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On the Split Mersenne and Mersenne-Lucas Hybrid Octonions

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Abstract

In this paper, we defined Split Mersenne and Mersenne-Lucas hybrid octonions and we obtained generating functions for these sequences. Further, we analyzed relations between Split Mersenne hybrid octonions and Split Mersenne-Lucas hybrid octonions. Also, we gave the Binet's formula for these two sequences and moreover, the well-known identities like Catalan, Cassini and d'Ocagne for these hybrid octonions.

Keywords: Cayley-Dickson Algebra, Octonions, Hybrid numbers, Mersenne sequence, Mersenne-Lucas sequence

MSC: 11B99

Introduction

The split- octonions can be obtained from the Cayley-Dickson construction by defining a multiplication on pair of quaternions.

$(a+\ell b)(c+\ell d)=\left(ac+\lambda d\bar{b}\right)+\ell(\bar{a}d+cb)$

where $\lambda=\ell^2, \ell$ is the imaginary unit.

If λ is chosen to be -1, we get the octonions. If it is taken to be $+1$ we get the split-octonions.

Let $\breve{O}$ be the split octonion algebra over the real number field $\mathbb{R}$. A natural basis of this algebra as a space over $\mathbb{R}$ is formed by the elements

$$e_0=1, e_1=i, e_2=j, e_3=k, e_4=\ell, e_5=\ell i, e_6=\ell j, e_7=\ell k.$$

For the properties about octonions and split octonions one can refer $[1,4,5,6]$

The Mersenne sequence $\{M_n\}$ are defined by the recurrence relation

$M_n=3M_{n-1}-2M_{n-2}$, $n\geq 2$, where $M_0=0, M_1=1$.

The Mersenne- Lucas sequence $\{ML_n\}$ are defined recurrently by

$ML_n=3ML_{n-1}-2ML_{n-2}$, $n\geq 2$, where $ML_0=2, ML_1=3$.

The Binet's formula for Mersenne and Mersenne-Lucas sequences are given by

$M_n=2^n-1$ and $ML_n=2^n+1$.

For detailed discussion of the Mersenne and Mersenne-Lucas numbers and their properties, see $[2,3]$

The Mersenne octonions and Mersenne-Lucas octonions are defined by

$$\widetilde{M_n} = \sum_{i=0}^{7} M_{n+i}e_i$$

$$= M_n e_0 + M_{n+1}e_1 + M_{n+2}e_2 + M_{n+3}e_3 + M_{n+4}e_4 + M_{n+5}e_5 + M_{n+6}e_6 + M_{n+7}e_7$$

$$\widetilde{ML_n} = \sum_{i=0}^{7} ML_{n+i}e_i$$

$$= ML_n e_0 + ML_{n+1}e_1 + ML_{n+2}e_2 + ML_{n+3}e_3 + M_{Ln+4}e_4 + ML_{n+5}e_5 + ML_{n+6}e_6 + \quad ML_{n+7}e_7$$

where $e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7$ are octonion basis.

The split Mersenne octonions and split Mersenne-Lucas octonions are defined as

$\widetilde{SM_n} = \sum_{s=0}^{7} M_{n+s}e_s$ and $\widetilde{SML_n} = \sum_{s=0}^{7} ML_{n+s}e_s$

The Binet's formula for split Mersenne octonions and split Mersenne-Lucas octonions has the form

$\widetilde{SM_n} = 2^n A^* - B^*$ and $\widetilde{SML_n} = 2^n A^* + B^*$

where $A^* = \sum_{s=0}^{7} 2^s e_s$, $B^* = \sum_{s=0}^{7} e_s$

The hybrid number is defined as $\mathcal{H} = a + bi + c\varepsilon + dh$, where $a, b, c, d \in \mathbb{R}$,

$i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = -hi = \varepsilon + i$.

The Mersenne hybrid numbers and Mersenne-Lucas hybrid numbers are defined as

$M\mathcal{H}_n = M_n + M_{n+1}i + M_{n+2}\varepsilon + M_{n+3}h$

$ML\mathcal{H}_n = ML_n + ML_{n+1}i + ML_{n+2}\varepsilon + ML_{n+3}h$

where i, ε, h are hybrid units.

