

Cordial Labeling on Few Graphs of Subdividedshell Graphs

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Abstract

In 1987, Cahit[1] implemented cordial labeling. Cordial labeling[1] is defined as a function $h: V(\theta) \rightarrow \{0,1\}$ in which each edge ab is assigned the label $|h(a) - h(b)|$ with the conditions $|v_h(0) - v_h(1)| \leq 1$ and $|e_h(0) - e_h(1)| \leq 1$ where $v_h(0)$ and $v_h(1)$ signify the number of vertices with 0's and 1's, similarly $e_h(0)$ and $e_h(1)$ signify the number of edges with 0's and 1's. We want to show that the graphs such as uniform sub-divided shell bow graph, uniform sub-divided shell flower graph, one point union of multiple sub-divided shell graph, sub-divided shell Graph with star graphs coupled to the apex and path vertices are cordial.

Keywords: Cordial labeling, Sub-divided Shell Graph, Sub-divided shell bow graph, Sub-divided shell flower graph, Multiple sub-divided shell graph.

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1. Introduction

Graph labeling is one among the most exciting areas of study. Labeling is the

process of assigning values to edges or vertices. Alexander Rosa [7] pioneered the notion of graceful labeling. A few labeling

approaches were introduced later. Cahit [1] suggested cordial labeling as one such kind of labeling. Cordial labeling [1] is defined as a function $h: V(\theta) \rightarrow \{0,1\}$ in which each edge ab is assigned the label $|h(a) - h(b)|$ with the conditions $|v_h(0) - v_h(1)| \leq 1$ and $|e_h(0) - e_h(1)| \leq 1$ where $v_h(0)$ and $v_h(1)$ signify the number of vertices with 0's and 1's, similarly $e_h(0)$ and $e_h(1)$ signify the number of edges with 0's and 1's. Cahit [2] demonstrated the cordiality for the complete graph iff $n \leq 3$, ladders, friendship graphs, paths, wheels and pinwheels. The shell graph was first introduced by Deb and Limaye [3], followed by subdivided shell graphs and subdivided shell flower graphs by Jeba Jesintha and Hilda [4][6]. For further information, refer Gallian's dynamic survey [5]. Shell graphs are used in a variety of fields including X-ray crystallography, radar communication and networks, coding theory and other domains [4].

We prove that the following graphs are cordial: uniform sub-divided shell bow graph, uniform sub-divided shell flower graph, one point union of multiple sub-divided shell graph, sub-divided shell

graph with star graphs coupled to the apex and the path vertices of shell graphs.

2. Definitions

As in the literature [5], this section contains few definitions.

Definition 2. 1

Create a cycle. The apex of C_n with $(n - 3)$ chords with a shared end point is defined to be the shell graph [3]. C is used to represent shell graphs $(n, n - 3)$. Fan graphs are another name for shell graphs.

Definition 2. 2

In each path in the shell graph is divided, we get a sub-divided shell graph [6]. We denote sub-divided shell graph as SSG.

Definition 2. 3

A double sub-divided shell graph of the same order is defined as sub-divided shell bow graph [6].

Definition 2. 4

One vertex union of t copies of sub-divided shell graph and t copies of the complete graph K_2 is defined as a subdivided shell flower graph [4].

3. Main Results

We prove few theorems about sub-divided shell graphs in this section.

Theorem 3.1

The graph of a uniform sub-divided shell bow graph is cordial.

Proof.

The uniform sub-divided shell bow graph, indicated as G , is created by joining two copies of a sub-divided shell graph with a as an apex vertex. The vertices of the first copy of the shell graph are $c_1, c_2, c_3, \dots, c_n$. The vertices of the first copy of the SSG are $c'_1, c'_2, c'_3, \dots, c'_{n-1}$. The vertices of the second copy of the shell graph are $d_1, d_2, d_3, \dots, d_n$. The vertices of the second copy of the SSG are $d'_1, d'_2, d'_3, \dots, d'_{n-1}$. Figure 1 illustrates the graph G . The graph G 's vertices and edges are specified as $|V(G)| = 4n - 1$, $|E(G)| = 6n - 4$. The graph vertex labeling is defined as μ is from $V(G)$ to $\{0,1\}$.

