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Concentration of Contaminants in Groundwater

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Abstract: Groundwater pollutes by leachate is the most harming ecological effect over the whole existence of a hazardous waste landfill (HWL). To date, the percolation of bacteria and viruses through landfill leachate into the groundwater table poses a potential risk and potential hazards towards the public health and the ecosystem, remains an aesthetic concern. In this paper, we developed a mathematical model to study the mean concentration of contaminants (bacteria and viruses) by leachate, employing generalized-dispersion technique. Solute transport in porous media is discussed by means of advection - dispersion equation. The dispersion and mean concentration are obtained by Generalized-Dispersion method . Results are obtained analytically and the numerical values are computed graphically.

Keywords: Landfill, Generalized-Dispersion model, Groundwater, Contaminants, Porous medium.

1. INTRODUCTION

Groundwater contamination is a problem which affects every individual [1]. Groundwater flow and transport analysis have been an important research topic in the last three decades. Micro-organic (MO) contaminants in groundwater can have adverse effects on both the environment and on human health. They enter the natural environment as a result of various processes, their presence in groundwater is the result of current anthropogenic activity and pollution loads from the past[14]. The transport of dissolved contaminants or suspended contaminants (bacteria and virus) by flowing water is of great significance to study the relation between environmental protection and resource utilization[13].

Municipal solid waste (MSW) has become one of the main factors, which adversely influence the environment. With the MSW problems being increased, the occurrence of groundwater contamination is also increasing accordingly. Hence the study

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of groundwater contamination resulting from MSW landfill leachate has become a focused issue nowadays [12].

The extent of contamination depends on the nature of the contaminant and hydrogeology of the area. Most of the contaminants occur in nature as either point sources or distributed sources. Examples of point source contamination are municipal waste sites (landfill), industrial discharges, leaks and spills etc. Distributed sources occur as a result of effluent from leaking sewers and septic tanks, oil and chemical pipelines.

Landfilling has long been the major disposal method for both domestic and industrial wastes. Bacteria and virus from sewage sludges, waste water, septic tanks and other sources can be transported from groundwater to drinking water wells. During this transport, bacteria and virus can be either irreversibly or reversibly stored on surface material [5]. Valsamy and Nirmala P.Ratchagar[11] developed a mathematical model to study the unsteady transport of bacteria and virus in groundwater.

This paper deals with the study of a viscous , incompressible, contaminated fluid (bacteria and virus) flowing between two parallel plates.The particles are assumed to be spherical in shape and uniform in size.The number density of the contaminated particles is taken constant.

The main objective of this paper is to study the concentration of contaminants following the generalized dispersion model[2] .They showed that an exact solution of the unsteady convection diffusion equation valid for all time can be developed by using the series expansion originally proposed by Gill [1].The generalized dispersion theory developed can be extended to consider the dispersion phenomena for a wide variety of flows which are too complex to solve analytically [4,6,8,9].

2. MATHEMATICAL FORMULATION

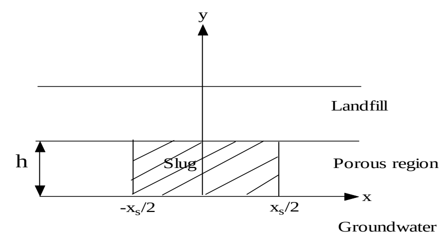

Figure1: Physical configuration

2.1 Dispersion

The concentration of contaminants(bacteria and virus) in the groundwater which diffuse in a fully developed flow, is given by

$$\frac{\partial C}{\partial t}+u\frac{\partial C}{\partial x}=D\left(\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}\right) \quad (1)$$

with the initial and boundary conditions,

(i) $$C(0,x,y)=\begin{cases} C_0, |x|\le \frac{x_s}{2} \\ 0, \ |x|>\frac{x_s}{2} \end{cases}$$

(ii) $$\frac{\partial C}{\partial y}(t,x,0)=\frac{\partial C}{\partial y}(t,x,h)=0$$

