JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 4194-4201
https://publishoa.com
ISSN: 1309-3452

Hybrid Gaussian Mersenne and Gaussian Mersenne - Lucas Sequences

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ABSTRACT

In this communication, we define Hybrid Gaussian Mersenne and Hybrid Gaussian Mersenne-Lucas sequences and also, we give some relations among them. Moreover, we verify some well-known identities like Catalan, Cassini, d'Ocagne and Honsberger for Hybrid Gaussian Mersenne and Mersenne-Lucas sequences.

Keywords Mersenne sequence, Mersenne-Lucas sequence, Hybrid numbers

2020 Mathematics Subject Classification 11B99

Introduction

In Number theory, recollect that a Mersenne number of order n, denoted by M_n, and it is a number of the form $2^n - 1$, where n is a positive integer. Some studies about Mersenne sequence have been worked in [3, 8] and Mersenne-Lucas sequence in [4, 5].

We observe that some Mersenne numbers are prime and search for Mersenne primes in an active field of Number theory, Computer science and Coding theory. We note that if M_n is prime then n is prime and the converse is not true. Some special sequences and its Gaussian form were studied in [1, 2, 7].

In this paper, we associate the hybrid number introduced by [6] with Mersenne and Mersenne-Lucas sequences. Also, given some relations among them. Some well-known identities also verified through Binet's formula.

The Mersenne sequence $\{M_n\}$ are defined by the recurrence relation

$$M_n = 3M_{n-1} - 2M_{n-2}\,,\ n \geq 2, \text{ where } M_0 = 0, M_1 = 1.$$

The Mersenne- Lucas sequence $\{ML_n\}$ are defined recurrently by

$$ML_n = 3ML_{n-1} - 2ML_{n-2}\,,\ n \geq 2, \text{ where } ML_0 = 2, ML_1 = 3.$$

The Binet's formula for Mersenne and Mersenne-Lucas sequences are given by

$$M_n = 2^n - 1 \text{ and } ML_n = 2^n + 1.$$

Gaussian Mersenne and Gaussian Mersenne-Lucas Sequences

The Gaussian Mersenne sequence GM_n is defined by the recurrence relation

$$GM_{n+1} = 3GM_n - 2GM_{n-1}, n \geq 1$$

with initial conditions $GM_0 = -\frac{1}{2}i$ and $GM_1 = 1$.

Also, $GM_n = M_n + iM_{n-1}$, where M_n is the nth Mersenne sequence.

The Gaussian Mersenne-Lucas sequence GML_n is defined by the recurrence relation

$$GML_{n+1} = 3GML_n - 2GML_{n-1}, n \geq 1$$

with initial conditions $GML_0 = 2 + \frac{3}{2}i$ and $GML_1 = 3 + 2i$.
And $GML_n = ML_n + iML_{n-1}$, where ML_n is the nth Mersenne-Lucas sequence.

Theorem 1

The Binet formula for the Gaussian Mersenne sequence and the Gaussian Mersenne-Lucas sequence are

$$GM_n = 2^{n-1}\mathcal{a} - \mathcal{b} \text{ and } GML_n = 2^{n-1}\mathcal{a} + \mathcal{b}$$

where $\mathcal{a} = (2+i), \mathcal{b} = (1+i)$.

Proof

$GM_n = M_n + iM_{n-1}$

$= (2^n - 1) + i(2^{n-1} - 1)$

$= (2^n + i2^{n-1}) - (1+i)$

$= 2^{n-1}(2+i) - (1+i)$

$= 2^{n-1}\mathcal{a} - \mathcal{b}$, where $\mathcal{a} = (2+i), \mathcal{b} = (1+i)$.

$GML_n = ML_n + iML_{n-1}$

$= (2^n + 1) + i(2^{n-1} + 1)$

$= (2^n + i2^{n-1}) + (1+i)$

$= 2^{n-1}(2+i) + (1+i)$

$= 2^{n-1}\mathcal{a} + \mathcal{b}$, where $\mathcal{a} = (2+i), \mathcal{b} = (1+i)$.

