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SIR Model with New Non-Linear Incidence and Treatment Rates with Application

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Abstract

In this article, we will propose SIR Model with new non-linear incidence and treatment rates for dynamic system, we study the stability of the proposed model in all equilibrium points of the model. Routh Hurwitz criteria and lyapunov function methods are used in this purpose. Will applied the propose SIR model to a real data that represent susceptible, infection and recovery of Corona epidemic (Covid-19) in Iraq for the period from 1 January to 31 March. In addition we estimate the model parameters by using least square root curve fit of Euler method. The endemic equilibrium points will be found and discuss the stability of the model at each equilibrium point and the trajectories of S,I and R are plotted to show the convergence of equilibrium points of the model .

Key words: SIR MODEL, Dynamical system, Epidemic models, Covid-19, Lyapunov function, Routh Hurwitz criteria.

1-Introduction

Basic molecular models for describing infectious disease transmission are presented in a series of three papers by W.O. Kermack and A.G. McKendrick in 1927, 1932, and 1933. The *SIS and SIR* model, due to Kermack and McKendrick [10],[11],[12], In 1981 Hethcote , Stech and van den Driessche studied the SIRS model [7] .

Many researchers have studied and analyzed different kinds of SIR models, in 2010 S.Pathak ,A.Maiti and G.P.Samanta studied the Rich dynamics of an SIR epidemic model [14] , In 2012 linhua zhou and Meng fan studied the dynamics of an SIR epidemic model with limited medical resources revisited [16], in 2013 Junhong Li and Ning cui studied the "dynamics Behavior for an SIRS Model with Nonlinear Incidence Rate and Treatment"[13]. in 2015

Olukayobe Adebimpe *et al.* studied the Stability Analysis of an SIR Model With Non-linear Incidence and Treatment rates[1]. In the same year Balram Dubey *et al.* studied the Dynamical proprties of SIR Model with Nonlinear Incidence and Treatment Rates [6]. In 2018 Dr.Mubarak Deeb studied the Role of Mathematical Models in Epidemiology [5]. In 2019 Amine *et al.* Analyze the Stochastic SIR Model with Vaccination and Nonlinear incidence Rate [3]. In 2019 Angel G.G *et al*. studied the Bifurcation Analysis of an SIR Model with Logistic Growth Nonlinear Incidence and Saturated Treatment [4]. In 2021 Fehaid Salem Alshammari and Muhammad Altaf Khan studied the dynamical behavior of a modified SIR model with nonlinear incidence and recovery rates [2].

2. Mathematical model

The entire population can be divided into three classes which are susceptible individuals; S, infected individuals ; I and excluded or recovered individuals ; R. Susceptible individuals are those who are healthy and can be infected and develop disease under the right conditions. The infected individuals are those who contracted the disease and are now infected with it, these individuals are able to transmit the disease to susceptible individuals through contacts. As time progresses, infected individuals lose the infection individuals are considered as recovered individuals which belong to recovered space (by auto recovery due to the body's immune response or by treatment). These recovered individuals are immune to infectious microbes and therefore do not infect again. The suggested SIR model can be formulated by the following system of differential equations:

$$\frac{dS}{dt} = \dot{S} = A - \delta_0 S - \frac{\alpha SI}{1 + \beta S + \gamma I + \lambda SI} \quad \ldots (1.a)$$

$$\frac{dI}{dt} = \dot{I} = \frac{\alpha SI}{1 + \beta S + \gamma I + \lambda SI} - \delta_0 I - \delta_1 I - \delta_2 I - \frac{wI}{1 + \theta I} \quad \ldots (1.b)$$

$$\frac{dR}{dt} = \dot{R} = \delta_2 I + \frac{wI}{1 + \theta I} - \delta_0 R \quad \ldots (1.c)$$

Where

$S(0) \geq 0$, $I(0) \geq 0$, $R(0) \geq 0$ and $\delta_3 = \delta_0 + \delta_1 + \delta_2$