Case 1: When n is even

$$\mu(a) = 1$$

$$\begin{aligned} \mu(c_j) & \text{ for } 1 \\ & = \begin{cases} 1; & j \equiv 1(\text{mod } 2) \\ 0; & j \equiv 0(\text{mod } 2) \end{cases} \leq j \leq n \\ \mu(c'_j) & \text{ for } 1 \\ & = \begin{cases} 0; & j \equiv 1(\text{mod } 2) \\ 1; & j \equiv 0(\text{mod } 2) \end{cases} \leq j \leq n-1 \\ \mu(d_j) & \text{ for } 1 \\ & = \begin{cases} 1; & j \equiv 1(\text{mod } 2) \\ 0; & j \equiv 0(\text{mod } 2) \end{cases} \leq j \leq n \\ \mu(d'_j) & \text{ for } 1 \\ & = \begin{cases} 0; & j \equiv 1(\text{mod } 2) \\ 1; & j \equiv 0(\text{mod } 2) \end{cases} \leq j \leq n-1 \end{aligned}$$

Case 2: When n is odd

$$\mu(a) = 0$$

$$\begin{aligned} \mu(c_j) & \text{ for } 1 \\ & = \begin{cases} 1; & j \equiv 1(\text{mod } 2) \\ 0; & j \equiv 0(\text{mod } 2) \end{cases} \leq j \leq n \\ \mu(c'_j) & \text{ for } 1 \\ & = \begin{cases} 1; & j \equiv 1(\text{mod } 2) \\ 0; & j \equiv 0(\text{mod } 2) \end{cases} \leq j \leq n-1 \\ \mu(d_j) & \text{ for } 1 \\ & = \begin{cases} 0; & j \equiv 1(\text{mod } 2) \\ 1; & j \equiv 0(\text{mod } 2) \end{cases} \leq j \leq n \\ \mu(d'_j) & \text{ for } 1 \\ & = \begin{cases} 1; & j \equiv 1(\text{mod } 2) \\ 0; & j \equiv 0(\text{mod } 2) \end{cases} \leq j \leq n-1 \end{aligned}$$

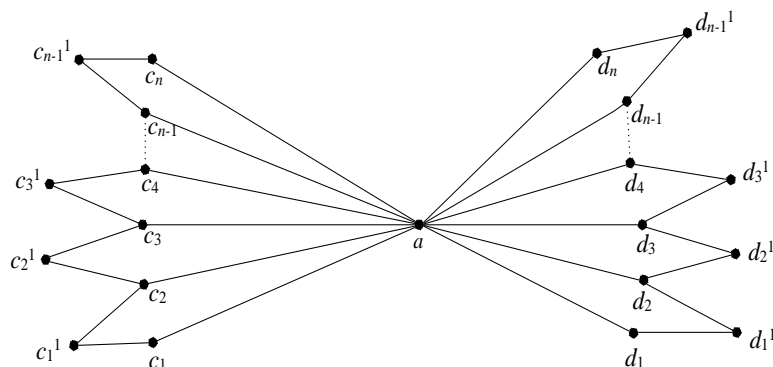


Figure 1: A graph of a uniform subdivided shell bow graph

The volume of vertices with identifiers 0's and 1's expressed as $v_\mu(0)$ and $v_\mu(1)$.

$$v_\mu(0) = 2n$$

$$v_\mu(1) = 2n - 1$$

The volume of edges with identifiers 0's and 1's expressed as $e_\mu(0)$ and $e_\mu(1)$.

$$e_\mu(0) = 3n - 2 = e_\mu(1)$$

As a result, the above labeling pattern meets both the requirements $|v_\mu(0) - v_\mu(1)| \leq 1$ and $|e_\mu(0) - e_\mu(1)| \leq 1$. Thus, subdivided shell bow graph admits cordiality.

Theorem 3.2

The graph of a uniform sub-divided shell flower graph is cordial.

Proof.

Let H stand for the sub-divided shell flower graph, which is defined as the union of t copies of the sub-divided shell graph with t pendent edges at a single point. The end vertices of the pendent edges attached to the apex vertex u are k_1, k_2, \dots, k_t . Let $l_1^1, l_2^1, l_3^1, \dots, l_n^1$ become the path vertices of the initial copy of the SSG's shell graph. Let the path vertices of the sub-divided shell network of the first copy of the SSG be $m_1^1, m_2^1, m_3^1, \dots, m_{n-1}^1$. Similarly, let $l_1^2, l_2^2, l_3^2, \dots, l_n^2$ be the path vertices of the second copy of SSG's shell graph and let the path vertices of the sub-divided shell network of the second copy

