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On Some Fixed Point Results in E – Algebra Fuzzy Metric Spaces

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ABSTRACT

In the existing literature, Banach contraction theorem as well as Meir-Keeler fixed point theorem were extended to E-Algebra fuzzy metric spaces. However, the existing extensions require strong additional assumptions. The purpose of this paper is to determine a class of E-Algebra fuzzy metric spaces in which both theorems remain true without the need of any additional condition. We demonstrate the wide validity of the new class.

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1. INTRODUCTION

Among the plethora of existing fixed point theorems, the Banach's contraction principle (BCP) is one of the very significant classical method in non-linear analysis. Many authors defined fuzzy metric space [1-5] in many ways. The most frequently cited one is that of George and Veeramani [6]. Several authors [17] obtained a Hausdorff topology by updating the definition of fuzzy metric space given by Kramosil and Michalek . It has been recently proved that the topology induced by a fuzzy metric space as per the definition of George and Veeramani are metrizable [6]. In order to apply BCP in fuzzy metric spaces, several research works [7, 8], [11-12] & [18-22] were carried out in numerous directions over these years. In

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the work of V.Gregori and A.Sapena [10], the Banach contraction Principle has been extended to complete fuzzy metric spaces by the generalized altering distance function. Following that Grabiec [9] illustrated a new definition of Cauchy sequence in fuzzy metric spaces and give the fuzzy version of the BCP. In 2020 the concept of fuzzy soft metric space is introduced by Kider [14] and many properties of this space are proved.

Algebra fuzzy metric space is coined by Kider in 2020 [15-16] by proving the basic properties of this space. It is shown that these fixed point theorems have numerous applications in the field of system of linear equation, solution of the differential equation and Integral equations. Consequently many general topology results are generalized to Algebra fuzzy topological space.

PRELIMINARIES

Definition : 2.1[15]

A binary operation $\oplus: [0,1] \times [0,1] \longrightarrow [0,1]$ is called a continuous *t*-conorm if it satisfies the following assertions:

(T 1) $\oplus$ is commutative and associative;

(T 2) $\oplus$ is continuous;

(T 3) $a \oplus 0 = a$ for all $a \in [0,1]$;

(T 4) $a \oplus b \leq c \oplus d$ when $a \leq c$ and $b \leq d$, with $a, b, c, d \in [0,1]$.

Definition : 2.2[14]

A Algebra fuzzy metric space is an ordered triple $(O, Q, \oplus)$ in which O is a nonempty set, $\oplus$ is a continuous *t*-conorm, and Q is a revised fuzzy set on $O \times O \times (0, \infty) \longrightarrow (0,1]$ such that

(AT 5) $Q(\kappa, \omega, t) \leq 1$;

(AT 6) $Q(\kappa, \omega, t) = 0$ if and only if $\kappa = \omega$;

(AT 7) $Q(\kappa, \omega, t) = Q(\omega, \kappa, t)$;

(AT 8) $Q(\kappa, z, t+s) \leq Q(\kappa, \omega, t) \oplus Q(\omega, z, s)$;

(AT 9) $Q(\kappa, \omega, -): (0, +\infty) \longrightarrow (0,1]$ is right continuous,

for all $\kappa, \omega, z \in O$ and $t, s > 0$.

In this paper, we will consider the following class of Algebra fuzzy metric spaces.

Definition : 2.3 ($E-$ Algebra fuzzy metric space).

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Let O denote a nonempty set, $\oplus$ refers to a continuous t-conorm, and Q serves as a Algebra fuzzy set on $O \times O \times (0, \infty) \longrightarrow (0,1]$ such that

(EAF 1) $(\kappa, \omega, t) \leq 1$;

(EAF 2) $Q(\kappa, \omega, t) = 0$ if and only if $\kappa = \omega$;

(EAF 3) $Q(\kappa, \omega, t) = Q(\omega, \kappa, t)$;

(EAF 4) $Q(\kappa, z, t + s) \leq Q(\kappa, \omega, t) \oplus Q(\omega, z, s)$;

(EAF 5) $Q(\kappa, \omega, -): (0, +\infty) \longrightarrow (0,1]$ is right continuous;.