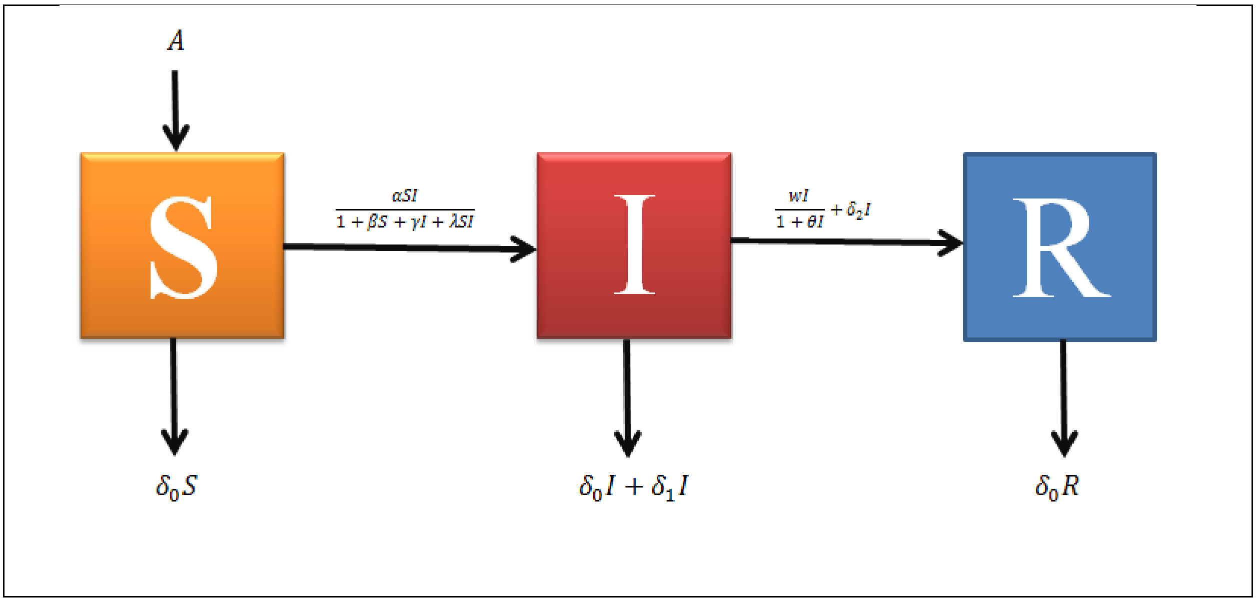

Figure 1: **SIR Model with new non-linear incidence and treatment rates**

The susceptible be recruited at a constant rate A (births rate) and δ_0 be the natural death rate of the population in each class. Where δ_1 represent the death rate of infected individuals due to infection and δ_2 be the natural recovery rate of infected individuals due to immunity (calculated by 1/recovery interval). In model (1.a, 1.b, 1.c) we consider the incidence rate as a function $f(S,I)=\frac{\alpha SI}{1+\beta S+\gamma I+\lambda SI}$, where α is the transmission rate, β is a measure of inhibition effect, such as preventive measure taken by susceptible individuals, and γ is a measure of inhibition effect such as treatment with respect to infected individuals ,and λ is mixing parameter of S and I.

The term $T(I)=\frac{wI}{1+\theta I}$ in (1.b) represent the treatment rate where w is a positive constant and θ is a constant taking the values depending into medical resource limitation .[15],[16]

2.1 Equilibrium Points of Mathematical model

In this section we will find Equilibrium points by solving the system (1.a, 1.b, 1.c) simultaneously and we obtain the following points:

1-$E_0(0,0,0)$ is the trivial Equilibrium points where $A=0$ only .

2-$E_1(S,0,0)$ is the Equilibrium points from equation (1.a) we get

$0=A-\delta_0 S-mS \quad then \quad S=\frac{A}{\delta_0} \quad and \quad E_1\left(\frac{A}{\delta_0},0,0\right)$ is Equilibrium points .