Split Mersenne Hybrid Octonions and Split Mersenne-Lucas Hybrid Octonions

For $n \geq 0$, the nth split Mersenne hybrid octonion sequence $\{\widetilde{SM\mathcal{H}_n}\}$ is defined by

$$\widetilde{SM\mathcal{H}_n} = \sum_{s=0}^{7} M\mathcal{H}_{n+s}e_s$$

$$= (M_n + iM_{n+1} + \varepsilon M_{n+2} + hM_{n+3})e_0 + +(M_{n+1} + iM_{n+2} + \varepsilon M_{n+3} + hM_{n+4})e_1$$

$$+(M_{n+2} + iM_{n+3} + \varepsilon M_{n+4} + hM_{n+5})e_2 + (M_{n+3} + iM_{n+4} + \varepsilon M_{n+5} + hM_{n+6})e_3$$

$$+(M_{n+4} + iM_{n+5} + \varepsilon M_{n+6} + hM_{n+7})e_4 + (M_{n+5} + iM_{n+6} + \varepsilon M_{n+7} + hM_{n+8})e_5$$

$$+(M_{n+6} + iM_{n+7} + \varepsilon M_{n+8} + hM_{n+9})e_6 + (M_{n+7} + iM_{n+8} + \varepsilon M_{n+9} + hM_{n+10})e_7$$

where i, ε, h are hybrid units and $e_0, e_1, e_2, e_3, e_4, e_5, e_6, e_7$ are split octonion basis.

$$= \widetilde{SM_n} + i\widetilde{SM_{n+1}} + \varepsilon\widetilde{SM_{n+2}} + h\widetilde{SM_{n+3}}$$

In the same way, we define the nth split Mersenne-Lucas hybrid octonion sequence $\{\widetilde{SML\mathcal{H}_n}\}$ by

$$\widetilde{SML\mathcal{H}}_n = \sum_{s=0}^{7} ML\mathcal{H}_{n+s} e_s$$

$$= (ML_n + iML_{n+1} + \varepsilon ML_{n+2} + hML_{n+3})e_0$$

$$+(ML_{n+1} + iML_{n+2} + \varepsilon ML_{n+3} + hML_{n+4})e_1$$

$$+(ML_{n+2} + iML_{n+3} + \varepsilon ML_{n+4} + hML_{n+5})e_2$$

$$+(ML_{n+3} + iML_{n+4} + \varepsilon ML_{n+5} + hML_{n+6})e_3$$

$$+(ML_{n+4} + iML_{n+5} + \varepsilon ML_{n+6} + hML_{n+7})e_4$$

$$+(ML_{n+5} + iML_{n+6} + \varepsilon ML_{n+7} + hML_{n+8})e_5$$

$$+(ML_{n+6} + iML_{n+7} + \varepsilon ML_{n+8} + hML_{n+9})e_6$$

$$+(ML_{n+7} + iML_{n+8} + \varepsilon ML_{n+9} + hML_{n+10})e_7$$

$$= \widetilde{SML_n} + i\widetilde{SML_{n+1}} + \varepsilon\widetilde{SML_{n+2}} + h\widetilde{SML_{n+3}}$$

Theorem 1

The generating functions for the split Mersenne hybrid octonions and the split Mersenne-Lucas hybrid octonions are

$$f(t) = \frac{\widetilde{SM\mathcal{H}}_0(1-3t) + \widetilde{SM\mathcal{H}}_1 t}{1-3t+2t^2}$$

and

$$g(t) = \frac{\widetilde{SML\mathcal{H}}_0(1-3t) + \widetilde{SML\mathcal{H}}_1 t}{1-3t+2t^2}$$

Proof

Let $f(t) = \sum_{n=0}^{\infty} \widetilde{SMH}_n t^n$

Multiplying this equation by $1, 3t, 2t^2$ respectively and summing these equations, we obtain

$(1-3t+2t^2)f(t)$

$$= \widetilde{SMH}_0 + \left(\widetilde{SMH}_1 - 3\widetilde{SMH}_0\right)t + \left(\widetilde{SMH}_2 - 3\widetilde{SMH}_1 + 2\widetilde{SMH}_0\right)t^2 + \cdots + \left(\widetilde{SMH}_n - 3\widetilde{SMH}_{n-1} + 2\widetilde{SMH}_{n-2}\right)t^n$$

$$= \widetilde{SMH}_0 + \left(\widetilde{SMH}_1 - 3\widetilde{SMH}_0\right)t + \sum_{n=2}^{\infty}\left(\widetilde{SMH}_n - 3\widetilde{SMH}_{n-1} + 2\widetilde{SMH}_{n-2}\right)t^n$$

$$= \frac{\widetilde{SM\mathcal{H}}_0(1-3t) + \widetilde{SM\mathcal{H}}_1 t}{1-3t+2t^2}$$

$\therefore f(t) = \frac{\widetilde{SM\mathcal{H}}_0(1-3t) + \widetilde{SM\mathcal{H}}_1 t}{1-3t+2t^2}$ is the generating functions for the split Mersenne hybrid octonions.