(EAF 6) For some $r > 0$, the family $\{Q(\kappa, \omega, -): (0, r) \longrightarrow (0,1];\ (\kappa, \omega) \in O^2\}$ is uniformly equicontinuous, for all κ, ω, z ∈ O, and t, s > 0. Then, the triple $(O, Q, \oplus)$ is called an $E-$ Algebra fuzzy metric space.

Remark : 2.4

Obviously, all $E-$ Algebra fuzzy metric space is a Algebra fuzzy metric space. So, all properties in Algebra fuzzy metric spaces remain true in $E-$ revised fuzzy metric spaces.

Definition : 2.5

Let $(O, Q, \oplus)$ be a E-Algebra fuzzy metric space. then,

(i) A sequence $\{k_n\}_n$ converges to $\kappa \in O$ if and only if $Q(k_n, \kappa, t) \longrightarrow 0$ as $n \longrightarrow +\infty$ for all $t > 0$;

(ii) A sequence $\{k_n\}_n$ in O is a Cauchy sequence if and only if for all $\varepsilon \in (0,1)$ and $t > 0$, there exists n_0 such that $Q(k_n, x_m, t) < \varepsilon$ for all $m, n \geq n_0$;

(iii) The E-Algebra fuzzy metric space is complete if every Cauchy sequence converges to some $x \in O$.

Lemma : 2.6

Let $(O, Q, \oplus)$ be an E− Algebra fuzzy metric space, $\bar{Q}$ be the continuous extension of Q up to $[0, \infty)$, and $\{k_n\}_n$ be a sequence in O such that $\lim_{n \to \infty} Q(k_n, k_{n+1}, t) = 0$, for all $t > 0$. then,

$$\lim_{n \to \infty} \bar{Q}(k_n, k_{n+1}, t) = 0. \qquad (2.1)$$

Proof.

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For all $x, \omega \in O$, function $t \mapsto Q(x, \omega, t)$ is positive, continuous, and non-increasing on $(0, +\infty)$, so $\bar{Q}$ is well defined.

Let $\{t_n\}_n$ be a monotonically increasing sequence of positive numbers, converging to 0, and $\{k_n\}_n$ be a sequence in O such that $\lim_{n\to\infty} Q(k_n, k_{n+1}, t) = 0$, for all $t > 0$, i.e., for all $t > 0$ and for all $\varepsilon > 0$, there exists $n_0 \in N$, for all $n \geq n_0$; $Q(k_n, k_{n+1}, t) > 1 - \varepsilon$.

From which it follows that for all $t > 0$ and for all $\varepsilon > 0$, there exists $n_0 \in N$: for all $n \geq n_0$, and for all $k \in N$,

$$\left\{\begin{matrix} \bar{Q}(k_n, k_{n+1}, 0) - \bar{Q}(k_n, k_{n+1}, 0) + Q(k_n, k_{n+1}, t_k) + Q(k_n, k_{n+1}, t_k) \\ -Q(k_n, k_{n+1}, t) \end{matrix}\right\} > 1 - \frac{\epsilon}{2} \quad (2.2)$$

Therefore, for all $t > 0$ and for all $\varepsilon > 0$, there exists $n_0 \in N$, for all $n \geq n_0$ and for all $k \in N$,

$$\bar{Q}(k_n, k_{n+1}, 0) > \left\{\begin{matrix} 1 - \frac{\epsilon}{2} - |\bar{Q}(k_n, k_{n+1}, 0) - Q(k_n, k_{n+1}, t_k)| \\ -|Q(k_n, k_{n+1}, t_k) - Q(k_n, k_{n+1}, t)| \end{matrix}\right\} \quad (2.3)$$

On the other hand, by the fact that $\lim_k Q(k_n, k_{n+1}, t) = \bar{Q}(k_n, k_{n+1}, 0)$ and assumption $(EAF\ 6)$, we deduce that there exists $t_0 > 0$ and for all $\varepsilon > 0$, there exists $n_0 \in N$, for all $n \geq n_0$ and for all $k_0 \in N$, such that

$$|\bar{Q}(k_n, k_{n+1}, 0) - Q(k_n, k_{n+1}, t_k)| \geq 1 - \frac{\epsilon}{2}, \quad (2.4)$$

$$|Q(k_n, k_{n+1}, t_k) - Q(k_n, k_{n+1}, t)| \geq 1 - \frac{\epsilon}{2}, \quad (2.5)$$

for all $k > K_0$ and $t < t_0$.