of the SSG be $m_1^2, m_2^2, m_3^2, \dots, m_{n-1}^2$. In general, let $l_1^t, l_2^t, l_3^t, \dots, l_n^t$ become the path vertices of the t^{th} copy of the SSG's shell graph. Let the path vertices of the subdivided shell network of the t^{th} copy of the SSG be $m_1^t, m_2^t, m_3^t, \dots, m_{n-1}^t$. Figure 2 depicts the graph for H . The graph H 's vertices and edges be defined as $|V(H)| = t(2n - 1) + 1 + k$, $|E(H)| = t(3n - 2) + k$. The graph H 's vertex labeling is specified as f is from $V(H)$ to $\{0,1\}$.

$$\begin{aligned} & \text{when } f(t_j) \\ & \equiv 1(\text{mod } 2) \end{aligned}$$

$$f(m_j^s) = \begin{cases} 0; & j \equiv 1(\text{mod } 2) \\ 1; & j \equiv 0(\text{mod } 2) \end{cases} \begin{matrix} \text{for } 1 \leq \\ j \leq n \\ , \\ 2 \leq s \leq \\ t - 1 \end{matrix}$$

$$\begin{aligned} & \text{when } f(t_j) \\ & \equiv 0(\text{mod } 2) \end{aligned}$$

$$f(k_j) = \begin{cases} 1; & j \equiv 1(\text{mod } 2) \\ 0; & j \equiv 0(\text{mod } 2) \end{cases} \begin{matrix} \text{for } 1 \leq j \\ \leq t \end{matrix}$$

Case 2. When n is an odd number

Case 1. When n is an even number

$$f(u) = 0$$

$$f(l_j^s) = \begin{cases} 1; & j \equiv 1(\text{mod } 2) \\ 0; & j \equiv 0(\text{mod } 2) \end{cases} \begin{matrix} \text{for } 1 \leq \\ j \leq n \\ , \\ 1 \leq s \leq t \end{matrix}$$

$$f(m_j^s) = \begin{cases} 1; & j \equiv 1(\text{mod } 2) \\ 0; & j \equiv 0(\text{mod } 2) \end{cases} \begin{matrix} \text{for } 1 \leq \\ j \leq n - 1 \\ , \\ 1 \leq s \leq t \end{matrix}$$

$$f(u) = 0$$

$$f(l_j^s) = \begin{cases} 1; & j \equiv 1(\text{mod } 2) \\ 0; & j \equiv 0(\text{mod } 2) \end{cases} \begin{matrix} \text{for } 1 \leq j \leq \\ n, \\ 1 \leq s \leq t \end{matrix}$$

$$f(m_j^s) = \begin{cases} 1; & j \equiv 1(\text{mod } 2) \\ 0; & j \equiv 0(\text{mod } 2) \end{cases} \begin{matrix} \text{for } 1 \leq j \leq \\ n - 1 \\ , \\ 1 \leq s \leq t \end{matrix}$$

$$f(k_j) = \begin{cases} 1; & j \equiv 1(\text{mod } 2) \\ 0; & j \equiv 0(\text{mod } 2) \end{cases} \begin{matrix} \text{for } 1 \leq j \leq t \end{matrix}$$