And let $g(t) = \sum_{n=0}^{\infty} \widetilde{SMLH}_n t^n$

Multiplying this equation by $1, 3t, 2t^2$ respectively and summing these equations, we obtain

$(1-3t+2t^2)g(t)$

JOURNAL OF ALGEBRAIC STATISTICS
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$$= \widetilde{SMLH}_0 + \left(\widetilde{SMLH}_1 - 3\widetilde{SMLH}_0\right)t + \left(\widetilde{SMLH}_2 - 3\widetilde{SMLH}_1 + 2\widetilde{SMLH}_0\right)t^2 + \cdots + \left(\widetilde{SMLH}_n - 3\widetilde{SMLH}_{n-1} + 2\widetilde{SMLH}_{n-2}\right)t^n$$

$$= \widetilde{SMLH}_0 + \left(\widetilde{SMLH}_1 - 3\widetilde{SMLH}_0\right)t + \sum_{n=2}^{\infty}\left(\widetilde{SMLH}_n - 3\widetilde{SMLH}_{n-1} + 2\widetilde{SMLH}_{n-2}\right)t^n$$

$$= \frac{\widetilde{SML\mathcal{H}}_0(1-3t) + \widetilde{SML\mathcal{H}}_1 t}{1-3t+2t^2}$$

$\therefore g(t) = \frac{\widetilde{SML\mathcal{H}}_0(1-3t)+\widetilde{SML\mathcal{H}}_1 t}{1-3t+2t^2}$ is the generating functions for split Mersenne-Lucas hybrid octonions.

Theorem 2 **(**Binet's formula**)**

For $n \geq 0$, the Binet's formulas for the split Mersenne hybrid octonions and split Mersenne-Lucas hybrid octonions are

$\widetilde{SM\mathcal{H}}_n = 2^n\alpha^*A^* - \beta^*B^*$ and $\widetilde{SML\mathcal{H}}_n = 2^n\alpha^*A^* + \beta^*B^*$

where $A^* = \sum_{s=0}^{7} 2^s e_s$, $B^* = \sum_{s=0}^{7} e_s$, $\alpha^* = (1 + 2i + 2^2\epsilon + 2^3h), \beta^* = (1 + i + \epsilon + h)$

Proof

$$\widetilde{SMH}_n = \widetilde{SM}_n + i\widetilde{SM}_{n+1} + \varepsilon\widetilde{SM}_{n+2} + h\widetilde{SM}_{n+3}$$
$$= (2^nA^* - B^*) + i(2^{n+1}A^* - B^*) + \epsilon(2^{n+2}A^* - B^*) + h(2^{n+3}A^* - B^*)$$
$$= 2^n(1 + 2i + 2^2\epsilon + 2^3h)A^* - (1 + i + \epsilon + h)B^*$$
$$= 2^n\alpha^*A^* - \beta^*B^*$$

$$\widetilde{SMLH}_n = \widetilde{SML}_n + i\widetilde{SML}_{n+1} + \varepsilon\widetilde{SML}_{n+2} + h\widetilde{SML}_{n+3}$$
$$= (2^nA^* + B^*) + i(2^{n+1}A^* + B^*) + \epsilon(2^{n+2}A^* + B^*) + h(2^{n+3}A^* + B^*)$$
$$= 2^n(1 + 2i + 2^2\epsilon + 2^3h)A^* + (1 + i + \epsilon + h)B^*$$
$$= 2^n\alpha^*A^* + \beta^*B^*$$

Lemma

$A^*B^* = 227 - 81e_1 + 59e_2 - 137e_3 - 193e_4 + 115e_5 - 25e_6 + 171e_7$

$B^*A^* = 227 + 87e_1 + 79e_2 + 155e_3 + 227e_4 - 49e_5 + 155e_6 + 87e_7$

Proof From the definition of A^* and B^*, and using the multiplication table for the basis of split octonions, we computed these results.