Hence, by relations (2.3) – (2.5), it yields for all $\varepsilon > 0$, there exists $n_0 \in N$, for all $n \geq n_0$, $\bar{Q}(k_n, k_{n+1}, 0) > 1 - \varepsilon$, and this means

$$\lim_{n\to\infty} \bar{Q}(k_n, k_{n+1}, t) = 0 \quad (2.6)$$

Which achieves the proof of the lemma.

3. Main Results

Theorem : 3.1

Let $(O, Q, \oplus)$ be a complete E− Algebra fuzzy metric space. Let $L: O \longrightarrow O$ be a Algebra fuzzy contractive mapping with the contractive constant k, i.e., there exists $k \in]0, 1[$ such that

$$Q(L\kappa, L\omega, t) \leq k(Q(\kappa, \omega, t)), \quad (3.1)$$

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for all $\kappa, \omega \in O$ and for all $t > 0$. then, L has a unique fixed point $\kappa^{\oplus}$. Furthermore, for all $\kappa \in O$, the sequence $\{L^n k\}$ converges to $\kappa^{\oplus}$.

Proof.

Let $\kappa \in O$ and $\kappa_n = L^n k(n \in N)$. Let $t > 0$ and $n \in N$.

By inequality (3.1), we obtain

$$Q(\kappa_{n+1}, \kappa_{n+2}, t) \leq k\big(Q(\kappa_n, \kappa_{n+1}, t)\big) \tag{3.2}$$

for all $t > 0$ and for all n in N, which deduce that

$$\lim_{n\to\infty} Q(\kappa_n, \kappa_{n+1}, t) = 0, \tag{3.3}$$

for all $t > 0$. Now, to prove that $\{k_n\}_n$ is a Cauchy sequence,

we assume to the contrary.

Since $t \mapsto Q(\kappa, \omega, t)$ is a non-increasing function, there exists $\varepsilon \in (0, 1)$ and there exists $\xi > 0$ such that for all $p \in N$, there exists $n_p(\geq p) < m_p \in N$ so that

$$Q\left(k_{m_p}, k_{n_p}, t\right) \geq \varepsilon, \tag{3.4}$$

for all $t < \xi$.

Let $t_0 < max\{\xi, r\}$. By virtue of limit (3.3) and the last relation, we can write that there exists $\varepsilon \in (0, 1)$, for all $p \in N$, there exists $n_p \geq p < m_p \in N$:

$$Q\left(k_{m_p}, k_{n_p}, t_0\right) \geq \varepsilon$$

$$Q\left(k_{m_{p-1}}, k_{n_p}, t_0\right) < \varepsilon. \tag{3.5}$$

Taking into account the continuity of the function $t \mapsto Q(\kappa, \omega, t)$ and the fact that $Q\left(k_{m_{p-1}}, k_{n_p}, t_0\right) < \varepsilon$, we can choose $q_0 \in N$ such that

$$Q\left(k_{m_{p-1}}, k_{n_p}, t_0 - \frac{1}{q_0}\right) < \varepsilon. \tag{3.6}$$

