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Operoid Groups Approach on Discrete Semigroups and Compact Semigroups

[1]D. LOHANAYAKI

Research Scholar,
P.G & Research Department of Mathematics, Government Arts College,
C. Mutlur, Chidambaram-608102.

[2]S. SIVAKUMAR

Assistant Professor,
P.G & Research Department of Mathematics, Government Arts College,
C. Mutlur, Chidambaram-608102.

ABSTRACT

In this paper we have studied relations between the theory of discrete groups and structure theory of topological groups and also developed the topological operoid groups acted on discrete semi groups to compact semi groups. Also we derive the basic properties of topological operoid group. We construct with a compact right topological semigroups from both the topological and the algebraic points of view.

KEYWORDS: operoid group-sub operoid group- homomorphism-discrete operoid group -compact operoid group.

1. INTRODUCTION

An operoid is a non-empty set $\mathcal{O}$ with a binary operation called product or multiplication. Formally, the operation is a mapping φ of $\mathcal{O} \times \mathcal{O}$ to $\mathcal{O}$ for $(x, y) \in \mathcal{O} \times \mathcal{O}$ we write xy instead of φ(x,y) and call xy the product of x and y. In general, the associativity of multiplication or the existence of the identity in $\mathcal{O}$ are not assumed. Given an operoid $\mathcal{O}$ with a topology $\mathcal{T}$ we will call it a topological operoid if the multiplication is jointly continuous. As in the case of semigroups, we define the mappings $\Re_a$ and $\mathcal{L}_a$ mappings of $\mathcal{O}$ to $\mathcal{O}$. $\Re_a = xa$ and $\mathcal{L}_a = ax$, for any a and x in $\mathcal{O}$. The mappings $\Re_a$ and $\mathcal{L}_a$ are called the right action and the left action by a on the operoid $\mathcal{O}$.

Definition 1.1 : An operoid $\mathcal{O}$ with a topology $\mathcal{T}$, is called topological operoid $\mathcal{T}_{\mathcal{O}}$ if the multiplication is jointly continuous.

Definition 1.2 : An element e of an operoid is called a right identity(left identity) if xe = x(xe=x), for each $x \in \mathcal{O}$, if $e \in \mathcal{O}$ is both a right identity and a left identity of an operoid $\mathcal{O}$, we say that e is an identity of $\mathcal{O}$.

Definition 1.3 : A suboperoid $\mathcal{S}$ of an $\mathcal{O}$ operoid is a non empty subset $\mathcal{S}$ of $\mathcal{O}$ closed under the multiplication in $\mathcal{O}$, In other words $mn \in \mathcal{S}$ for all $m,n \in \mathcal{S}$. Then the multiplication in $\mathcal{S}$ is restriction to $\mathcal{S}$ of the multiplication in $\mathcal{O}$.

Definition 1.4 : A mapping $g : \mathcal{S} \rightarrow \mathcal{T}$ of operoids $\mathcal{S}$ and $\mathcal{T}$ is called a homomorphism if it satisfies g(m,n) = g(m)g(n) for all $m,n \in \mathcal{S}$. If e is the identity of $\mathcal{S}$, then g(e) is the identity of $\mathcal{T}$ for every homomorphism g of $\mathcal{S}$ to $\mathcal{T}$.

Note : An operoid can have atmost one identity.

Theorem 1.5 : Let $\mathcal{O}$ be an operoid with the discrete topology and αo the Čech-Stone compactification of the discrete space $\mathcal{O}$. Then the product operation in $\mathcal{O}$ can be extended to a product operation in αo in such a way that αo becomes a right topological operoid. In such a way that the left action on αo by any element of $\mathcal{O}$ be continuous. Further more,

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under the last condition this extension is unique, and if $\mathcal{O}$ has a left(right) identity e, then e is also a left(right) identity of the operoid αo.

Proof : Let $\mathcal{L}_a$ be the left action by a on $\mathcal{O}$. Since $\mathcal{L}_a$ is a continuous mapping of $\mathcal{O}$ to $\mathcal{O}$

Now we can extend it to a continuous mapping of $\alpha\mathcal{O}$ to $\alpha\mathcal{O}$. Also we denote the later mapping by $\mathcal{L}_a$ and put av = $\mathcal{L}_a$ (v) for each $v \in \alpha\mathcal{O}$.

Thus the product av in $\alpha\mathcal{O}$ is defined for each $a \in \mathcal{O}$ and each $v \in \alpha\mathcal{O}$

Fix $v \in \alpha\mathcal{O}$ and put $\mathcal{R}_v(x) = xV$ for each $x \in \mathcal{O}$, In this way a mapping $\Re_v$ is defined on $\mathcal{O}$, with values in $\alpha\mathcal{O}$.

Since $\mathcal{O}$ is discrete, $\Re_v$ is continuous .

Therefore $\Re_v$ can be extended to $\alpha\mathcal{O}$, we denote the extension also by $\mathcal{R}_v$.

Now for any u in $\alpha\mathcal{O}$, put $uv = \Re_v(u)$.

The definition of the product operation is complete and, since the mapping $\Re_v$ is continuous for every $v \in \alpha\mathcal{O}$, $\alpha\mathcal{O}$ with this product operation is a right topological operoid.

Then the identities follows from the continuity of $\mathcal{L}_a$ and $\Re_a$, for each $a \in \mathcal{O}$.

The above construction of the extension of the multiplication $\mathcal{O}$ in ever $\alpha\mathcal{O}$.

This shows the extension is unique.