(iii) $$C(t,\infty,y)=\frac{\partial C}{\partial x}(t,\infty,y)=0$$

(2)

where, C_0 is the concentration of the initial slug input of length x_s

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where u =

$$\frac{c_0}{x}\left[\frac{Sinh[\sqrt{x}(y-1)]-Sinh[\sqrt{x}y]}{Sinh[\sqrt{x}]}+1\right]$$

$$+$$

$$\frac{4c_0}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n+1}Sin(2n+1)\pi y\,\frac{(1+Gx_1)^2 e^{x_1 t}}{x_1(1+Gx_1)^2+SG}$$

$$+\ \frac{(1+Gx_2)^2 e^{x_1 t}}{x_2(1+Gx_2)^2+SG}$$

Introducing non-dimensionless variables,

$$\theta=\frac{C}{C_0};\qquad X=\frac{Dx}{h^2\bar{u}};\qquad X_s=\frac{Dx_s}{h^2\bar{u}};\qquad Y=\frac{y}{h};\qquad \tau=\frac{Dt}{h^2};\qquad u^*=\frac{u}{\bar{u}};\qquad Pe=\frac{h\bar{u}}{D};$$

The equation (1) becomes,

$$\frac{\partial\theta}{\partial\tau}+U\frac{\partial\theta}{\partial X_1}=\frac{1}{Pe^2}\frac{\partial^2\theta}{\partial X_1^{\,2}}+\frac{\partial^2\theta}{\partial Y^2}\quad(3)$$

where, $\frac{1}{Pe^2}=\frac{D^2}{h^2\bar{u}^2}$ and $u^*=\frac{u}{\bar{u}}$

$$U=\frac{c_0}{Q^2 s}\left[\frac{SinhQ(y-1)-SinhQy}{SinhQ}+1\right]$$

Axial coordinate moving with the average velocity of flow is defined as $x_1=x-\bar{u}t$ which in dimensionless form is by

$X_1=X-\tau$, where, $X_1=\frac{Dx_1}{h^2\bar{u}}$

The non-dimensional initial and boundary conditions of (3) takes the form

(i) $\qquad \theta(0,x,y)=\begin{cases}1, & |X_1|\le\frac{X_s}{2}\\ 0, & |X_1|>\frac{X_s}{2}\end{cases}$

(ii) $\qquad \frac{\partial\theta}{\partial y}(\tau,X_1,0)=\frac{\partial\theta}{\partial Y}(\tau,X_1,1)=0$

(iii) $\qquad \theta(\tau,\infty,y)=\frac{\partial\theta}{\partial X_1}(\tau,\infty,Y)=0$

(4)

Following Gill and Sankarasubramanian(1970), the solution to equation (3) can be written as a series expansion in the form

$$\theta(\tau,X_1,Y)=\theta_m(\tau,X_1)+\sum_{k=1}^{\infty}f_k(\tau,Y)\frac{\partial^k\theta_m}{\partial X_1^{\,k}}$$

(5)

where, θ_m is the dimensionless cross sectional average concentration, given by

$$\theta_m(\tau,X_1)=\int_0^1\theta(\tau,X_1,Y)\,dY$$

(6)

Integrating equation (21) with respect to Y in [0,1] and substituting for θ_m we get ,

$$\frac{\partial\theta_m}{\partial\tau}=\frac{1}{Pe^2}\frac{\partial^2\theta_m}{\partial X_1^2}-\frac{\partial}{\partial X_1}\int_0^1 U\theta\,dY$$

(7)

The generalized dispersion model with time dependent dispersion coefficient can be written as

$$\frac{\partial\theta_m}{\partial\tau}=K_1\frac{\partial\theta_m}{\partial X_1}+K_2\frac{\partial^2\theta_m}{\partial X_1^2}+K_3\frac{\partial^3\theta_m}{\partial X_1^3}+\ldots\ldots\ldots\ldots\ldots$$

(8)