Theorem 2

The generating function for the Gaussian Mersenne sequence and the Gaussian Mersenne-Lucas sequence are

$$f(t) = \frac{2t + (3t-1)i}{4t^2 - 6t + 2}$$

and

$$g(t) = \frac{4 - 6t + (3-5t)i}{4t^2 - 6t + 2}$$

Proof

Let $f(t) = \sum_{n=0}^{\infty} GM_n t^n$

Multiplying this equation by $1, -3t, 2t^2$ respectively and summing these equations, we obtain

$$(1 - 3t + 2t^2)f(t) = GM_0 + (GM_1 - 3GM_0)t + (GM_2 - 3GM_1 + 2GM_0)t^2 + \cdots + (GM_n - 3GM_{n-1} + 2GM_{n-2})t^n$$

$$= GM_0 + (GM_1 - 3GM_0)t$$

$$f(t) = \frac{GM_0 + (GM_1 - 3GM_0)t}{1 - 3t + 2t^2}$$

$$= \frac{-\frac{1}{2}i + \left(1 + \frac{3}{2}i\right)t}{1 - 3t + 2t^2}$$

$$= \frac{2t + (3t-1)i}{4t^2 - 6t + 2}$$

Similarly, let $g(t) = \sum_{n=0}^{\infty} GML_n t^n$

Multiplying this equation by $1, -3t, 2t^2$ respectively and summing these equations, we obtain

$$(1 - 3t + 2t^2)g(t)$$
$$= GML_0 + (GML_1 - 3GML_0)t + (GML_2 - 3GML_1 + 2GML_0)t^2 + \cdots + (GML_n - 3GML_{n-1} + 2GML_{n-2})t^n$$

$$= GML_0 + (GML_1 - 3GML_0)t$$

$$g(t) = \frac{GML_0 + (GML_1 - 3GML_0)t}{1 - 3t + 2t^2}$$

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https://publishoa.com
ISSN: 1309-3452

$$= \frac{(2+\frac{3}{2}i) + \left[3+2i-3(2+\frac{3}{2}i)\right]t}{1-3t+2t^2}$$

$$= \frac{4-6t+(3-5t)i}{4t^2-6t+2}$$

Theorem 3 (Catalan's Identity for Gaussian Mersenne and Gaussian Mersenne-Lucas sequences)

For any nonzero positive integers n, k we have

$$GM_{n+k}GM_{n-k} - GM_n{}^2 = 2^n ab(1 - 2^{-(k+1)}ML_{2k})$$

$$GML_{n+k}GML_{n-k} - GML_n{}^2 = 2^n ab(2^{-(k+1)}ML_{2k} - 1)$$

Proof

$$GM_{n+k}GM_{n-k} - GM_n{}^2 = (2^{n+k-1}a - b)(2^{n-k-1}a - b) - (2^{n-1}a - b)^2$$

$$= 2^n ab\left[1 - 2^{-(k+1)} - 2^{k-1}\right]$$

$$= 2^n ab\left[1 - 2^{-(k+1)}(1 + 2^{2k})\right]$$

$$= 2^n ab\left[1 - 2^{-(k+1)}ML_{2k}\right]$$

$$GML_{n+k}GML_{n-k} - GML_n{}^2 = (2^{n+k-1}a + b)(2^{n-k-1}a + b) - (2^{n-1}a + b)^2$$

$$= 2^n ab\,[2^{-(k+1)} + 2^{k-1} - 1]$$

$$= 2^n ab\left[2^{-(k+1)}(1 + 2^{2k}) - 1\right]$$

$$= 2^n ab\left[2^{-(k+1)}ML_{2k} - 1\right]$$

Theorem 4 (Cassini's Identity for Gaussian Mersenne and Gaussian Mersenne-Lucas sequences)

Let $n \geq 1$. Then we have

$$GM_{n+1}GM_{n-1} - GM_n{}^2 = -2^{n-2}ab$$

$$GML_{n+1}GML_{n-1} - GML_n{}^2 = 2^{n-2}ab$$

Theorem 5 (d'Ocagne's Identity for Gaussian Mersenne and Gaussian Mersenne-Lucas sequences)

For all $m, n \in \mathbb{Z}$ we have

$$GM_mGM_{n+1} - GM_{m+1}GM_n = -2^{m-1}abM_{n-m}$$

$$GML_mGML_{n+1} - GML_{m+1}GML_n = 2^{m-1}abM_{n-m}$$

Proof

$$GM_mGM_{n+1} - GM_{m+1}GM_n = (2^{m-1}a - b)(2^n a - b) - (2^m a - b)(2^{n-1}a - b)$$

$$= 2^{n-1}ab(1-2) - 2^{m-1}ab(1-2)$$

$$= 2^{m-1}ab(1 - 2^{n-m})$$

$$= -2^{m-1}abM_{n-m}$$

$$GML_mGML_{n+1} - GML_{m+1}GML_n = (2^{m-1}a + b)(2^n a + b) - (2^m a + b)(2^{n-1}a + b)$$