Main Results

Theorem 3

Let n be any positive integer then

$2\widetilde{SM\mathcal{H}}_{n-1} + \widetilde{SM\mathcal{H}}_{n+1} = 3\widetilde{SM\mathcal{H}}_n$

Proof

$2\widetilde{SM\mathcal{H}}_{n-1} + \widetilde{SM\mathcal{H}}_{n+1} = 2(2^{n-1}\alpha^*A^* - \beta^*B^*) + (2^{n+1}\alpha^*A^* - \beta^*B^*)$

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 2, 2022, p. 3318-3325
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$$= 3(2^n\alpha^*A^* - \beta^*B^*)$$

$$= 3\widetilde{SM\mathcal{H}}_n$$

Theorem 4

Let n be any positive integer then

$i.\widetilde{SM\mathcal{H}_n}^2 - \widetilde{SML\mathcal{H}_n}^2 = -2^{n+1}[\alpha^*\beta^*A^*B^* + \beta^*\alpha^*B^*A^*]$
$ii.\widetilde{SM\mathcal{H}_n}^2 + \widetilde{SML\mathcal{H}_n}^2 = 2[2^{2n}(\alpha^*A^*)^2 + (\beta^*B^*)^2]$

Proof

$i.\widetilde{SM\mathcal{H}_n}^2 - \widetilde{SML\mathcal{H}_n}^2 = (2^n\alpha^*A^* - \beta^*B^*)^2 - (2^n\alpha^*A^* + \beta^*B^*)^2$

$$= 2^{2n}(\alpha^*)^2(A^*)^2 - 2^n\beta^*\alpha^*B^*A^* - 2^n\alpha^*\beta^*A^*B^* + (\beta^*)^2(B^*)^2 - 2^{2n}(\alpha^*)^2(A^*)^2 - (\beta^*)^2(B^*)^2 - 2^n\beta^*\alpha^*B^*A^* - 2^n\alpha^*\beta^*A^*B^*$$

$$= -2^{n+1}\alpha^*\beta^*A^*B^* - 2^{n+1}\beta^*\alpha^*B^*A^*$$

$$= -2^{n+1}[\alpha^*\beta^*A^*B^* + \beta^*\alpha^*B^*A^*]$$

$ii.\widetilde{SM\mathcal{H}_n}^2 + \widetilde{SML\mathcal{H}_n}^2 = (2^n\alpha^*A^* - \beta^*B^*)^2 + (2^n\alpha^*A^* + \beta^*B^*)^2$

$$= 2^{2n}(\alpha^*)^2(A^*)^2 - 2^n\beta^*\alpha^*B^*A^* - 2^n\alpha^*\beta^*A^*B^* + (\beta^*)^2(B^*)^2 + 2^{2n}(\alpha^*)^2(A^*)^2 + (\beta^*)^2(B^*)^2 + 2^n\beta^*\alpha^*B^*A^* + 2^n\alpha^*\beta^*A^*B^*$$

$$= 2[2^{2n}(\alpha^*A^*)^2 + (\beta^*B^*)^2]$$

Theorem 5

Let m, n be any positive integers then

$i.\widetilde{SM\mathcal{H}}_{m+n} + 2^n\widetilde{SM\mathcal{H}}_{m-n} = ML_n\widetilde{SM\mathcal{H}}_m$
$ii.\widetilde{SML\mathcal{H}}_{m+n} + 2^n\widetilde{SML\mathcal{H}}_{m-n} = ML_n\widetilde{SML\mathcal{H}}_m$

Proof

$i.\widetilde{SM\mathcal{H}}_{m+n} + 2^n\widetilde{SM\mathcal{H}}_{m-n} = (2^{m+n}\alpha^*A^* - \beta^*B^*) + 2^n(2^{m-n}\alpha^*A^* - \beta^*B^*)$

$$= 2^{m+n}\alpha^*A^* - 2^n\beta^*B^* + 2^m\alpha^*A^* - \beta^*B^*$$
$$= 2^m\alpha^*A^*(1 + 2^n) - \beta^*B^*(1 + 2^n)$$
$$= ML_n(2^m\alpha^*A^* - \beta^*B^*)$$

$$= ML_n\widetilde{SM\mathcal{H}}_m$$

$ii.\widetilde{SML\mathcal{H}}_{m+n} + 2^n\widetilde{SML\mathcal{H}}_{m-n} = (2^{m+n}\alpha^*A^* + \beta^*B^*) + 2^n(2^{m-n}\alpha^*A^* + \beta^*B^*)$

$$= 2^{m+n}\alpha^*A^* + 2^n\beta^*B^* + 2^m\alpha^*A^* + \beta^*B^*$$
$$= 2^m\alpha^*A^*(1 + 2^n) + \beta^*B^*(1 + 2^n)$$
$$= ML_n(2^m\alpha^*A^* + \beta^*B^*)$$