Introducing equations (5) and (8) in (7) and making use of the boundary condition (ii) of (4) gives

$$K_1\frac{\partial\theta_m}{\partial X_1}+K_2\frac{\partial^2\theta_m}{\partial X_1^2}+K_3\frac{\partial^3\theta_m}{\partial X_1^3}+\ldots\ldots\ldots\ldots\ldots$$

$$=\frac{1}{Pe^2}\frac{\partial^2\theta_m}{\partial X_1^2}-\frac{\partial}{\partial X_1}\int_0^1 U\left(\theta_m(\tau,x_1)+f_1(\tau,y)\frac{\partial\theta_m}{\partial x_1}+f_2(\tau,y)\frac{\partial^2}{\partial x}\right.$$

(9)

Comparing the coefficient of

$\frac{\partial^k\theta_m}{\partial X_1^{\,k}}$ $(k=1,2,3,\ldots\ldots)$, we get

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$$K_1 = -\int_0^1 U dY$$
(10)

$$K_2(\tau) = \frac{1}{Pe^2} - \int_0^1 U f_1(\tau, y) dY \qquad (11)$$

$$K_3(\tau) = -\int_0^1 U f_2(\tau, y) dY, \ldots\ldots\ldots.. \qquad (12)$$

Substituting equation (5) in (3)

$$\frac{\partial(\theta_m(\tau, X_1))}{\partial\tau} + f_1(\tau, y)\frac{\partial\theta_m}{\partial X_1}(\tau, X_1) + f_2(\tau, y)\frac{\partial^2\theta_m}{\partial X_1^2}(\tau, X_1) + \ldots\ldots\ldots)$$

$$+U \left(\frac{\partial(\theta_m(\tau, X_1))}{\partial\tau} + f_1(\tau, y)\frac{\partial\theta_m}{\partial X_1}(\tau, X_1) + f_2(\tau, y)\frac{\partial^2\theta_m}{\partial X_1^2}(\tau, X_1) + \ldots\ldots\ldots\right)$$

$$= \frac{1}{Pe^2}\frac{\partial^2(\theta_m(\tau, X_1))}{\partial X_1} + f_1(\tau, y)\frac{\partial\theta_m}{\partial X_1}(\tau, X_1) + f_2(\tau, y)\frac{\partial^2\theta_m}{\partial X_1^2}(\tau, X_1) + \ldots\ldots\ldots)$$

$$+ \frac{\partial^2(\theta_m(\tau, X_1))}{\partial Y^2} + f_1(\tau, y)\frac{\partial\theta_m}{\partial X_1}(\tau, X_1) + f_2(\tau, y)\frac{\partial^2\theta_m}{\partial X_1^2}(\tau, X_1) + \ldots\ldots\ldots) \qquad (13)$$

Following Gill and Sankarasubramanian(1970) and noting

$$\frac{\partial^{k+1}\theta_m}{\partial\tau\partial X_1^k} = \sum_{i=1}^{\infty} K_i(\tau)\frac{\partial^{k+i}\theta_m}{\partial X_1^{k+i}} \qquad for\ k = 1,2,3,\ldots\ldots.$$

we get,

$$\left[\frac{\partial f_1}{\partial\tau} - \frac{\partial^2 f_1}{\partial Y^2} + U + k_1(\tau)\right]\frac{\partial\theta_m}{\partial X_1} + \left[\frac{\partial f_2}{\partial\tau} - \frac{\partial^2 f_2}{\partial Y^2} + Uf + k_1(\tau) f_1 + k_2(\tau) - \frac{1}{Pe^2}\right]\frac{\partial^2\theta_m}{\partial X_1^2}$$

$$+\sum_{k=1}^{\infty}\left[\frac{\partial f_{k+2}}{\partial\tau} - \frac{\partial^2 f_{k+2}}{\partial Y^2} + Uf_{k+1} + k_1(\tau) f_{k+1} + \left(k_2(\tau) - \frac{1}{Pe^2}\right) f_k + \right.$$
$$= 0 \qquad (14)$$

with $f_0 = 1$. Equating the coefficients of $\frac{\partial^k\theta_m}{\partial X_1^k}$ $(k = 1,2,3,\ldots\ldots\ldots)$ to zeros, a set of differential equations are obtained:

