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Total Chromatic Number of Some Classes of Graphs

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ABSTRACT

A total coloring of a graph is a coloring to the elements (vertices and edges) of thegraph G, for which any adjacent vertices or edges and incident elements are colored differently. The total chromatic number of G is the minimum number of colors that needed in a *total coloring*. In this paper, we have determined the total chromatic number for pan graph, double wheel graph and double crown graph.

1. INTRODUCTION

In this paper, we have chosen finite, simple and undirected graphs. Let G = (V (G), E(G)) be a graph with the vertex set $V(G)$ and the edge set $E(G)$respectively. In 1965, the concept of total coloring was introduced by Behzad [1]and Vizing [8]. Also he conjecture that for any simple graph G can be total colored with $\Delta(G)+1 \le \chi''(G) \le \Delta(G)+2$, where $\Delta(G)$ is the maximum degreeof the graph G.This conjecture is called as the Total Coloring Conjecture(TCC).Let $f: V(G) \cup E(G) \to C$ be a *total coloring* of G, where C is set of colors and satisfies the given conditions:

(a) $f(a) \ne f(b), \forall a, b \in V(G)$ are any two adjacent vertices

(b) $f(e_1) \ne f(e_2), \forall e_1, e_2 \in E(G)$ are two any adjacent edges and

(c) $f(a) \ne f(e), \forall e \in E(G)$ is incident with any vertex $a \in V(G)$

The minimum number of colors needed in a total coloring of G is called the *total chromatic number* of G, and it is denoted by $\chi''(G)$. Rosenfeld [6] and Vijayaditya [7] verified the total coloring conjecture, for any graph G with $\Delta \le 3$. In Borodin [2] verified the total coloring conjecture(TCC)for the maximaum degree $\Delta \ge 9$ in planar graphs. In recent era, total coloring have beenextensively studied in different families of graph. Muthuramakrishnanand Jayaraman [4] have prove that the total chromatic number ofmiddle, total graph of star, square graph of bistar graph and line graphof bistar.

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Jayaraman and Muthuramakrishnan [5] have prove thatthe total chromatic number of twig graph, splitting and shadow graphof comb graph. Jayaraman and Muthuramakrishnan [3] prove thatthe total chromatic number of line, middle, total and splitting graph of double star graph.

2. PRELIMINARIES

Definition 2.1. The n- Pan graph P_n is the graph obtained by joining a cycle graph C_n to a singleton graph K_1 with a bridge.

Definition 2.2. The *double wheel graph*, denoted by DW_n is the graph obtained by joining all vertices of a disjoint union of two cycles C_n to an external vertex. That is, $2C_n + K_1$.

Definition 2.3. The *double Crown graph* C_n^{++} is the graph obtained from the cycle C_n by attaching two pendent edges at each vertex of C_n.

In this paper, we have discussed the total coloring of pan graph, double wheel graph and double crown graphand also obtained the total chromatic number of pan graph,double wheel graph and double crown graph.

3. MAIN RESULTS

Theorem 3.1. Let P_n be the n-pan graph of order $n \geq 3$. Then its total chromatic number is $\chi''(P_n) = 4$.

Proof. $V(P_n) = \{u\} \cup \{u_i : 1 \leq i \leq n\}$ and $E(P_n) = \{e\} \cup \{e_i : 1 \leq i \leq n\}$, where $\{e_i : 1 \leq i \leq n\}$ is an edge $\{v_i v_{i+1} : 1 \leq i \leq n\}$ and the edge $\{e\}$ is an edge $\{uu_1\}$ respectively.

Define the total coloring $c : V(P_n) \cup E(P_n) \to \{1, 2, 3, 4\}$ as follows. We assign the total coloring to all the vertices and edges as given below. We consider the following two cases.