3- $E_2(S,I,0)$ is the equilibrium point from equation (1.c) we get

$$\delta_2 I + \frac{wI}{1+\theta I} = 0 \qquad \ldots(2)$$

Since $I \neq 0$ $\textit{Then}\ \delta_2 + \frac{w}{1+\theta I} = 0$ $\quad$ *we get condition*

$I = \frac{-(\delta_2+w)}{\delta_2\theta}$ For founded $E_2(S,I,0)$

Substitute (2) in (1.b) we get

$$\frac{\alpha SI}{1+\beta S+\gamma I+\lambda SI} = \delta_0 I + \delta_1 I \qquad \ldots(3)$$

Substitute (3) in (1.a) we get

$$0 = A - \delta_0 S - \delta_0 I - \delta_1 I$$

Impels that

$$S = \frac{A}{\delta_0} - \frac{(\delta_0+\delta_1)I}{\delta_0} \qquad \ldots(4)$$

$$S = b_1 + b_2 I \qquad where \quad b_1 = \frac{A}{\delta_0} \quad and \quad b_2 = -\frac{(\delta_0+\delta_1)}{\delta_0}$$

Substitute (4) in (1.b) we get

$$A_1 I^3 + A_2 I^2 + A_3 I + A_4 = 0 \qquad \ldots(5)$$

Where

$$A_1 = \delta_3\theta\lambda b_2$$
$$A_2 = \delta_3\lambda b_2 + \delta_3\theta(\beta b_2+\gamma+\lambda b_1) + w\lambda b_2 - \alpha b_2\theta$$
$$A_3 = \delta_3(\beta b_2+\gamma+\lambda b_1) + \delta_3\theta(1+\beta b_1) + w(\beta b_2+\gamma+\lambda b_1) - \alpha b_2 - \alpha b_1\theta$$
$$A_4 = \left(\delta_3(1+\beta b_1)\right) + \left(w(1+\beta b_1)\right) - (\alpha b_1)$$

Then the solution of equation (5) is $I_{1,2,3} = \{I_1, I_2, I_3\}$ we get $E_{2,1}(S_1,I_1,0)$, $E_{2,2}(S_2,I_2,0)$ *and* $E_{2,3}(S_3,I_3,0)$ are equilibrium points of equation (1.a, 1.b, 1.c) .

4-$E_{3,\tau}(S_\tau,I_\tau,R_\tau)$ *where* $\tau = 1,2,3,4$ are endemic equilibrium points of system , Now from equation (1.c) we get

$0 = \delta_2 I + \frac{wI}{1+\theta I} - \delta_0 R$ Impels that

$$R = \frac{\delta_2 I}{\delta_0} + \frac{wI}{\delta_0(1+\theta I)} \qquad \ldots(6)$$

Now from (1.b) we get

$$0 = \frac{\alpha SI}{1+\beta S+\gamma I+\lambda SI} - \delta_0 I - \delta_1 I - \delta_2 I - \frac{wI}{1+\theta I}$$

$$where \quad \delta_3 = \delta_0 + \delta_1 + \delta_2 \quad and\ I \neq 0$$

We get

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$$S = \frac{(\delta_3 + \delta_3\gamma I + \delta_3\theta I + \delta_3\theta\gamma I^2 + w + w\gamma I)}{(\alpha + \alpha\theta I - \delta_3\beta - \delta_3\lambda I - \delta_3\theta\beta I - \delta_3\theta\lambda I^2 - w\beta - w\lambda I\ )} \quad \ldots(7)$$

Now by adding (1.a), (1.b) and (1.c) we get

$$A - \delta_0 S - \delta_0 I - \delta_1 I - \delta_0 R = 0 \quad \ldots (8)$$

Now substitute (6) and (7) in (8) we get

$$A_1 I^4 + A_2 I^3 + A_3 I^2 + A_4 I + A_5 = 0 \quad (9)$$

Where

$A_1 = {\delta_3}^2\theta^2\lambda$

$A_2 = -A\delta_3\theta^2\lambda - (\delta_0\delta_3\theta^2\gamma - {\delta_3}^2\theta\lambda + \delta_3\theta(\alpha\theta - \delta_3\lambda - \delta_3\theta\beta - w\lambda) - w\delta_3\theta\lambda)$