$$= ML_n\widetilde{SML\mathcal{H}}_m$$

Theorem 6

Let n be any positive integer then

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 2, 2022, p. 3318-3325
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$i.2S\widetilde{M\mathcal{H}}_n S\widetilde{ML\mathcal{H}}_n + 2S\widetilde{M\mathcal{H}}_{n-1} S\widetilde{ML\mathcal{H}}_{n-1}$
$= 2^{2n-1}(\alpha^*)^2(A^*)^2 ML_2 - 4(\beta^*)^2(B^*)^2 + 3ML_1 2^n[\alpha^*\beta^*A^*B^* - \beta^*\alpha^*B^*A^*]$
$ii.2S\widetilde{M\mathcal{H}}_n S\widetilde{ML\mathcal{H}}_n - 2S\widetilde{M\mathcal{H}}_{n-1} S\widetilde{ML\mathcal{H}}_{n-1}$
$= 2^{2n-1}(\alpha^*)^2(A^*)^2 M_2 + 2^n M_1[\alpha^*\beta^*A^*B^* - \beta^*\alpha^*B^*A^*]$

Proof

i.$2S\widetilde{M\mathcal{H}}_n S\widetilde{ML\mathcal{H}}_n + 2S\widetilde{M\mathcal{H}}_{n-1} S\widetilde{ML\mathcal{H}}_{n-1}$
$= 2(2^n\alpha^*A^* - \beta^*B^*)(2^n\alpha^*A^* + \beta^*B^*) + 2(2^{n-1}\alpha^*A^* - \beta^*B^*)(2^{n-1}\alpha^*A^* + \beta^*B^*)$
$= 2[2^{2n}(\alpha^*)^2(A^*)^2 - 2^n\beta^*\alpha^*B^*A^* + 2^n\alpha^*\beta^*A^*B^* - (\beta^*)^2(B^*)^2 + 2^{2n-2}(\alpha^*)^2(A^*)^2 - (\beta^*)^2(B^*)^2 - 2^{n-1}\beta^*\alpha^*B^*A^* + 2^{n-1}\alpha^*\beta^*A^*B^*]$
$= 2^{2n-1}(\alpha^*)^2(A^*)^2 ML_2 - 4(\beta^*)^2(B^*)^2 + 3ML_1 2^n[\alpha^*\beta^*A^*B^* - \beta^*\alpha^*B^*A^*]$

ii.$2S\widetilde{M\mathcal{H}}_n S\widetilde{ML\mathcal{H}}_n - 2S\widetilde{M\mathcal{H}}_{n-1} S\widetilde{ML\mathcal{H}}_{n-1}$
$= 2(2^n\alpha^*A^* - \beta^*B^*)(2^n\alpha^*A^* + \beta^*B^*) - 2(2^{n-1}\alpha^*A^* - \beta^*B^*)(2^{n-1}\alpha^*A^* + \beta^*B^*)$
$= 2^{2n+1}(\alpha^*)^2(A^*)^2 - 2^{n+1}\beta^*\alpha^*B^*A^* + 2^{n+1}\alpha^*\beta^*A^*B^* - 2(\beta^*)^2(B^*)^2 - 2^{2n-1}(\alpha^*)^2(A^*)^2 + 2(\beta^*)^2(B^*)^2 + 2^n\beta^*\alpha^*B^*A^* - 2^n\alpha^*\beta^*A^*B^*$

$= 2^{2n-1}(\alpha^*)^2(A^*)^2 M_2 + 2^n M_1[\alpha^*\beta^*A^*B^* - \beta^*\alpha^*B^*A^*]$

Theorem 7

Let m, n, r be any positive integers then

$$S\widetilde{M\mathcal{H}}_{m+n} S\widetilde{ML\mathcal{H}}_{m+r} - S\widetilde{M\mathcal{H}}_{m+r} S\widetilde{ML\mathcal{H}}_{m+n} = -2^{m+n} M_{r-n}[\alpha^*\beta^*A^*B^* + \beta^*\alpha^*B^*A^*]$$

Proof

$S\widetilde{M\mathcal{H}}_{m+n} S\widetilde{ML\mathcal{H}}_{m+r} - S\widetilde{M\mathcal{H}}_{m+r} S\widetilde{ML\mathcal{H}}_{m+n}$

$= (2^{m+n}\alpha^*A^* - \beta^*B^*)(2^{m+r}\alpha^*A^* + \beta^*B^*) - (2^{m+r}\alpha^*A^* - \beta^*B^*)(2^{m+n}\alpha^*A^* + \beta^*B^*)$