By virtue of assumptions (*EAF 4*) and (*EAF* 4) and relations (3.5) and (3.6), it follows that

$$\varepsilon \leq Q\left(k_{m_p}, k_{n_p}, t_0\right)$$

$\leq$

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$$Q\left(k_{m_p}, k_{n_{p-1}}, \frac{1}{q_0}\right) \oplus Q\left(k_{m_{p-1}}, k_{n_p}, t_0 - \frac{1}{q_0}\right) \quad (3.7)$$

$$\leq \bar{Q}\left(k_{m_p}, k_{n_{p-1}}, 0\right) \oplus \varepsilon$$

So, according to assumptions (*EAF* 2)-(*EAF* 3), limit (3.3), and Lemma 2.6, one has

$$lim_{p\to\infty} Q\left(k_{m_p}, k_{n_p}, t_0\right) = \varepsilon. \quad (3.8)$$

Suppose that for all $p_1 \geq 0$, there exists $p \geq p_1$ such that $Q\left(k_{m_{p+1}}, k_{n_{p+1}}, t_0\right) \geq \varepsilon$ means, having in mind relations (3.1) and (3.8), that the sequence $\{k_n\}_n$ has two subsequences $\left\{k_{n_p}\right\}_p$ and $\left\{k_{m_p}\right\}_p$ verifying

$$lim_{p\to\infty} Q\left(k_{m_p}, k_{n_p}, t_0\right) = lim_{p\to\infty} Q\left(k_{m_{p+1}}, k_{n_{p+1}}, t_0\right) = \varepsilon, \quad (3.9)$$

(for the sake of simplicity, we have saved the same notation for the subsequence).

Now, we suppose that there exists $p_1 \geq 0$ such that $Q\left(k_{m_{p+1}}, k_{n_{p+1}}, t_0\right) < \varepsilon$ for all $p \geq p_1$. We claim that $lim_{p\to\infty} Q\left(k_{m_{p+1}}, k_{n_{p+1}}, t_0\right) = \varepsilon$.

Suppose not, i.e., there exists $\alpha > 0$ and two subsequences $\left\{k_{n_p}\right\}_p$ and $\left\{k_{m_p}\right\}_p$ verifying

$$Q\left(k_{m_{p+1}}, k_{n_{p+1}}, t_0\right) < \alpha + \varepsilon, \quad (3.10)$$

for all $p \in N$.

Having $q \in N$ satisfying $Q\left(k_{m_{p+1}}, k_{n_{p+1}}, t_0 - \left(\frac{1}{q}\right)\right) < \alpha + \varepsilon$, we obtain

$$\varepsilon \leq Q\left(k_{m_p}, k_{n_p}, t_0\right)$$

$$\leq Q\left(k_{m_p}, k_{n_{p+1}}, \frac{1}{2q}\right) \oplus Q\left(k_{m_{p+1}}, k_{n_{p+1}}, t_0 - \right.$$

1q ⊕ Qknp+1, knp, 12q

$$\leq \bar{Q}\left(k_{m_p}, k_{n_{p+1}}, 0\right) \oplus \alpha + \varepsilon \oplus \bar{Q}\left(k_{n_{p+1}}, k_{n_p}, 0\right) \longrightarrow \alpha + \varepsilon, \quad (3.11)$$

as $p \longrightarrow \infty$. This is a contradiction. Then,

$$lim_p Q\left(k_{m_{p+1}}, k_{n_{p+1}}, t_0\right) = \varepsilon. \quad (3.12)$$

Relations (3.8), (3.9), and (3.12) drive to a clear contradiction with condition (3.1). So,

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$\{k_n\}_n$ is a Cauchy sequence in the complete E-Algebra fuzzy metric space O and we deduce that there exists $\kappa^{\oplus} \in O$ such that

$$lim_p \; Q(k_n, \kappa^{\oplus}, t) = 0, \tag{3.13}$$

for all $t > 0$, and by relation (3.1), we obtain

$$Q(Lk_n, L\kappa^{\oplus}, t) \leq k(Q(k_n, \kappa^{\oplus}, t)), \tag{3.14}$$

for all $n \in N$ and for all $t > 0$.

Passing to the limit, having in mind the limit in (3.13), it follows that $Q(\kappa^{\oplus}, L\kappa^{\oplus}, t) = 0$, which, with assumption (AF 2) and relation (3.1), means that $\kappa^{\oplus}$ is the unique fixed point of mapping L. This achieves the proof.