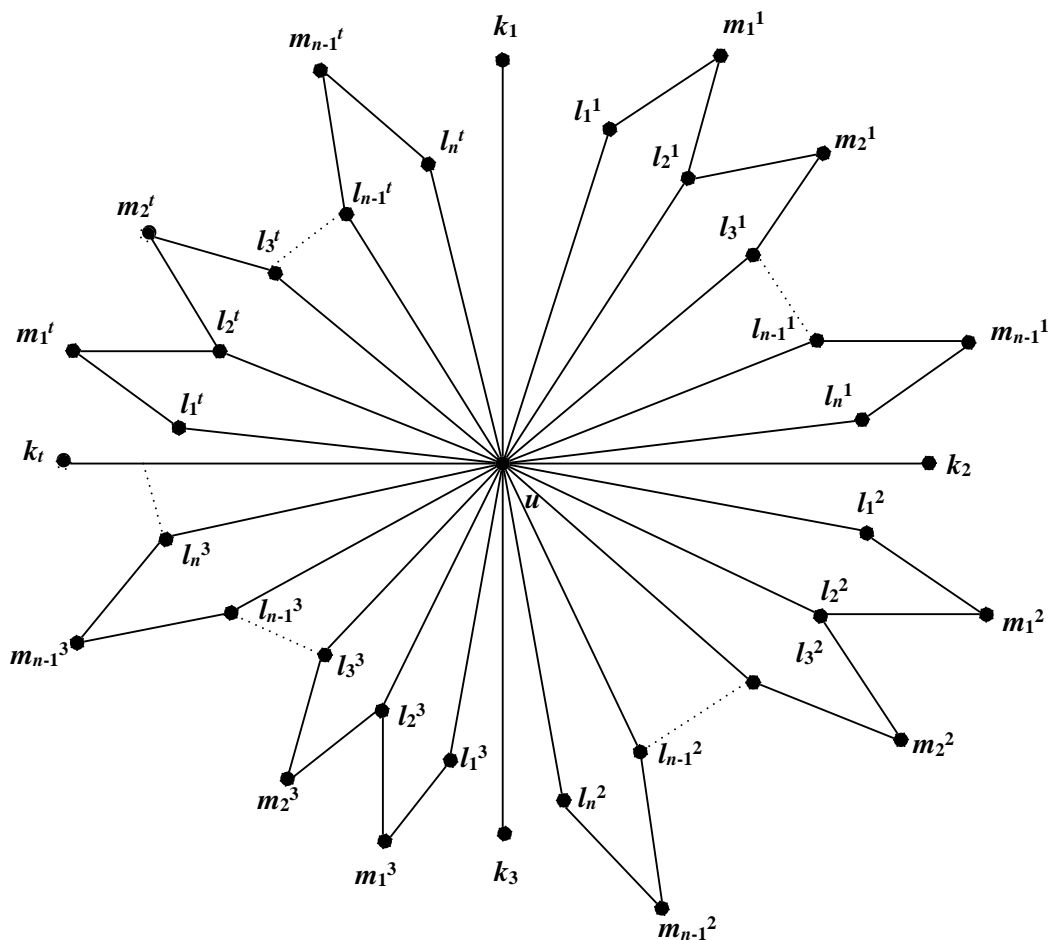


Figure 2: Subdivided shell flower graph

Label the number of vertices and edges as follows:

$$e_f(0) = (3n - 2) \frac{t}{2} = e_f(1)$$

Case a. When n, t even; n odd and t even

$$v_f(0) = (n - 1)t + \frac{t}{2} + 1$$

$$v_f(1) = (n - 1)t + \frac{t}{2}$$

Case b. When n is eventis odd

$$v_f(0) = (n - 1)t + \left\lfloor \frac{t}{2} \right\rfloor + 1 = v_f(1)$$

$$e_f(0) = (3n - 2) \frac{t}{2} = e_f(1)$$

Case c. When n, t is odd

$$\begin{aligned} v_f(0) &= (n - 1)t + \left\lfloor \frac{t}{2} \right\rfloor \\ &+ 1 \\ &= v_f(1) \end{aligned}$$

$$\begin{aligned} e_f(0) &= \left\lfloor \frac{3n - 2}{2} \right\rfloor t \\ &+ \left\lfloor \frac{t}{2} \right\rfloor \end{aligned}$$

$$\begin{aligned} e_f(1) &= \left\lfloor \frac{3n - 2}{2} \right\rfloor t + \left\lfloor \frac{t}{2} \right\rfloor \\ &+ 1 \end{aligned}$$

As a result, the above labeling pattern meets the conditions $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. Thus, Sub-divided shell flower graph admits cordial labeling.

Theorem 3.3

One point union of multiple sub-divided shell graph of same order is cordial.

Proof.

Joining m number of SSG to the apex u yields a multiple sub-divided shell graph G .

Let $a_1^1, a_2^1, a_3^1, \dots, a_n^1$ become the path vertices of the initial copy of the SSG's shell graph. Let $b_1^1, b_2^1, b_3^1, \dots, b_{n-1}^1$ become the path vertices of the sub-divided shell graph of initial copy of the SSG.

Similarly, let $a_1^2, a_2^2, a_3^2, \dots, a_n^2$ become the path vertices of the second copy of the SSG's shell graph and let $b_1^2, b_2^2, b_3^2, \dots, b_{n-1}^2$ be the path vertices of the sub-divided shell graph of second copy of the SSG.