$= 2^{2m+n+r}(\alpha^*)^2(A^*)^2 - 2^{m+r}\beta^*\alpha^*B^*A^* + 2^{m+n}\alpha^*\beta^*A^*B^* - (\beta^*)^2(B^*)^2 - 2^{2m+n+r}(\alpha^*)^2(A^*)^2 - (\beta^*)^2(B^*)^2 + 2^{m+n}\beta^*\alpha^*B^*A^* - 2^{m+r}\alpha^*\beta^*A^*B^*$

$= 2^{m+n}(1 - 2^{r-n})[\alpha^*\beta^*A^*B^* + \beta^*\alpha^*B^*A^*]$

$= -2^{m+n} M_{r-n}[\alpha^*\beta^*A^*B^* + \beta^*\alpha^*B^*A^*]$

Theorem 8

Let m, n be any positive integers such that $m \geq n$ then

$i.S\widetilde{M\mathcal{H}}_m S\widetilde{ML\mathcal{H}}_n - S\widetilde{ML\mathcal{H}}_m S\widetilde{M\mathcal{H}}_n = 2^{n+1}[2^{m-n}\alpha^*\beta^*A^*B^* - \beta^*\alpha^*B^*A^*]$
$ii.S\widetilde{M\mathcal{H}}_m S\widetilde{ML\mathcal{H}}_n + S\widetilde{ML\mathcal{H}}_m S\widetilde{M\mathcal{H}}_n = 2[2^{m+n}(\alpha^*A^*)^2 - (\beta^*B^*)^2]$

Proof

i.$S\widetilde{M\mathcal{H}}_m S\widetilde{ML\mathcal{H}}_n - S\widetilde{ML\mathcal{H}}_m S\widetilde{M\mathcal{H}}_n$
$= (2^m\alpha^*A^* - \beta^*B^*)(2^n\alpha^*A^* + \beta^*B^*) - (2^m\alpha^*A^* + \beta^*B^*)(2^n\alpha^*A^* - \beta^*B^*)$

$= 2^{m+n}(\alpha^*)^2(A^*)^2 - 2^n\beta^*\alpha^*B^*A^* + 2^m\alpha^*\beta^*A^*B^* - (\beta^*)^2(B^*)^2 - 2^{m+n}(\alpha^*)^2(A^*)^2 - (\beta^*)^2(B^*)^2 + 2^n\beta^*\alpha^*B^*A^* - 2^m\alpha^*\beta^*A^*B^*$

$= 2^{n+1}[2^{m-n}\alpha^*\beta^*A^*B^* - \beta^*\alpha^*B^*A^*]$

ii.$S\widetilde{M\mathcal{H}}_m S\widetilde{ML\mathcal{H}}_n + S\widetilde{ML\mathcal{H}}_m S\widetilde{M\mathcal{H}}_n$

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 2, 2022, p. 3318-3325
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$= (2^m\alpha^*A^* - \beta^*B^*)(2^n\alpha^*A^* + \beta^*B^*) + (2^m\alpha^*A^* + \beta^*B^*)(2^n\alpha^*A^* - \beta^*B^*)$

$= 2^{m+n+1}(\alpha^*A^*)^2 - 2(\beta^*B^*)^2$

$= 2[2^{m+n}(\alpha^*A^*)^2 - (\beta^*B^*)^2]$

Theorem 9

Let n, r, s be any positive integers then

$$S\widetilde{M\mathcal{H}}_{n+r}S\widetilde{ML\mathcal{H}}_{n+s} - S\widetilde{M\mathcal{H}}_{n+s}S\widetilde{ML\mathcal{H}}_{n+r} = 2^{n+r}ML_{s-r}[\alpha^*\beta^*A^*B^* - \beta^*\alpha^*B^*A^*]$$

Proof

$S\widetilde{M\mathcal{H}}_{n+r}S\widetilde{ML\mathcal{H}}_{n+s} - S\widetilde{M\mathcal{H}}_{n+s}S\widetilde{ML\mathcal{H}}_{n+r}$

$= (2^{n+r}\alpha^*A^* - \beta^*B^*)(2^{n+s}\alpha^*A^* + \beta^*B^*) - (2^{n+s}\alpha^*A^* + \beta^*B^*)(2^{n+r}\alpha^*A^* - \beta^*B^*)$