$$\frac{\partial f_1}{\partial\tau} = \frac{\partial^2 f_1}{\partial Y^2} - U - K_1(\tau) \qquad (15)$$

$$\frac{\partial f_2}{\partial\tau} = \frac{\partial^2 f_2}{\partial Y^2} - Uf_1 - K_1(\tau) f_1 - K_2(\tau) + \frac{1}{Pe^2} \qquad (16)$$

$$\frac{\partial f_{k+2}}{\partial\tau} = \frac{\partial^2 f_{k+2}}{\partial Y^2} - Uf_{k+1} - K_1(\tau) f_{k+1} - \left(K_2(\tau) - \frac{1}{Pe^2}\right) f_k - \sum_{i=3}^{k+2} K_i f_{k+2-i} \qquad (17)$$

To evaluate $K_i's$ we should know $f_k's$. Therefore we have to solve equation (14) subject to the initial and boundary conditions,

(i) $f_k(0.y) = 0$ (18)

(ii) $\frac{\partial f_k}{\partial Y}(\tau, 0) = 0$ (19)

(iii) $\frac{\partial f_k}{\partial Y}(\tau, 1) = 0$ (20)

(iv) $\int_0^1 f_k(\tau, Y) dY = 0$, for k=1,2,3,……. (21)

From equation (10), we get K_1 as

$K_1(\tau) = 0$ (22)

Equation (11) implies

$$K_2(\tau)=\frac{1}{Pe^2}-\int_0^1 Uf_1 dY$$

Let

$$f_1=f_{10}(y)+f_{11}(\tau,y)$$

(23)

where, $f_{10}(y)$ corresponds to an infinitely wide slug which is independent of τ satisfies

(i) $f_{11}=-f_{10}(Y)\quad at\ \tau=0$

(ii) $\frac{\partial f_{11}}{\partial Y}=0\quad at\ Y=0$

(iii) $\frac{\partial f_{11}}{\partial Y}=0\quad at\ Y=1$

(iv) $\int_0^1 f_{11}dY=0$

(24)

Substituting (23) in (15) gives $\frac{d^2 f_{10}(Y)}{dY^2}=U(Y)$ and $\frac{\partial f_{11}}{\partial \tau}=\frac{\partial^2 f_{11}}{\partial Y^2}$ is the well-known heat conduction equation which is solved by separation of variables.

Hence the solution of f_1 is given by

$$f_1=\frac{1}{B}\left[\frac{c_0}{xSinh\sqrt{x}}\left(\frac{Sinh\sqrt{x}y}{x}Cosh\sqrt{x}\,\frac{Cosh\sqrt{x}}{x}Sinh\sqrt{x}+\frac{Sinh\sqrt{x}y}{x}+\frac{y^2}{2}Sinh\sqrt{x}\right)+\right]$$

$$\frac{4c_0}{\pi^3}\sum_{n=1}^{\infty}(A)\frac{(-Sin(2n+1)\pi y)}{(2n+1)^3}-\frac{By^2}{2}+c_1y+c_2+\sum_{m=1}^{\infty}A_m Cos(\lambda_m Y)e^{-\lambda_m^2\tau}$$

(25)

where,

$$A=\frac{(1+Gx_1)^2e^{x_1t}}{x_1(1+Gx_1)^2+SG}+\frac{(1+Gx_2)^2e^{x_1t}}{x_2(1+Gx_2)^2+SG}$$

$$B=\frac{c_0}{xSinh\sqrt{x}}\left[\frac{2-2Cosh\sqrt{x}+\sqrt{x}Sinh\sqrt{x}}{\sqrt{x}}+\frac{8c_0}{\pi^2}\sum_{n=1}^{\infty}\frac{A}{(2n+1)^2}\right.$$

$$c_1=-\frac{1}{B}\frac{c_0}{xSinh\sqrt{x}}\left(\frac{Cosh\sqrt{x}-1}{\sqrt{x}}\right)+\frac{4c_0}{\pi^2}\sum_{n=1}^{\infty}\frac{-A}{(2n+1)^2}$$