In general, let $a_1^m, a_2^m, a_3^m, \dots, a_n^m$ be the path vertices of the shell graph of m^{th} of the SSG. Let $b_1^m, b_2^m, b_3^m, \dots, b_{n-1}^m$ be the path vertices of the sub-divided shell graph of m^{th} of the SSG.

Figure 3 illustrates the for G . Number of vertices and edges should be calculated as follows: $|V(G)| = m(2n - 1) + 1$, $|E(G)| = m(3n - 2)$.

The graph's vertex labeling is defined as g is from $V(G)$ to $\{0,1\}$.

$$g(u) = \begin{cases} 1; & \text{when } n, m \text{ even; } n, m \text{ odd; } n \text{ odd and } m \text{ even} \\ 0; & \text{otherwise} \end{cases}$$

$$g(a_k^h) = \begin{cases} 1; & k \equiv 1 \pmod{2} \\ 0; & k \equiv 0 \pmod{2} \end{cases} \quad \begin{array}{l} \text{for } 1 \leq k \leq n, 1 \leq h \leq m, \\ \text{when } g(a_k^h) \equiv 1 \pmod{2} \end{array}$$

$$g(a_k^h) = \begin{cases} 0; & k \equiv 1 \pmod{2} \\ 1; & k \equiv 0 \pmod{2} \end{cases} \quad \begin{array}{l} \text{for } 1 \leq k \leq n-1, 2 \leq h \leq m-1, \\ \text{when } g(a_k^h) \equiv 0 \pmod{2} \end{array}$$

$$g(b_k^h) = \begin{cases} 0; & k \equiv 1 \pmod{2} \\ 1; & k \equiv 0 \pmod{2} \end{cases} \quad \begin{array}{l} \text{for } 1 \leq k \leq n, 1 \leq h \leq m, \\ g(a_k^h) \equiv 1 \pmod{2} \end{array}$$

$$g(b_k^h) = \begin{cases} 1; & k \equiv 1 \pmod{2} \\ 0; & k \equiv 0 \pmod{2} \end{cases} \quad \begin{array}{l} \text{for } 1 \leq k \leq n-1, 2 \leq h \leq m-1, \\ g(a_k^h) \equiv 0 \pmod{2} \end{array}$$