$= 2^{2n+r+s}(\alpha^*)^2(A^*)^2 - 2^{n+s}\beta^*\alpha^*B^*A^* + 2^{n+r}\alpha^*\beta^*A^*B^* - (\beta^*)^2(B^*)^2 - 2^{2n+r+s}(\alpha^*)^2(A^*)^2 + (\beta^*)^2(B^*)^2 + 2^{n+r}\beta^*\alpha^*B^*A^* - 2^{n+s}\alpha^*\beta^*A^*B^*$

$= -2^{n+r}\beta^*\alpha^*B^*A^*(1 + 2^{s-r}) + 2^{n+r}\alpha^*\beta^*A^*B^*(1 + 2^{s-r})$

$= 2^{n+r}ML_{s-r}[\alpha^*\beta^*A^*B^* - \beta^*\alpha^*B^*A^*]$

Theorem 10 (Catalan's Identity)

Let $n \geq 0, r \geq 0$ be integers such that $r \leq n$ then we have

$S\widetilde{M\mathcal{H}}_{n+r}S\widetilde{M\mathcal{H}}_{n-r} - {S\widetilde{M\mathcal{H}}_n}^2 = -2^{n-r}M_r[2^r\alpha^*\beta^*A^*B^* - \beta^*\alpha^*B^*A^*]$

$S\widetilde{ML\mathcal{H}}_{n+r}S\widetilde{ML\mathcal{H}}_{n-r} - {S\widetilde{ML\mathcal{H}}_n}^2 = 2^{n-r}M_r[2^r\alpha^*\beta^*A^*B^* - \beta^*\alpha^*B^*A^*]$

where $A^* = \sum_{s=0}^{7} 2^s e_s$, $B^* = \sum_{s=0}^{7} e_s$

Proof

$S\widetilde{M\mathcal{H}}_{n+r}S\widetilde{M\mathcal{H}}_{n-r} - {S\widetilde{M\mathcal{H}}_n}^2$

$= (2^{n+r}\alpha^*A^* - \beta^*B^*)(2^{n-r}\alpha^*A^* - \beta^*B^*) - (2^n\alpha^*A^* - \beta^*B^*)^2$

$= 2^{2n}(\alpha^*)^2(A^*)^2 - 2^{n-r}\beta^*\alpha^*B^*A^* - 2^{n+r}\alpha^*\beta^*A^*B^* + (\beta^*)^2(B^*)^2 - 2^{2n}(\alpha^*)^2(A^*)^2 - (\beta^*)^2(B^*)^2 + 2^n\beta^*\alpha^*B^*A^* + 2^n\alpha^*\beta^*A^*B^*$

$= -2^n\alpha^*\beta^*A^*B^*(2^r - 1) + 2^{n-r}\beta^*\alpha^*B^*A^*(2^r - 1)$

$= -2^{n-r}M_r[2^r\alpha^*\beta^*A^*B^* - \beta^*\alpha^*B^*A^*]$

$S\widetilde{ML\mathcal{H}}_{n+r}S\widetilde{ML\mathcal{H}}_{n-r} - {S\widetilde{ML\mathcal{H}}_n}^2$

$= (2^{n+r}\alpha^*A^* + \beta^*B^*)(2^{n-r}\alpha^*A^* + \beta^*B^*) - (2^n\alpha^*A^* + \beta^*B^*)^2$

$= 2^{2n}(\alpha^*)^2(A^*)^2 + 2^{n-r}\beta^*\alpha^*B^*A^* + 2^{n+r}\alpha^*\beta^*A^*B^* + (\beta^*)^2(B^*)^2 - 2^{2n}(\alpha^*)^2(A^*)^2 - (\beta^*)^2(B^*)^2 - 2^n\beta^*\alpha^*B^*A^* - 2^n\alpha^*\beta^*A^*B^*$

$= 2^n\alpha^*\beta^*A^*B^*(2^r - 1) - 2^{n-r}\beta^*\alpha^*B^*A^*(2^r - 1)$

$= 2^{n-r}M_r[2^r\alpha^*\beta^*A^*B^* - \beta^*\alpha^*B^*A^*]$

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Theorem 11 (Cassini's Identity)

For any integer $n \geq 0$, we have

$$\widetilde{SM\mathcal{H}}_{n+1}\widetilde{SM\mathcal{H}}_{n-1} - \widetilde{SM\mathcal{H}}_n{}^2 = 2^{n-1}(\beta^*\alpha^*B^*A^* - 2\alpha^*\beta^*A^*B^*)$$

$$\widetilde{SML\mathcal{H}}_{n+1}\widetilde{SML\mathcal{H}}_{n-1} - \widetilde{SML\mathcal{H}}_n{}^2 = 2^{n-1}(2\alpha^*\beta^*A^*B^* - \beta^*\alpha^*B^*A^*)$$

Proof By substituting $r = 1$ in Catalan's identity, we get these results.