Case(i): When *n* is odd

$c(u) = 2$

For $1 \leq i \leq n-1$

$$c(u_i) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$$

$c(v_n) = 3$

$$c(e_i) = \begin{cases} 3, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$$

$c(e_n) = 2$, $c(e) = 4$

Case(ii): When *n* is even

$c(u) = 4$

For $1 \leq i \leq n$

$$c(u_i) = \begin{cases} 1, & \text{if } i \text{ is odd} \\ 2, & \text{if } i \text{ is even} \end{cases}$$

$$c(e_i) = \begin{cases} 3, & \text{if } i \text{ is odd} \\ 4, & \text{if } i \text{ is even} \end{cases}$$

$c(e_n) = 2$, $c(e) = 4$

By applying the above method of coloring pattern, the pan graph is properly total colored with 4 colors. Hence the total chromatic number of pan graph is 4.

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Example 1: Consider the pan graph with 5 vertices

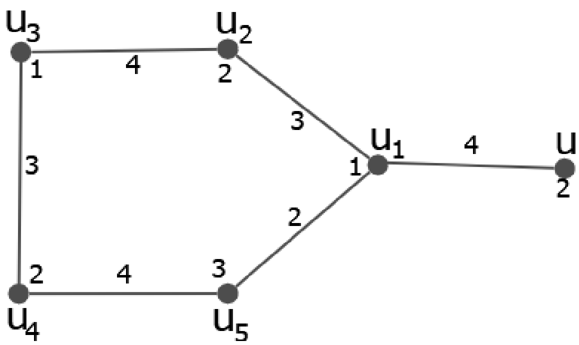

Figure 1: Total coloring of pan graph

Example 2: Consider the pan graph with 6 vertices

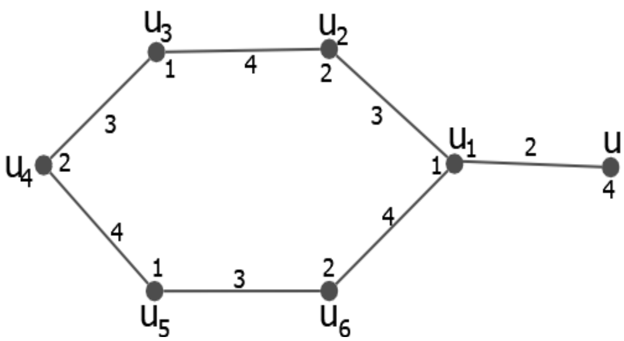

Figure 2: Total coloring of pan graph

Theorem 3.2. Let C_n^{++} be the double crown graph. Then its total chromatic number is $\chi''(C_n^{++}) = 5$.

Proof. Let C_n^{++} be double crown graph with $3n$ vertices and $3n$ edges. The vertex set and edge set are given by $V(C_n^{++}) = \{v_i : 1 \le i \le n\} \bigcup \{v_{i,j} : 1 \le i \le n \text{ and } j = 1,2\}$ and $E(C_n^{++}) = \{e_i = v_i v_{i+1} : 1 \le i \le n;\ i+1$ taken modulo $n\}$ $\bigcup \{e_{i,j} = (v_i, v_{i,j}) : 1 \le i \le n \text{ and } j = 1,2\}$.

Now we construct the total coloring $c : V(C_n^{++}) \bigcup E(C_n^{++}) \to \{1, 2, 3, 4, 5\}$. We assign the colors to these vertices and edges as follows. We consider the following two cases.

Case(i): When n is even, For $1 \le i \le n$,

$$c(v_i) = \begin{cases} 1, & \text{if } i \equiv 1 (\text{mod } 2) \\ 2, & \text{if } i \equiv 0 (\text{mod } 2) \end{cases}$$

For $1 \le i \le n$ and $j = 1,2$,

$c(v_i v_j) = 3$

For $1 \le i \le n,\ i+1$ taken modulo n,

$$c(e_i) = \begin{cases} 3, & \text{if } i \equiv 1 (\text{mod } 2) \\ 4, & \text{if } i \equiv 0 (\text{mod } 2) \end{cases}$$

For $1 \le i \le n$ and $j = 1$,

$c(e_{i,j}) = 5$

For $1 \le i \le n$ and $j = 2$,

$$c(e_{i,j}) = \begin{cases} 2, & \text{if } i \equiv 1 (\text{mod } 2) \\ 1, & \text{if } i \equiv 0 (\text{mod } 2) \end{cases}$$

Therefore, the double crown graph is total colored with 5 colors. Hence the total chromatic number of double crown graph is 5.