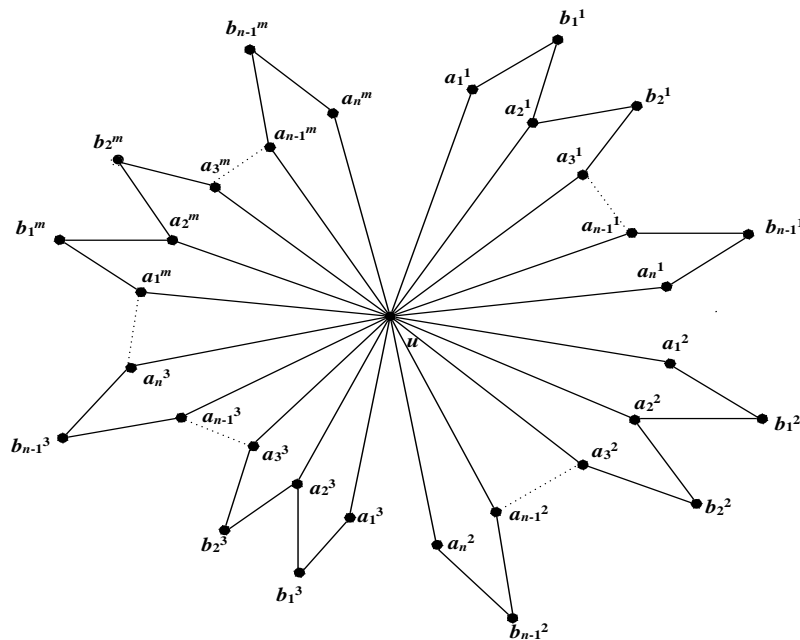


Figure 3: One point union of multiple sub-divided shell graph

Labeling the quantity of vertices and edges
 be

Case 1. When n, m is odd

$$v_g(0) = n(m + 1) - \left\lfloor \frac{n}{2} \right\rfloor = v_g(1)$$

$$e_g(0) = \left\lfloor \frac{n}{2} \right\rfloor (3m + 1) = e_g(1)$$

**Case 2. When n, m is even;
 When n is even, m is odd**

$$v_g(0) = n(m + 1)$$

$$v_g(1) = n(m + 1) - 1$$

$$e_g(0) = \frac{n}{2} (3m + 1) = e_g(1)$$

Case 3. When n is odd, m is even

$$v_g(0) = n(m + 1) - \left\lfloor \frac{n}{2} \right\rfloor = v_g(1)$$

$$e_g(0) = \left\lfloor \frac{n(3m + 1)}{2} \right\rfloor$$

$$e_g(1) = \left\lfloor \frac{n(3m + 1)}{2} \right\rfloor + 1$$

As a result, the above labeling pattern meets the conditions $|v_g(0) - v_g(1)| \leq 1$ and $|e_g(0) - e_g(1)| \leq 1$. Thus, Cordial labeling is possible with the multiple sub-divided shell graph.

Theorem 3.4

Sub-divided shell Graph with uniform star graphs coupled to the apex and path vertices admits cordiality.

Proof.

Let X be the graph formed by connecting the apex and path vertices of the shell graph with uniform star graphs [8]. The following is a description of the graph X . Let v represent the graph's apex. Let c_1, c_2, \dots, c_n denote the path vertices of the SSG's shell graph, and let d_1, d_2, \dots, d_{n-1} denote the path vertices of the SSG's split shell graph. The star graph associated to the vertex c_1 is $s_1^1, s_2^1, s_3^1, \dots, s_r^1$. Similarly, the star graph associated to the vertex c_2 is $s_1^2, s_2^2, s_3^2, \dots, s_r^2$. In general, the star associated to the vertex c_n is $s_1^n, s_2^n, s_3^n, \dots, s_r^n$. The star graph associated to the apex vertex v is t_1, t_2, \dots, t_r . The graph X has $|V(X)| = 2n + r(n + 1)$,

$|E(X)| = 3n + r(n + 1) - 2$. The graph's vertex labeling is defined as \emptyset is from $V(X)$ to $\{0,1\}$.

$$\emptyset(t_k) = \begin{cases} 0; & k \equiv 1(\text{mod } 2) \\ 1; & k \equiv 0(\text{mod } 2) \end{cases} \quad \text{for } 1 \leq i \leq r$$

Case 2. When n, s is odd and n

Case 1. When n, r is even and n odd, r even

even, r odd
 $\emptyset(v) = 0$

$$\emptyset(v) = \begin{cases} 1; & \text{if } n, s \text{ is even} \\ 0; & \text{otherwise} \end{cases}$$

$$\emptyset(c_k) = \begin{cases} 1; & k \equiv 1(\text{mod } 2) \\ 0; & k \equiv 0(\text{mod } 2) \end{cases} \quad \text{for } 1 \leq k \leq n$$

$$\emptyset(c_k) = \begin{cases} 1; & k \equiv 1(\text{mod } 2) \\ 0; & k \equiv 0(\text{mod } 2) \end{cases}$$

$$\emptyset(d_k) = \begin{cases} 0; & k \equiv 1(\text{mod } 2) \\ 1; & k \equiv 0(\text{mod } 2) \end{cases} \quad \text{for } 1 \leq k \leq n-1$$

$$\emptyset(d_k) = \begin{cases} 0; & k \equiv 1(\text{mod } 2) \\ 1; & k \equiv 0(\text{mod } 2) \end{cases}$$

$$\text{for } 1 \leq k \leq n \quad \text{When } i \equiv 1, 2(\text{mod } 4)$$

$$\emptyset(s_k^i) = \begin{cases} 1; & k \equiv 1(\text{mod } 2) \\ 0; & k \equiv 0(\text{mod } 2) \end{cases} \quad \text{for } 1 \leq k \leq r-1$$

When $i \equiv 0, 3(\text{mod } 4)$

$$\emptyset(s_k^i) = \begin{cases} 0; & k \equiv 1(\text{mod } 2) \\ 1; & k \equiv 0(\text{mod } 2) \end{cases} \quad \text{for } 2 \leq k \leq r$$

$$\text{When } i \equiv 1(\text{mod } 2)$$

$$\emptyset(s_k^i) = \begin{cases} 1; & k \equiv 1(\text{mod } 2) \\ 0; & k \equiv 0(\text{mod } 2) \end{cases}$$

$$\text{for } 1 \leq k \leq r$$

$$\emptyset(t_k) = \begin{cases} 0; & k \equiv 1(\text{mod } 2) \\ 1; & k \equiv 0(\text{mod } 2) \end{cases} \quad \text{for } 1 \leq k \leq r$$