Suppose that $\chi : \alpha\mathcal{O} \times \alpha\mathcal{O} \rightarrow \alpha\mathcal{O}$ is a mapping whose restriction to $\mathcal{O} \times \mathcal{O}$ coincides with the multiplication in $\mathcal{O}$ and which makes continuous all right actions $\Re_a^*$ with $a \in \alpha\mathcal{O}$ and all left actions $\mathcal{L}_a^*$ with $a \in \mathcal{O}$ where $\Re_a^*(x) = \chi(x,a)$ and $\mathcal{L}_a^*(x) = \chi(a,x)$ for each $x \in \alpha\mathcal{O}$.

By the assumption, the left actions $\mathcal{L}_a^*$ and $\mathcal{L}_a$ coincide on the dense subset X of the space $\alpha\mathcal{O}$ for each $a \in \alpha\mathcal{O}$.

Hence it follows that $av = \chi(a,v)$ for all $a \in \mathcal{O}$.

Thus $\Re_v^*(a) = \Re_v(a)$ for each $a \in \mathcal{O}$ and the density argument together with the continuity of the right actions $\Re_v^*$ and $\Re_v$ for $v \in \alpha\mathcal{O}$ implies that these actions coincide on $\alpha\mathcal{O}$.

Therefore $uv = \Re_v^*(u) = \chi(u,v)$ for all $u,v \in \alpha\mathcal{O}$.

Thus χ coincides with the multiplication in $\alpha\mathcal{O}$.

Theorem 1.6 : Let $\mathfrak{s}$ and $\mathcal{T}$ be compact right topological operoids, $\mathfrak{D}$ a dense suboperoid of $\mathfrak{s}$ and g a continuous mapping of the space $\mathfrak{s}$ to the space $\mathcal{T}$ satisfying following conditions:

(a) The left action $\mathcal{L}_a$ is continuous on $\mathfrak{s}$, for every $a \in \mathfrak{D}$,
(b) The restriction of g to $\mathfrak{D}$ is a homomorphism of $\mathfrak{D}$ to $\mathcal{T}$,
(c) The left action $\mathcal{L}_{g(a)}$ is continuous on $\mathcal{T}$, for every $a \in \mathfrak{D}$. Then g is a homomorphism of $\mathfrak{s}$ to the space $\mathcal{T}$.

Proof : For each $a \in \mathfrak{D}$, $g \circ \mathcal{L}_a$ and $\mathcal{L}_{g(a)} \circ g$ are continuous mapping of $\mathfrak{s}$ to the space $\mathcal{T}$ coincideing on the dense subset $\mathfrak{D}$ of $\mathfrak{s}$.

Therefore, they coincide on the whole of $\mathfrak{s}$.

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That is $g(an) = g(a)\, g(n)$, for all $a \in \mathfrak{D}$ and $n \in \mathfrak{s}$.

However $g(an) = g(\mathfrak{R}_n(a))$, and $g(a)\, g(n) = \mathfrak{R}\, g(n)\, (g(a))$.

Thus the mappings $g \circ \mathfrak{R}_n$ and $\mathfrak{R}\, g(n) \circ g$ coincide on the dense subset $\mathfrak{D}$ of $\mathfrak{s}$.

Since the mappings $g \circ \mathfrak{R}_n$ and $\mathfrak{R}\, g(n) \circ g$ are continuous.

We conclude that they coincide on $\mathfrak{s}$.

That is $g(m)\, g(n) = g(mn)$ for all $m,n \in \mathfrak{s}$.

Thus g is a homomorphism.

Theorem 1.7 : Let θ is a discrete semigroup, then $\alpha\theta$ is right topological semigroup.

Proof : Since the multiplication in $\alpha\theta$ defined in theorem 1.5 is associative .

That is $(uv)w = u(vw)$ for every u,v and w in $\alpha\theta$

We have $u\,(vw) = \mathfrak{R}_{vw}(u)$ and $(uv)w = \mathfrak{R}_w(\mathfrak{R}_v(u))$.

Since all the right actions on $\alpha\theta$ are continuous.

It suffices to show that the mappings $\mathfrak{R}_{vw}$ and $\mathfrak{R}_w \circ \mathfrak{R}_v$ coincide on θ.

Let $a \in \theta$, then $\mathfrak{R}_w(\mathfrak{R}_v(a)) = (av)w = \mathfrak{R}_w(\mathcal{L}_a(v))$ while

$$\mathfrak{R}_{vw}(a) = a(vw) = \mathcal{L}_a(\mathfrak{R}_w(v))$$

Since $\mathcal{L}_a$ and $\mathfrak{R}_w$ are continuous on $\alpha\theta$, it suffices to show that $\mathcal{L}_a \circ \mathfrak{R}_w$ and $\mathfrak{R}_w \circ \mathcal{L}_a$ coincide on θ.

Take any $b \in \theta$, then $\mathcal{L}_a(\mathfrak{R}_w(b)) = a(bw) = \mathcal{L}_a(\mathcal{L}_b(w))$

While $\mathfrak{R}_w(\mathcal{L}_a(b)) = (ab)w = \mathcal{L}_{ab}(w)$.

Since the mappings $\mathcal{L}_a$, $\mathcal{L}_b$ and $\mathcal{L}_{ab}$ are continuous on $\alpha\theta$.

It remains to check that $\mathcal{L}_a(\mathcal{L}_b(c)) = \mathcal{L}_{ab}(c)$ for each $c \in \theta$.

Since the product operation in θ is associative

we have $\mathcal{L}_a(\mathcal{L}_b(w)) = a(bc) = (ab)c = \mathcal{L}_{ab}(c)$, for each $c \in \theta$.

Hence the product operation in $\alpha\theta$ is associative and $\alpha\theta$ is a right topological semigroup.

Conclusion:

In this paper, relations between the theory of discrete groups and structure theory of topological groups and also developed to the topological operoid groups

Conflict of interest:

The authors declare that there is no conflicts of interest associated with this publication.

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