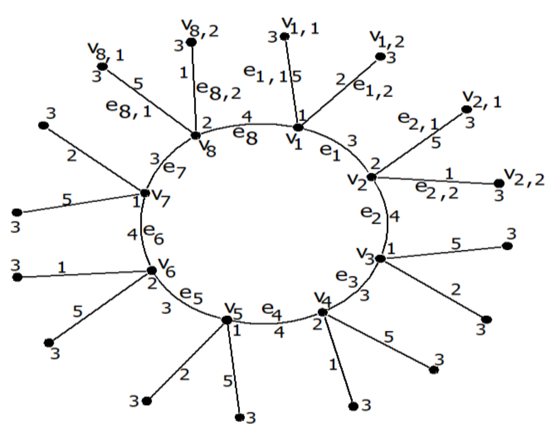

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Figure 3: Total coloring of double crown graph $\boldsymbol{C_8^{++}}$

Case(ii): When n is odd

For this case, the vertices and edges are colored as in case(i), except these vertices and edges

$c(v_n) = 3,\ c(v_{i,j}) = 3,\ c(v_{n,1}) = 4$

$c(v_{n,2}) = 4,\ c(e_{1,1}) = 4,\ c(e_{n,1}) = 1$

The vertices v_n, $v_{n,1}$ and $v_{n,2}$ are colored with the colors 3, 4 and 4. The edges e_n, $e_{1,1}$ and $e_{n,1}$ are assigned with the colors 5, 4 and 1.

By applying the above method of coloring pattern, the graph double crown graph is properly total colored with 5 colors. Therefore the double crown graph is total colored with 5 colors. Hence the total chromatic number of double crown graph is 5.

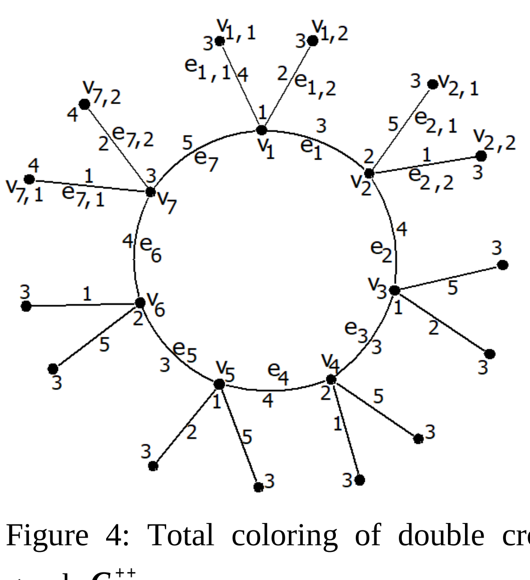

Figure 4: Total coloring of double crown graph $\boldsymbol{C_7^{++}}$

Theorem 3.3: Let DW_n be the double wheel graph. Then its total chromatic number is $\chi''(DW_n) = 2n+1,\ n \geq 4$.

Proof. Let DW_n be the double wheel graph with $2n+1$ vertices and $2n$ edges. Let v_0 be the apex vertex and let $\{u_1, u_2, ..., u_n\}$ and $\{v_1, v_2, ..., v_n\}$ be the rim vertices of DW_n. In DW_n, the vertex set and the edge set is given by

$$V(DW_n) = \{v_0\} \cup \{v_i : 1 \leq i \leq n\} \cup \{v_i' : 1 \leq i \leq n\}$$

and

$$E(DW_n) = \{e_i : 1 \leq i \leq n\} \cup \{e_i' : 1 \leq i \leq n\} \cup \{e_i'' : 1 \leq i \leq n\} \cup \{e_i''' : 1 \leq i \leq$$

,where $e_i\,(1 \leq i \leq n)$ is an edge $v_0 v_i (1 \leq i \leq n)$, $e_i' (1 \leq i \leq n)$ is an edge $v_0 v_i' (1 \leq i \leq n)$, $e_i'' (1 \leq i \leq n-1)$ is an edge $v_i v_{i+1} (1 \leq i \leq n, i+1,$ taken modulo $n)$, $e_i''' (1 \leq i \leq n)$ is an edge $v_i' v_{i+1}' (1 \leq i \leq n,\ i+1$ taken modulo n).