When $i \equiv 0(\text{mod } 2)$

$$\emptyset(s_k^i) = \begin{cases} 0; & k \equiv 1(\text{mod } 2) \\ 1; & k \equiv 0(\text{mod } 2) \end{cases}$$

$$\text{for } 2 \leq k \leq r$$

- 1

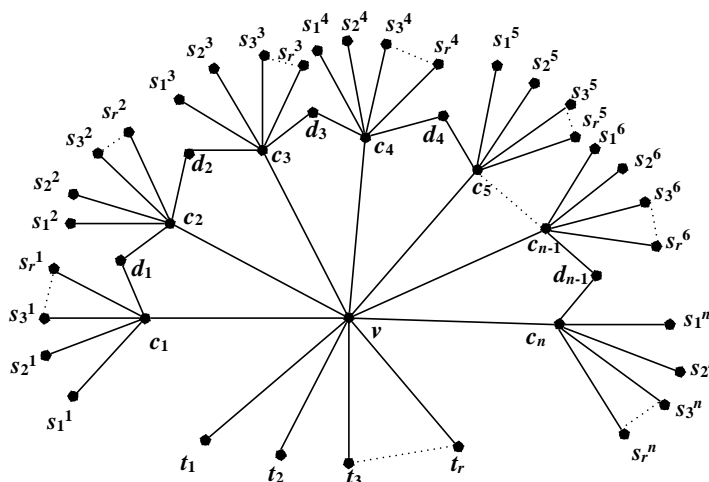


Figure 4. Sub-divided shell graph with star attached coupled at the apex and path vertices.

Label the number of vertices and edges as follows:

Case a. When n, r is even

$$v_{\emptyset}(0) = n + \frac{r}{2}(n + 1) = v_{\emptyset}(1)$$

$$e_{\emptyset}(0) = \frac{3n}{2} + \frac{r}{2}(n + 1) - 1 = e_{\emptyset}(1)$$

Case b. When n, r is odd

$$v_{\emptyset}(0) = n + \frac{r}{2}(n + 1) = v_{\emptyset}(1)$$

Subcase:1 When $n = 5$ and $r = 2i - 1$

$$e_{\emptyset}(0) = \left\lfloor \frac{3n}{2} \right\rfloor + \frac{r}{2}(n + 1)$$

$$e_{\emptyset}(1) = \left\lfloor \frac{3n}{2} \right\rfloor + \frac{r}{2}(n + 1) - 1$$

Subcase:2 When $n \geq 3$

$$e_{\emptyset}(0) = \left\lfloor \frac{3n}{2} \right\rfloor + \frac{r}{2}(n + 1) - 1$$

$$e_{\emptyset}(1) = \left\lfloor \frac{3n}{2} \right\rfloor + \frac{r}{2}(n + 1)$$

Case c. When n is odd, r is even

$$v_{\emptyset}(0) = n + \frac{r}{2}(n + 1) = v_{\emptyset}(1)$$

$$e_{\emptyset}(0) = \left\lfloor \frac{3n}{2} \right\rfloor + \frac{r}{2}(n + 1) - 1$$

$$e_{\emptyset}(1) = \left\lfloor \frac{3n}{2} \right\rfloor + \frac{r}{2}(n + 1)$$

Case d. When n is even, r is odd

$$v_{\emptyset}(0) = n + \left\lfloor \frac{r(n + 1)}{2} \right\rfloor + 1$$

$$v_{\emptyset}(1) = n + \left\lfloor \frac{r(n + 1)}{2} \right\rfloor$$

$$e_{\emptyset}(0) = \frac{3n}{2} + \left\lfloor \frac{r(n + 1)}{2} \right\rfloor$$

$$e_{\emptyset}(1) = \frac{3n}{2} + \left\lfloor \frac{r(n + 1)}{2} \right\rfloor - 1$$

As a result, the above labeling pattern meets the requirements $|v_{\emptyset}(0) - v_{\emptyset}(1)| \leq 1$ and $|e_{\emptyset}(0) - e_{\emptyset}(1)| \leq 1$. Thus, subdivided shell graph with uniform star graphs coupled to the apex and path vertices admits cordiality.

4. Conclusion

We showed that the uniform subdivided shell bow graphs, uniform sub-

divided shell flower graphs, one point union of multiple sub-divided shell graphs, sub-divided shell graphs with star graphs coupled to the apex and path vertices are cordial in this work.

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