$$c_2=-\frac{1}{B}\frac{c_0}{xSinh\sqrt{x}}\left(\frac{1-Cosh\sqrt{x}}{x\sqrt{x}}+\frac{1}{6}Sinh\sqrt{x}\right)+\frac{4c_0}{\pi^4}\sum_{n=1}^{\infty}\frac{-1}{(2n+1)^4}-$$

$$+\frac{1}{B}\frac{c_0}{xSinh\sqrt{x}}\left(\frac{Cosh\sqrt{x}-1}{x\sqrt{x}}\right)+\frac{4c_0}{\pi^4}\sum_{n=1}^{\infty}\frac{A}{(2n+1)^4}$$

$$A_m=2\left\{\left(\frac{1}{B}\left\{\frac{c_0}{XSinh\sqrt{x}}\left(\frac{\sqrt{x}\left(Cosh^2\sqrt{x}+Sinh\sqrt{x}-Cosh\sqrt{x}\right)}{x}\right.\right.\right.\right.$$

$$\left(\frac{-Cos(\lambda_m)}{(\lambda_m)^2}\right)-\left(\frac{1}{B}\left\{\frac{c_0}{xSinh\sqrt{x}}\left(\frac{\sqrt{x}(Cosh\sqrt{x}-1)}{x}\right)+\frac{4c_0}{\pi^2}\sum_{n=1}^{\infty}\right.\right.$$

$$+\frac{1}{B}\left\{\frac{c_0}{xSinh\sqrt{x}}\left(\frac{-\sqrt{x}(1+Cosh\sqrt{x})}{x}\right)+4c_0\sum_{n=1}^{\infty}(A)(-1)^{2n+1}\right\}\Bigg($$

$$\frac{1}{B}\left\{\frac{c_0}{X\sinh\sqrt{x}}\left(\frac{1-Cosh\sqrt{x}}{x}\right)+4c_0\sum_{n=1}^{\infty}(A)\right\}\left(\frac{1}{(\lambda_m)^4}\right)$$

(26)

and $\lambda_m=m\pi$.

Therefore substituting f_1 in equation (11) gives the solution of K_2.

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Similarly $K_3(\tau)$, $K_4(\tau),...$ are obtained and we found that $K_i(\tau), i > 2$ are negligibly small compared to $K_2(\tau)$.
The dispersion model (8) takes the form

$$\frac{\partial \theta_m}{\partial \tau} = K_2 \frac{\partial^2 \theta_m}{\partial X_1^2} \qquad (27)$$

3.Mean Concentration

The solution to equation (27) can be solved with the initial and boundary conditions

$$\theta_m(0, X_1) = \begin{cases} 1, |X_1| \leq \frac{X_s}{2} \\ 0, |X_1| > \frac{X_s}{2} \end{cases} \quad (28)$$

can be obtained using Fourier Transform[10] as

$$\theta_m(0, X_1) = \frac{1}{2}\left[erf\left(\frac{\frac{X_s}{2} + X_1}{2\sqrt{T}} \right) + erf\left(\frac{\frac{X_s}{2} - X_1}{2\sqrt{T}} \right) \right]$$

(29)
where,

$$T = \int_0^\tau K_2(y)dy \quad and \quad erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-z^2} dz$$

4. RESULTS AND DISCUSSIONS

In this paper we have studied the concentration of contaminants consisting a mixture of solid phase (bacteria and virus) and fluid phase. Results of dispersion coefficient and mean concentration are obtained analytically and the numerical values have been computed using MATHEMATICA 8.0.

Figure 2 shows that the dispersion coefficient is greatest for bacteria(G>1) when compared with virus(G<1) and the fluid (G=1).

Figure 3 shows the mean concentration θ_m with axial distance X_1 for the contaminated phase and fluid phase. It depicts the marked variation of concentration with the axial distance and the dimensionless time. The curve is bell shaped and perfectly symmetrical about the origin. Also it is apparent from the figure the mean concentration is greater for bacteria compared with fluid and virus.