$$\begin{aligned} A_3 = &-A\delta_3\theta\lambda + A\theta(\alpha\theta - \delta_3\lambda - \delta_3\theta\beta - w\lambda) - (\delta_0\delta_3\theta\gamma \\ &+ \theta(\delta_0\delta_3\gamma + \delta_0\delta_3\theta + \delta_0 w\gamma) + \delta_3(\alpha\theta - \delta_3\lambda - \delta_3\theta\beta - w\lambda) \\ &+ \left(\delta_3\theta(\alpha - \delta_3\beta - w\beta)\right) + w(\alpha\theta - \delta_3\lambda - \delta_3\theta\beta - w\lambda)) \end{aligned}$$

$$\begin{aligned} A_4 = &A(\alpha\theta - \delta_3\lambda - \delta_3\theta\beta - w\lambda) + \left(A\theta(\alpha - \delta_3\beta - w\beta)\right) \\ &- ((\delta_0\delta_3\gamma + \delta_0\delta_3\theta + \delta_0 w\gamma) + \left(\theta(\delta_0\delta_3 + \delta_0 w)\right) \\ &+ \left(\delta_3(\alpha - \delta_3\beta - w\beta)\right) + \left(w(\alpha - \delta_3\beta - w\beta)\right)) \end{aligned}$$

$A_5 = A(\alpha - \delta_3\beta - w\beta) - (\delta_0\delta_3 + \delta_0 w)$

Is an equation (9) of fourth order polynomial and we can find values of I and through the methods and can simply use the calculator program to calculate once the parameters. Then $I = \{I_\tau\}_{\tau=1}^{4}$ From equation (7) and (6) we get $S = \{S_\tau\}_{\tau=1}^{4}$, $R = \{R_\tau,\}_{\tau=1}^{4}$ then $E_{4,\tau}(S_\tau, I_\tau, R_\tau)\ where\ \tau = 1,2,3$ are equilibrium points of SIR model.

2.2. Reproduction number $\mathcal{R}_0$

By using equilibrium point $E_1\left(\frac{A}{\delta_0}, 0,0\right)$ the value of basic reproduction number

$$\mathcal{R}_0 = \frac{\alpha A}{(\delta_0 + \beta A)(\delta_3 + w)} \quad \ldots(10)$$

If $\mathcal{R}_0 > 1$ then the epidemic is spread, In case when $\mathcal{R}_0 < 1$ then the epidemic damped and If $\mathcal{R}_0 = 1$ then the epidemic is settlement then

2.3. Stability of equilibrium point by using Routh Hurwitz criteria

To find eigenvalue by solving $\left| J - \Psi \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right| = 0$ then

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$$\left|\begin{bmatrix} j_{11} & j_{12} & j_{13} \\ j_{21} & j_{22} & j_{23} \\ j_{31} & j_{32} & j_{33} \end{bmatrix} - \Psi \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right| = 0$$

$$\left\|\begin{bmatrix} -\left(\delta_0 + \frac{\begin{pmatrix} \alpha I\left((1+\beta S+\gamma I+\lambda SI)\right) \\ -\alpha SI\left((\beta+\lambda I)\right) \end{pmatrix}}{(1+\beta S+\gamma I+\lambda SI)^2}\right) - \Psi & -\frac{\begin{pmatrix} \alpha S(1+\beta S+\gamma I+\lambda SI) \\ -\alpha SI(\gamma+\lambda S) \end{pmatrix}}{(1+\beta S+\gamma I+\lambda SI)^2} & 0 \\ \frac{\begin{pmatrix} \alpha I\left((1+\beta S+\gamma I+\lambda SI)\right) \\ -\alpha SI\left((\beta+\lambda I)\right) \end{pmatrix}}{(1+\beta S+\gamma I+\lambda SI)^2} & \frac{\begin{pmatrix} \alpha S(1+\beta S+\gamma I+\lambda SI) \\ -\alpha SI(\gamma+\lambda S) \end{pmatrix}}{(1+\beta S+\gamma I+\lambda SI)^2} - \delta_0 - \delta_1 - \delta_2 - \left(\frac{w(1+\theta I) - w\theta I}{(1+\theta I)^2}\right) - \Psi & 0 \\ 0 & \delta_2 + \left(\frac{w(1+\theta I) - w\theta I}{(1+\theta I)^2}\right) & -\delta_0 - \Psi \end{bmatrix}\right\| = 0 \quad \ldots (11)$$