Now we construct a total coloring $c : V(DW_n) \cup E(DW_n) \to \{1,\ 2,\ 3, ..., 2n+1\}$. The total coloring is obtained by coloring these vertices and edges as follows.

For $1 \leq i \leq n$,

$c(v_0) = 2n+1$, $c(v_i') = i$,

$c(e_i) = i$

$$c(v_i) = \begin{cases} i+1, & \text{if } (i+1) \not\equiv 0 \bmod (2n+1) \\ 2n+1, & \text{otherwise} \end{cases}$$

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$$c(e_i^{'}) = \begin{cases} n+i, & \text{if } (n+i) \not\equiv 0 \bmod (2n+1) \\ 2n+1, & \text{otherwise} \end{cases}$$

$$c(e_i^{''}) = \begin{cases} n+i, & \text{if } (n+i) \not\equiv 0 \bmod (2n+1) \\ 2n+1, & \text{otherwise} \end{cases}$$

For $1 \le i \le n-1$,

$$c(e_i^{'''}) = \begin{cases} i+2, & \text{if } (i+2) \not\equiv 0 \bmod (2n+1) \\ 2n+1, & \text{otherwise} \end{cases}$$

$$c(e_n^{'''}) = 2$$

By applying the above method of coloring pattern, the graph double wheel is properly total colored with 2n+1 colors. Therefore the double wheel graph is total colored with 2n+1 colors. Hence the total chromatic number of double wheel graph is 2n+1.

Example 3: Consider the double wheel graph with 7 vertices

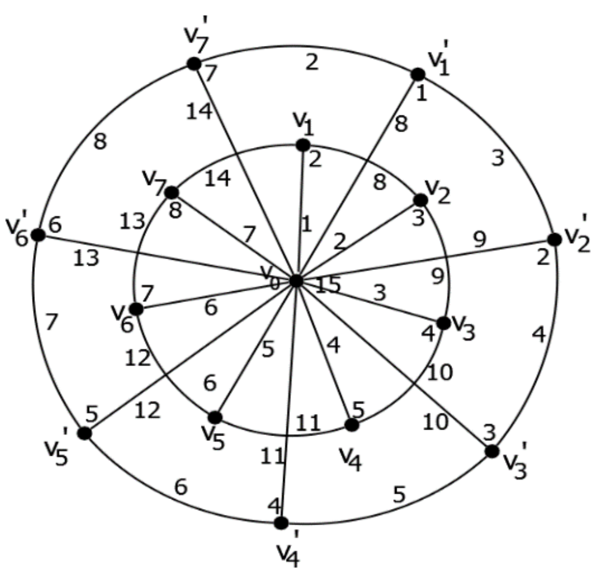

Figure 5: Total coloring of double wheel graph DW_7

REFERENCES

1. M. Behzad, Graphs and their chromatic numbers, Doctoral Thesis, Michigan StateUniversity, 1965.
2. O. V. Borodin, On the total coloring of planar graphs, J. Reine Angew. Math.394 (1969), 180-185.
3. G. Jayaraman and D. Muthuramakrishnan, Total Chromatic Number of Double Star Graph Families, *Jour of Adv Research in Dynamical & control system.*, 10 (5)(2018), 631-635.
4. D. Muthuramakrishnan and G. Jayaraman, Total chromatic number of star and Bistar graphs, *International Journal of Pure and Applied Mathematics*, 117(21)(2017), 699-708.
5. D. Muthuramakrishnan and G. Jayaraman, Totalcoloring of certain graphs, Advances and Applications in Discrete Mathematics, 27(1), (2021) 31-38.
6. M. Rosenfeld, On the total coloring of certain graphs, Israel J. Math. 9 (1971), 396-402.
7. N. Vijayaditya, On total chromatic number of a graph, J. London Math. Soc. (2)3 (1971), 405-408.
8. V. G. Vizing, Some unsolved problems in graph theory, Russian MathematicalSurvey 23(6) (1968), 125-141.

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