Theorem 12 (d'Ocagne's Identity)

Let m, n be any positive integers then

$$\widetilde{SM\mathcal{H}}_m\widetilde{SM\mathcal{H}}_{n+1} - \widetilde{SM\mathcal{H}}_{m+1}\widetilde{SM\mathcal{H}}_n = 2^m\alpha^*\beta^*A^*B^* - 2^n\beta^*\alpha^*B^*A^*$$

$$\widetilde{SML\mathcal{H}}_m\widetilde{SML\mathcal{H}}_{n+1} - \widetilde{SML\mathcal{H}}_{m+1}\widetilde{SML\mathcal{H}}_n = 2^n\beta^*\alpha^*B^*A^* - 2^m\alpha^*\beta^*A^*B^*$$

Proof

$$\widetilde{SM\mathcal{H}}_m\widetilde{SM\mathcal{H}}_{n+1} - \widetilde{SM\mathcal{H}}_{m+1}\widetilde{SM\mathcal{H}}_n$$

$$= (2^m\alpha^*A^* - \beta^*B^*)(2^{n+1}\alpha^*A^* - \beta^*B^*) - (2^{m+1}\alpha^*A^* - \beta^*B^*)(2^n\alpha^*A^* - \beta^*B^*)$$

$$= 2^{m+n+1}(\alpha^*)^2(A^*)^2 - 2^{n+1}\beta^*\alpha^*B^*A^* - 2^m\alpha^*\beta^*A^*B^* + (\beta^*)^2(B^*)^2 - 2^{m+n+1}(\alpha^*)^2(A^*)^2 - (\beta^*)^2(B^*)^2 + 2^n\beta^*\alpha^*B^*A^* + 2^{m+1}\alpha^*\beta^*A^*B^*$$

$$= 2^m\alpha^*\beta^*A^*B^*(2-1) + 2^n\beta^*\alpha^*B^*A^*(1-2)$$

$$= 2^m\alpha^*\beta^*A^*B^* - 2^n\beta^*\alpha^*B^*A^*$$

$$\widetilde{SML\mathcal{H}}_m\widetilde{SML\mathcal{H}}_{n+1} - \widetilde{SML\mathcal{H}}_{m+1}\widetilde{SML\mathcal{H}}_n$$

$$= (2^m\alpha^*A^* + \beta^*B^*)(2^{n+1}\alpha^*A^* + \beta^*B^*) - (2^{m+1}\alpha^*A^* + \beta^*B^*)(2^n\alpha^*A^* + \beta^*B^*)$$

$$= 2^{m+n+1}(\alpha^*)^2(A^*)^2 + 2^{n+1}\beta^*\alpha^*B^*A^* + 2^m\alpha^*\beta^*A^*B^* + (\beta^*)^2(B^*)^2 - 2^{m+n+1}(\alpha^*)^2(A^*)^2 - (\beta^*)^2(B^*)^2 - 2^n\beta^*\alpha^*B^*A^* - 2^{m+1}\alpha^*\beta^*A^*B^*$$

$$= 2^m\alpha^*\beta^*A^*B^*(1-2) + 2^n\beta^*\alpha^*B^*A^*(2-1)$$

$$= 2^n\beta^*\alpha^*B^*A^* - 2^m\alpha^*\beta^*A^*B^*$$

References

1. Culbert. C, Cayley-Dickson Algebras and Loops, Journal of Generalized Lie Theory and Applications, (1) (2007) no. 1, 1-17.
2. Malini Devi. B, Devibala. S, A View on the Mersenne and Mersenne-Lucas Numbers, GIS Science Journal, (8) 8 (2021), 784-789.
3. Malini Devi. B, Devibala. S, Some Sums Formulae for Products of terms of Mersenne and Mersenne-Lucas Numbers, Adalya Journal, (10) 8 (2021)74-79.
4. Serodio. R, On octonionic polynomials, Adv. Appl. Clifford Algebras, (17) (2007), no. 2, 245-258.
5. Tian. Y, Matrix representations of octonions and their applications, Adv. Appl. Clifford Algebras, (10) (2000), no. 1, 61-90.
6. http://en.wikipedia.org/wiki/Split-octonion