By solving the linear system (11) at each Equilibrium point we obtain

1- At the point $E_0(0,0,0)$ we obtain the following polynomial

$$\Psi^3 + f_{1,0}\Psi^2 + f_{2,0}\Psi + f_{3,0} = 0 \quad \ldots (12)$$

Where

$$f_{1,0} = 2\delta_0 + (\delta_3 + w)$$

$$f_{2,0} = {\delta_0}^2 + \delta_0(\delta_3 + w) + \delta_0(\delta_3 + w)$$

$$f_{3,0} = {\delta_0}^2(\delta_3 + w)$$

Then by Routh Hurwitz criteria this Equilibrium point $E_0(0,0,0)$ is stable if $f_{1,0} > 0$ and $f_{3,0} > 0$ with condition $f_{1,0}f_{2,0} - f_{3,0} > 0$

2-At the point $E_1\left(\frac{A}{\delta_0}, 0{,}0\right)$ we obtain the following polynomial

$$\Psi^3 + f_{1,1}\Psi^2 + f_{2,1}\Psi + f_{3,1} = 0 \quad \ldots (13)$$

Where

$$f_{1,1} = 2\delta_0 - \frac{v_3}{v_1}$$

$$f_{2,1} = {\delta_0}^2 - 2\delta_0\frac{v_3}{v_1}$$

$$f_{3,1} = {\delta_0}^2\frac{v_3}{v_1}$$

Then by Routh Hurwitz criteria $E_1\left(\frac{A}{\delta_0}, 0{,}0\right)$ is stable if $f_{1,1} > 0$ and $f_{3,1} > 0$ with condition $f_{1,1}f_{2,1} - f_{3,1} > 0$

3-At the points $E_{2,1}(S_1, I_1, 0)$, $E_{2,2}(S_2, I_2, 0)$ *and* $E_{2,3}(S_3, I_3, 0)$, $E_{3,\tau}(S_\tau, I_\tau, R_\tau)$ *where* $\tau =$ 1,2,3,4 we obtain the following polynomial

$$\Psi^3 + f_{1,s}\Psi^2 + f_{2,s}\Psi + f_{3,s} = 0 \quad , \quad s = 2{,}3 \quad \ldots (14)$$

Where

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$f_{1,s} = -(j_{1,1} + j_{2,2} + j_{3,3})$

$f_{2,s} = j_{1,1}j_{2,2} + j_{1,1}j_{3,3} + j_{2,2}j_{3,3} - j_{1,2}j_{2,1}$

$f_{3,s} = -(j_{1,1}j_{2,2}j_{3,3} - j_{1,2}j_{2,1}j_{3,3})$

Then by Routh Hurwitz criteria$E_{2,1}(S_1, I_1, 0)$, $E_{2,2}(S_2, I_2, 0)$ *and* $E_{2,3}(S_3, I_3, 0)$, $E_{3,\tau}(S_\tau, I_\tau, R_\tau)$ *where* $\tau = 1,2,3,4$ Are stable if $f_{1,s} > 0$ and $f_{3,s} > 0$ with condition $f_{1,s}f_{2,s} - f_{3,s} > 0$ where $s = 2,3$.

2.4 Lyapunov Function

We choose $V(S, I, R) = \frac{1}{2}(S^2 + I^2 + R^2)$

$$\frac{dV}{dt} = \dot{V}(S(t), I(t), R(t)) = S\dot{S} + I\dot{I} + R\dot{R}$$

$$\frac{dV}{dt} = AS - \delta_0(S^2 + I^2 + R^2) - \delta_1 I^2 + \delta_2(IR - I^2) + \frac{\alpha SI}{1 + \beta S + \gamma I + \lambda SI}(I - S) + (R - I)\frac{wI}{1 + \theta I} \; ...(15)$$

If $\frac{dv}{dt} \leq 0$ then the model is stable.

If $\frac{dv}{dt} > 0$ then the model is unstable.