Theorem : 3.2 (E-Algebra fuzzy Meir-Keeler fixed point theorem).

Let $(O, Q, \oplus)$ be a complete E− Algebra fuzzy metric space. Let $L: O \longrightarrow O$ be a E-Algebra fuzzy Meir-Keeler type mapping, i.e., for all $\varepsilon \in (0,1)$, there exists $\delta > 0$ such that

$$\varepsilon - \delta > Q(\kappa, \omega, t) \geq \varepsilon \Longrightarrow Q(L\kappa, L\omega, t) < \varepsilon, \tag{3.15}$$

for all $\kappa, \omega \in O$ and for all $t > 0$. Then, L has a unique fixed point $\kappa^{\oplus}$. Furthermore, for all $\kappa \in O$, the sequence $\{L^n k\}$ converges to $\kappa^{\oplus}$.

Proof.

Let $\kappa \in O$ and $\kappa_n = L^n \kappa \; (n \in N)$ and $t > 0$. Obviously, we have

$$Q(\kappa, L\kappa, t) - \delta > Q(\kappa, L\kappa, t) \geq Q(\kappa, L\kappa, t), \tag{3.16}$$

for all $\delta > 0$, and due to relation (3.15), we obtain $Q(L2\kappa, L\kappa, t) < Q(\kappa, L\kappa, t)$.

Recursively, we obtain a sequence $\{Q(\kappa_n, \kappa_{n+1}, t)\}_n$ in $[0,1]$ verifying

$$Q(\kappa_n, \kappa_{n+1}, t) > Q(\kappa_{n+1}, \kappa_{n+2}, t), \tag{3.17}$$

for all $n \in N$. It is a bounded non-increasing sequence. then, there exists a function $u: (0, \infty) \longrightarrow [0,1]$ such that

$$\lim_{n \to +\infty} Q(\kappa_n, \kappa_{n+1}, t) = \inf_{n \in N} Q(\kappa_n, \kappa_{n+1}, t) = u(t), \tag{3.18}$$

for all $t > 0$. We claim that $u(t) = 0$, for all $t > 0$.

Suppose not, i.e., there exists $t_0 > 0$ such that $u(t_0) \in (0, 1)$.

By the limit in (3.18), for all $\delta \in (0, u(t_0))$, there exists $n_0 \in N$ such that

$$u(t_0) - \delta > Q\ (\kappa_n, \kappa_{n+1}, (t_0)) \geq u(t_0), \tag{3.19}$$

for all $n \geq n_0$, which, with condition (3.15), implies that $Q(\kappa_{n+1}, \kappa_{n+2}, (t_0)) \geq u(t_0)$.

This is a clear contradiction with (3.18).

Therefore,

$$lim_n\ Q(\kappa_n, \kappa_{n+1}, t) = 0, \tag{3.20}$$

for all $t > 0$.

Now, we follow, exactly, the same lines as in the proof of theorem 3.1 to deduce that $\{k_n\}_n$ is a Cauchy sequence in the complete E-Algebra fuzzy metric space O, which deduce that there exists $\kappa^{\oplus} \in O$ such that

$$lim_n\ Q(\kappa^{\oplus}, \kappa_n, t) = 0. \tag{3.21}$$

On the other hand, for all $n \in N$ and all $\delta \in (0, Q(\kappa^{\oplus}, \kappa_n, t))$, we have

$$Q(\kappa^{\oplus}, \kappa_n, t) - \delta > Q(\kappa^{\oplus}, \kappa_n, t) \geq Q(\kappa^{\oplus}, \kappa_n, t). \tag{3.22}$$

Condition (3.15) assures that

$$0 \leq Q(L\kappa^{\oplus}, L\kappa_n, t) < Q(\kappa^{\oplus}, \kappa_n, t), \tag{3.23}$$

which, with the limit in (3.21), gives $lim_n\ Q(\kappa^{\oplus}, \kappa_n, t) = 0$, and finally

$$\kappa^{\oplus} = L\kappa^{\oplus}. \tag{3.24}$$

For the uniqueness, we assume that there exists $w^{\oplus}(\neq k^{\oplus}) \in O$ such that $w^{\oplus} = Lw^{\oplus}$.