Figures 4 and 5 represents the graph of mean concentration versus dimensionless time τ for contaminated phase and fluid phase outside the slug (X_s = 0.02) and (X_1 =0.1) and inside the slug (X_s = 0.02) and (X_1 =0.05). In general for both the cases, we observe a gradual decrease in θ_m ,for increasing time τ . Also for very smaller values of τ , the concentration outside the slug and inside the slug shows a rapid decrease.

5. CONCLUSION

Groundwater contamination by pathogenic bacteria and viruses has long been recognized as a serious hazard to human health[3]. An analytical approach for analyzing two dimensional contaminant transport from a landfill has been described in detail by advection - dispersion equation. The behavior of dispersion coefficient K_2 increases with increase in time. Regarding the mean concentration, for smaller values of time τ , concentration θ_m shows a rapid decrease for both contaminated fluid phase.

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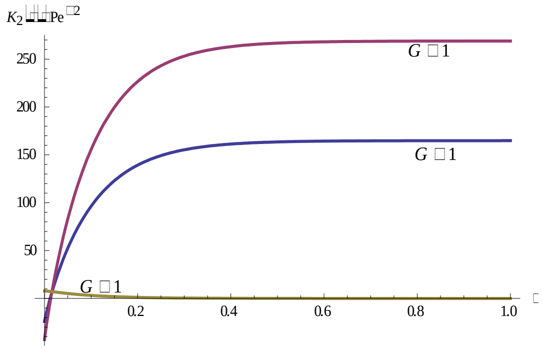


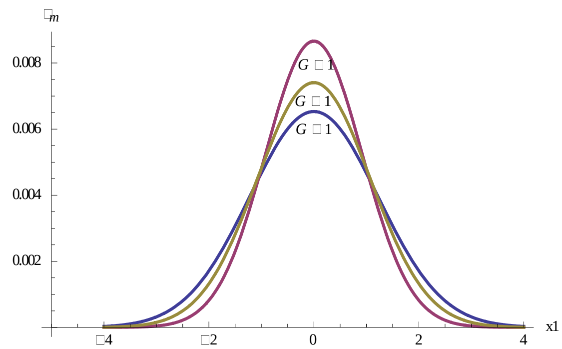
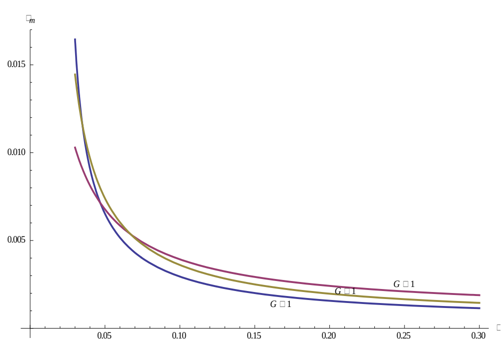


Figure 2:Dispersion coefficient varying with\Figure 3: Mean concentration varying along axial
dimensionless time for contaminated phasedistance at different time for fluid and
and fluid phase Contaminatedphase



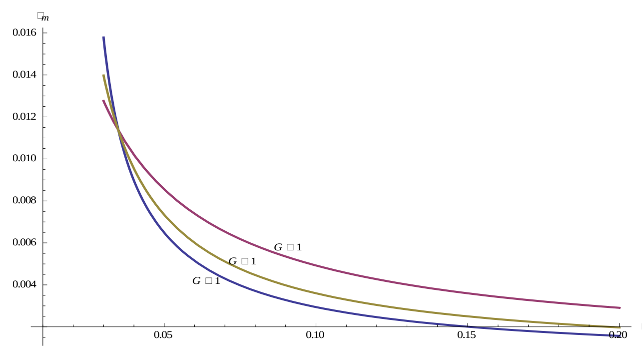


Figure 4: Mean concentration (outside the Figure 5: Mean concentration (inside the slug)
slug) varying with dimensionless time for varying with dimensionless time for contaminated
contaminated and fluid phase and fluid phase