3. Application

In this section, the stability of the equilibrium points in the dynamic system (1.a ,1.b, 1.c), which was previously mentioned, which will be applied to real data representing the numbers of people exposed to infection with the Corona epidemic, people infected with this epidemic, and people who have recovered and who have been cleared of the disease in interval (1 January to 31 March) from 2022, and this dynamic system will contain nonlinear functions, including: nonlinear function incidence rate and nonlinear treatment rate, and this system will also contain normal births and deaths and deaths of the Corona epidemic (covid-19)

Matlab software was used to make a program to find the estimated values of some parameters given in the dynamic system, and the stability of the dynamic system will be checked by Euler method that will be explained in the later steps.

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3.1 Data description

The data studied in this thesis represent the daily records of the Corona epidemic in Iraq [8] in addition to the average daily number of births, the daily rate of natural deaths and corona deaths as well, due to the importance of knowing the behavior of these data and studying the stability of these data through the dynamic system (1.a, 1.b, 1.c), especially in The waves that occur from one period to another, and Figure (2) represents the daily chronology of the real data of the Corona epidemic (Covid-19) in Iraq in the interval (1 January to 31 March).

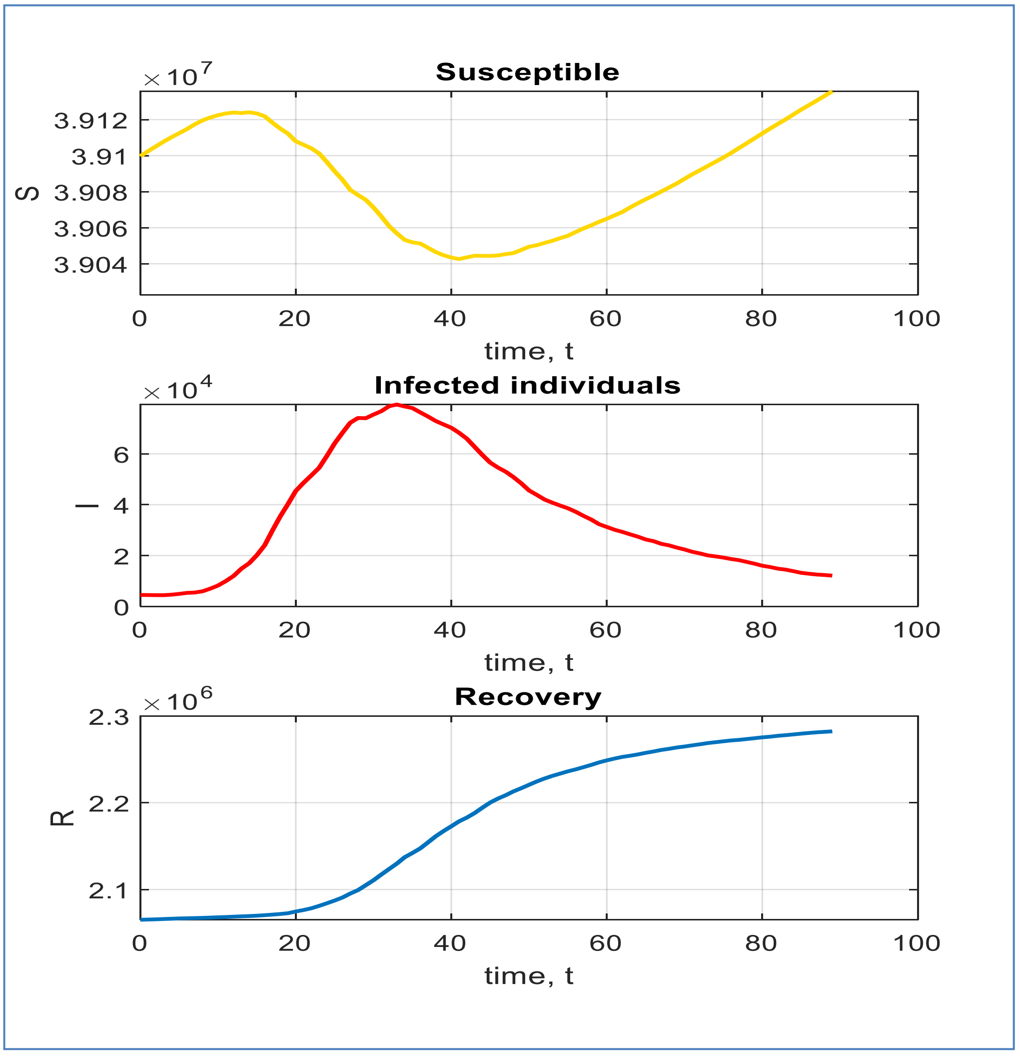

Figure 2: The real data of the (Covid-19) for the SIR Model in Iraq in the interval (1 January to 31 March)