It is clear that for all,

$\delta \in \left(0, Q(k^{\oplus}, w^{\oplus}, t)\right), Q(k^{\oplus}, w^{\oplus}, t) - \delta > Q(k^{\oplus}, w^{\oplus}, t) \geq Q(k^{\oplus}, w^{\oplus}, t)$.

Hence, by (3.15), $Q(Lk^{\oplus}, L\, w^{\oplus}, t) < Q(k^{\oplus}, w^{\oplus}, t)$ or $Q(k^{\oplus}, w^{\oplus}, t) < Q(k^{\oplus}, w^{\oplus}, t)$, a contradiction, and this achieves the proof.

Now, we give the following corollary.

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Corollary : 3.3

Let (O, d) be a complete metric space, and L a Meir-Keeler mapping on O, i.e., for each $\varepsilon > 0$, there exists $\delta > 0$ such that for all $\kappa, \omega \in O$,

$$\varepsilon \leq d(\kappa, \omega) < \varepsilon + \delta \Longrightarrow d(L(\kappa), L(\omega)) < \varepsilon. \tag{3.25}$$

Let Q be a function on $O \times O \times (0, +\infty)$ defined by

$$Q(\kappa, \omega, t) = \frac{d(\kappa,\omega)}{t+1 + d(\kappa,\omega)}. \tag{3.26}$$

then,

(1) $(O, Q, -)$ is an $E-$ Algebra fuzzy metric space, where $-$ is the product t-conorm.

(2) For all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\varepsilon - \delta > Q(\kappa, \omega, t) \geq \varepsilon \Longrightarrow Q(L\kappa, L\omega, t) < \varepsilon, \tag{3.27}$$

for all $\kappa, \omega \in O$ and for all $t > 0$.

Proof.

$(O, Q, -)$ is a E-Algebra fuzzy metric space and $\{t \mapsto Q(\kappa, \omega, t);\ \kappa, \omega \in O\}$ is a set of functions with common Lipschitz constant "1".

So, it is uniformly equicontinuous, this means that $(O, Q, -)$ is an $E-$ Algebra fuzzy metric space.

For the second assumption, it suffices to see that For all $\varepsilon > 0$, $\delta \in (0, \varepsilon)$, $t > 0$ and all $\kappa, \omega \in O$, we have

$$\varepsilon - \delta > Q(\kappa, \omega, t) \geq \varepsilon \Leftrightarrow \varepsilon - \delta > \frac{d(\kappa, \omega)}{t + 1 + d(\kappa, \omega)} \geq \varepsilon$$

$$\Leftrightarrow \varepsilon - \delta > \frac{1}{1+(t+1)\left(\frac{1}{d(\kappa,\omega)}\right)} \geq \varepsilon$$

$$\Leftrightarrow (t+1)\left(\frac{1}{\varepsilon} - 1\right) \geq d(\kappa, \omega) > (t + 1)\left(\frac{1}{\varepsilon - \delta} - 1\right) \tag{3.28}$$

Let $\varepsilon_0 = (t + 1)\left(\frac{1}{\varepsilon} - 1\right)$ and $\delta_{\varepsilon_0} > 0$ such that

$$\varepsilon_0 \geq d(\kappa, \omega) > \varepsilon_0 + \delta_{\varepsilon_0} \Longrightarrow d(L\kappa, L\omega) > \varepsilon_0. \tag{3.29}$$

Now, we choose δ in (3.28) such that $(t + 1)\left(\frac{1}{\varepsilon - \delta} - 1\right) > (t + 1)\left(\frac{1}{\varepsilon} - 1\right) +$

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δ_{ε_0}. Therefore, using relations (3.28) and (3.29), it follows that

$$\begin{aligned} \varepsilon - \delta > Q(\kappa,\omega,t) \geq \varepsilon & \\ \Rightarrow (t+1)\left(\frac{1}{\varepsilon}-1\right) & \\ > d(\kappa,\omega) & \\ \geq (t+1)\left(\frac{1}{\varepsilon}-1\right)+\delta_{\varepsilon_0} & \\ \Rightarrow d(L\kappa,L\omega) > (t+1)\left(\frac{1}{\varepsilon}-1\right) & \\ \Rightarrow \frac{d(\kappa,\omega)}{t+1+d(L\kappa,L\omega)} < \varepsilon & \\ \Rightarrow Q(L\kappa,L\omega,t) < \varepsilon, & \qquad (3.30) \end{aligned}$$

and this achieves the proof.

