin\left\{\left(l+\frac{\lambda+\varepsilon}{2}+1\right)(\theta-w)\right\}}{\sin\frac{\theta-w}{2}}\right]\sin w\,dw$$

now, putting $\theta-w=u'$, we have

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$$M_{32} = O\left[e^{-\gamma} \int_{-\gamma^{-1}}^{\gamma^{-\Delta}} \frac{|\eta_1(u')|}{u'} \left\{ \sum_{l=0}^{\infty} \frac{\gamma' \sin lu'}{l\,!} \cos\left(\frac{\lambda+\varepsilon}{2}+1\right)u' + \sum_{l=0}^{\infty} \frac{\gamma' \cos lu'}{l\,!} \sin\left(\frac{\lambda+\varepsilon}{2}+1\right)u' \right\} du' \right]$$

$= M_{3,2,1} + M_{3,2,2}$ 'say'

then

$$M_{3,2,1} = O\left\{ \int_{-\gamma^{-1}}^{\gamma^{-\Delta}} \frac{|\eta_1(u')|}{u'} \frac{\sin(\gamma \sin u')}{\gamma(1-\cos u')} du' \right\}$$

we have, if $0 < \Delta < \Delta^1 < \frac{1}{2}$, by second mean value theorem

$$M_{3,2,1} = O\left\{ \frac{1}{e^{\gamma^{\Delta}} 2\sin^2\left(\frac{\gamma^{\Delta}}{2}\right)} \int_{-\gamma^{-1}}^{\gamma^{-\Delta^1}} \frac{|\eta_1(u')|}{u'} \sin(\gamma \sin u')\, d\, u' \right\}$$

$$= O\left\{ \int_{-\gamma^{-1}}^{\gamma^{-\Delta^1}} \frac{|\eta_1(u')|}{u'} du' \right\},$$

using integrating by parts and lemma, we have

$$M_{3,2,1} = O\left\{ O\left[\frac{1}{\log \frac{1}{u'}} \right]_{-\gamma^{-1}}^{\gamma^{-\Delta^1}} + \int_{-\gamma^{-1}}^{\gamma^{-\Delta^1}} O\left(\frac{1}{u' \log \frac{1}{u'}} \right) du' \right\}$$

$$= O\left(\frac{1}{\log \gamma}\right) + O\left(\left[\log\, \log \frac{1}{u'} \right]_{-\gamma^{-1}}^{\gamma^{-\Delta^1}} \right)$$

$$= O(1) + O(1)$$

$= O(1)$ as $\gamma \to \infty$.

Similarly, we have

$M_{3,2,2} = O(1)$ and

hence $M_{32} = O(1)$.

On the lines of M_{32}, we obtain

$M_{31} = O(1)$.

We also observe that under the hypothesis of the theorem, we have

$M_{33} = O(1)$.

Therefore, we obtain

$$M_3 = O(1) \qquad \ldots \qquad (12)$$

Combining (8), (9), (10), (11) and (12),
we obtain (7).
This completes the proof of the theorem.

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