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3.2. Data modeling

In this section, we will model the time series of daily records of the Corona epidemic in Iraq and use Matlab software to estimate some parameter values of the SIR model of the dynamic system by making use of the (lsqcurvefit) function in Matlab. Errors are found depending on the estimated values that we will get in the Euler method.

we will show when the dynamical system is stable based on the Routh Hurwitz criteria and the Lyapunov function for both the values . $A, \delta_0, \delta_1, \delta_2, S_0, I_0, R_0, N_0$ Parameters were obtained from [8],[9] most medical article mention that the incubation period equal two weeks then $\delta_2 = \frac{1}{14}$ and $\alpha, \beta, \gamma, \lambda, w, \theta$ parameters are estimated by solving the Euler formula using the least square curve fitting that guessed by the Matlab program by the function $(lsqcurvefit)$. The criterion used for this model is Akaike information criterion (AIC).

The following table (1) is for dynamic system parameters

parameters	Value	parameters	Value
A	3535.971377	λ	1.234513497837479e − 09
δ_0	$1.52399 * 10^{-05}$	w	$1.559565168453541 * 10^4$
δ_1	$4.75326 * 10^{-05}$	θ	$2.629612019106355 * 10^6$
δ_2	0.071428571428571	S_0	39099707.14
α	40.363793053343095	I_0	4519.929714
β	$5.102313603984587 * 10^2$	R_0	2065176.162
γ	$4.357131450606011 * 10^4$	N_0	41169403.23
resnorm	$7.166372764778849 * 10^8$	AIC	$1.448124363871285 * 10^3$

Table (1): Estimated values of model parameter

From equation (10) we get Reproduction number $\mathcal{R}_0 = 5.072467870213507 * 10^{-6}$, and from equation (9) we can get the values of I numerically, and then we find the values of S and R from equations (6) and (7) we get

$\boldsymbol{E}_{3,1}(-2.418794221910655 * 10^{14}, 5.156179111555119 * 10^{10}, 2.416672733868437 * 10^{14})$

$\boldsymbol{E}_{3.2}(3.385628932939827 * 10^7, 4.224286209370679 * 10^4, 1.979903557889571 * 10^8)$

$\boldsymbol{E}_{3.3}(2.320166000401431 * 10^8, 0.778579032237307, 4.038318205486847 * 10^3)$

$\boldsymbol{E}_{3.4}(-0.001927420704492, -3.802848747296032 * 10^{-7}, 2.320206416439007 * 10^8)$

Where

$A_1 = 43.630090857629654$, $A_2 = -2.249647474247501 * 10^{12}$

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$A_3 = 9.503322168403021 * 10^{16}$, $A_4 = -7.398947392686461 * 10^{16}$
The following table (2) shows the values of $f_{1,3}, f_{2,3}, f_{3,3}$ are found through equation (14) for all equilibrium points $E_{3,i}$, $i = 1,2,3,4$ will be as follows:

$f_{1,3}(E_{3,i})$	$f_{2,3}(E_{3,i})$	$f_{3,3}(E_{3,i})$
0.006914698881248	0.000000227981736	0.000000000001872
0.006922783462636	0.000000824220328	0.000000000010957
-0.007586931933926	-0.000000231944932	-0.000000000001769
$1.820404678531864*10^{19}$	$-1.195401278271067 * 10^{33}$	$-1.821779594071995 * 10^{28}$

Table 2: The values of $f_{1,3}, f_{2,3}, f_{3,3}$

Then from Routh Hurwitz criteria this points are stable if $f_{1,3} > 0$ and $f_{3,3} > 0$ with condition $f_{1,3}f_{2,3} - f_{3,3} > 0$, The following table shows The following table (3) shows the stability of the model by Routh Hurwitz criteria .