4. Application

The purpose of this section is to give an example of the existence of a solution for an integral equation, where we can apply theorem 3.1 to get its solution. For such integral equations, we refer the reader to where the authors provide a common solution for a system of two integral equations.

Consider the integral equation,

$\kappa(r) = g(r) + \int_0^r F(r,s,\kappa(s))\,ds$, for all $r \in [0,I], I > 0$, (3.31)

and Banach space $C([0,I],R)$ of all continuous functions defined on $[0,I]$ equipped with infimum conorm

$$\|\kappa\| = \inf_{r\in[0,1]} |\kappa(r)|, \kappa \in C([0,I],R), \qquad (3.32)$$

with induced metric

$$d(\kappa,\omega) = \inf_{r\in[0,1]} |\kappa(r)-\omega(r)|. \qquad (3.33)$$

Now, consider the E-Algebra fuzzy metric space with product t−conorm as

$Q(\kappa,\omega,t) = \frac{d(\kappa,\omega)}{t+d(\kappa,\omega)}$, for all $\kappa,\omega \in C([0,I],R), t > 0$.
(3.34)

According to standard Algebra fuzzy metric space and the corresponding metric space have same topologies. So, Algebra fuzzy metric space defined in (3.34) is complete.

Theorem : 4.1

Consider the integral operator L on $C([0,I],R)$ as

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$$L\kappa(r) = g(r) + \int_0^r F(r, s, \kappa(s))\, ds. \tag{3.35}$$

Suppose that there exists $f: [0, I] \times [0, I] \longrightarrow [0, \infty)$ such that $f \in L^1([0, I], R)$ and suppose that F satisfies the following condition:

$$|F(s, r, \kappa(r)) - F(s, r, \omega(r))| \leq f(r, s)|\kappa(s) - \omega(s)|, \tag{3.36}$$

for all $\kappa, \omega \in C([0, I], R)$ and for all $r, s \in [0, I]$ where

$$\inf_{r \in [0,1]} \int_0^r F(r, s)\, ds \geq k > 0. \tag{3.37}$$

Then, the integral equation (3.31) has a unique solution.

Proof.

Let $\kappa, \omega \in C([0, I], R)$ and consider

$$|L\kappa(r) - L\omega(r)| \leq \int_0^r |F(r, s, \kappa(s)) - F(r, s, \omega(s))| ds$$

$$\leq \int_0^r F(r, s)|(\kappa(s) - \omega(s)| ds$$

$$\leq kd(\kappa, \omega). \tag{3.38}$$

So,

$$d(L\kappa, L\omega) \leq kd(\kappa, \omega). \tag{3.39}$$

Using (40), we can write

$$Q(\kappa, \omega, t) = \frac{t}{d(\kappa, \omega)}$$

$$\leq \frac{t + d(L\kappa, L\omega) - t}{t}$$

$$\leq \left(\frac{d(L\kappa, L\,\omega)}{t}\right) \leq k\left(\frac{d(\kappa, \omega)}{t}\right)$$

$$\leq k(Q(\kappa, \omega, t). \tag{3.40}$$

Since all the conditions of theorem 3.1 hold, (3.31) has a unique solution.

5. CONCLUSION

In this paper, we prove that the Banach contraction theorem as well as the Meir-Keeler fixed point theorem remain true in E-Algebra fuzzy metric spaces with only a slight modification in the definition of Algebra fuzzy spaces. Last, in this paper, we give some results to illustrate the broad validity of our results.

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