$E_{3,i}$	Stability
$E_{3,1}$	neglect because it is negative
$E_{3,2}$	Stable
$E_{3,3}$	Unstable
$E_{3,4}$	neglect because it is negative

Table 3: Stability by Routh Hurwitz criteria

The following table (4) shows the stability of the model by Lyapunov function, we obtained the following results through the equation (15) we get:

$E_{3,i}$	Lyapunov function $\dot{V}(E_{3,i})$	Stability
$E_{3,1}$	$-8.931301165369385 * 10^{9}$	neglect because it is negative
$E_{3,2}$	$-7.459963671863079 * 10^{-5}$	Stable
$E_{3,3}$	$3.733727984473082 * 10^{2}$	Unstable
$E_{3,4}$	$1.220703125 * 10^{-4}$	neglect because it is negative

Table 4: Stability by Table 3: Stability by Lyapunov function

Figure (3) shows the graph of the dynamic system trajectories at the initial points mentioned in Table (1), showing the trajectories approach to the stable equilibrium point $E_{3,2}$ by Euler's method.

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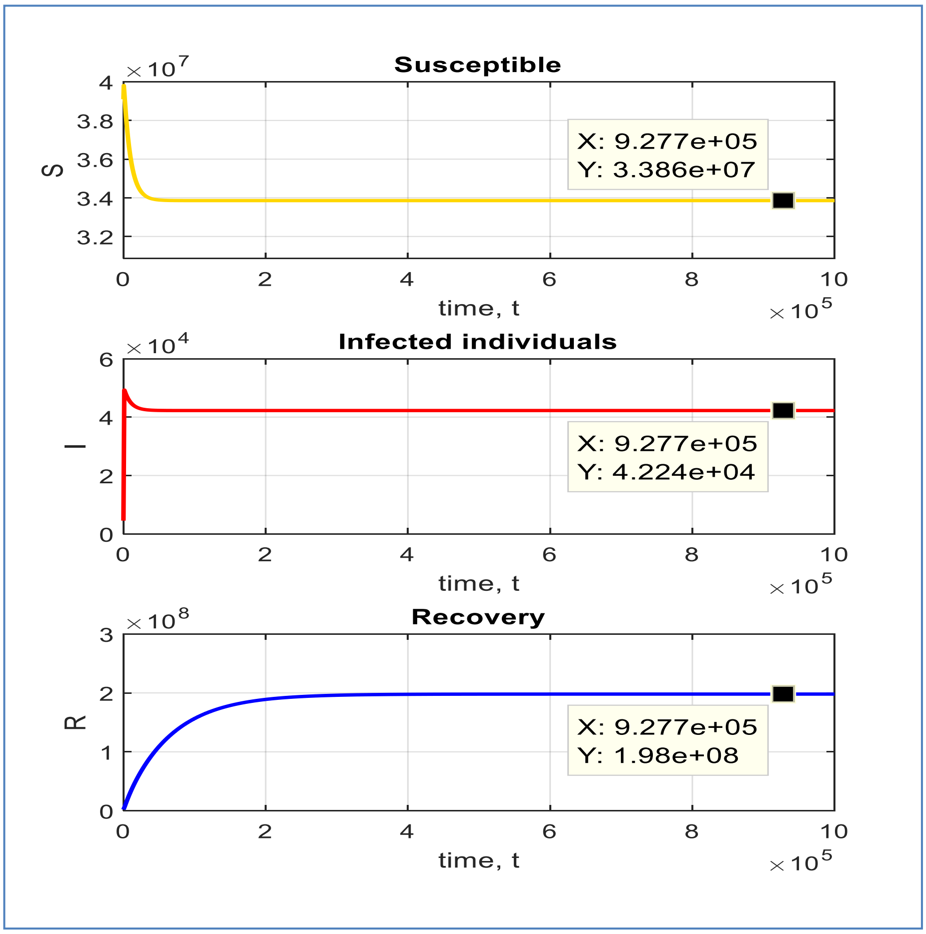

Figure 3:- trajectories plot of stable equilibrium point $E_{3,2}$

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