$$= -2^{n-1}ab(1-2) + 2^{m-1}ab(1-2)$$

$$= 2^{m-1}ab(2^{n-m} - 1)$$

$$= 2^{m-1}abM_{n-m}$$

Theorem 6

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ISSN: 1309-3452

For $n \geq 1$, we have

$$GM_n{}^2 + GM_{n+1}{}^2 = 2^{2n-2}a^2ML_2 + 2b^2 - 2^n ab ML_1$$

$$GML_n{}^2 + GML_{n+1}{}^2 = 2^{2n-2}a^2ML_2 + 2b^2 + 2^n ab ML_1$$

Proof

$$\begin{aligned} GM_n{}^2 + GM_{n+1}{}^2 &= (2^{n-1}a - b)^2 + (2^n a - b)^2 \\ &= 2^{2n-2}a^2 - 2^n ab + b^2 + 2^{2n}a^2 + b^2 - 2^{n+1}ab \\ &= 2^{2n-2}a^2(2^2+1) + 2b^2 - 2^n ab(1+2) \\ &= 2^{2n-2}a^2ML_2 + 2b^2 - 2^n ab ML_1 \end{aligned}$$

$$\begin{aligned} GML_n{}^2 + GML_{n+1}{}^2 &= (2^{n-1}a + b)^2 + (2^n a + b)^2 \\ &= 2^{2n-2}a^2 + 2^n ab + b^2 + 2^{2n}a^2 + b^2 + 2^{n+1}ab \\ &= 2^{2n-2}a^2(2^2+1) + 2b^2 + 2^n ab(1+2) \\ &= 2^{2n-2}a^2ML_2 + 2b^2 + 2^n ab ML_1 \end{aligned}$$

Theorem 7

For $n \geq 1$, we have

i. $GM_{n+1} + GM_{n-1} = 2^{n-2}aML_2 - 2b$
ii. $GM_{n+1} + GM_n = 2^{n-1}aML_1 - 2b$
iii. $GML_{n+1} + GML_{n-1} = 2^{n-2}aML_2 + 2b$
iv. $GML_{n+1} + GML_n = 2^{n-1}aML_1 + 2b$

Proof

i. $$\begin{aligned} GM_{n+1} + GM_{n-1} &= (2^n a - b) + (2^{n-2}a - b) \\ &= 2^{n-2}a(1+2^2) - 2b \\ &= 2^{n-2}aML_2 - 2b \end{aligned}$$

ii. $$\begin{aligned} GM_{n+1} + GM_n &= (2^n a - b) + (2^{n-1}a - b) \\ &= 2^{n-1}a(1+2) - 2b \\ &= 2^{n-1}aML_1 - 2b \end{aligned}$$

iii. $$\begin{aligned} GML_{n+1} + GML_{n-1} &= (2^n a + b) + (2^{n-2}a + b) \\ &= 2^{n-2}a(1+2^2) + 2b \\ &= 2^{n-2}aML_2 + 2b \end{aligned}$$

iv. $$\begin{aligned} GML_{n+1} + GML_n &= (2^n a + b) + (2^{n-1}a + b) \\ &= 2^{n-1}a(1+2) + 2b \\ &= 2^{n-1}aML_1 + 2b \end{aligned}$$

Theorem 8

For $n \geq 1$, we have

i. $GM_n + GML_n = 2^n a$
ii. $GM_n GML_n = M_{2n} - M_{2n-2} + 2iM_{2n-1}$

Proof

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https://publishoa.com
ISSN: 1309-3452

i. $GM_n + GML_n = (2^{n-1}\mathscr{a} - \mathscr{b}) + (2^{n-1}\mathscr{a} + \mathscr{b})$

$= 2^n\mathscr{a}$

ii. $GM_nGML_n = (2^{n-1}\mathscr{a} - \mathscr{b})(2^{n-1}\mathscr{a} + \mathscr{b})$

$= 2^{2n-2}\mathscr{a}^2 - \mathscr{b}^2$

$= 2^{2n-2}(3 + 4i) - 2i$

$= (2^{2n} - 1) - (2^{2n-2} - 1) + 2i(2^{2n-1} - 1)$

$= M_{2n} - M_{2n-2} + 2iM_{2n-1}$

Hybrid Gaussian Mersenne and Gaussian Mersenne-Lucas Sequences

The Mersenne hybrid numbers and Mersenne-Lucas hybrid numbers are defined as

$MH_n = M_n + M_{n+1}i + M_{n+2}\varepsilon + M_{n+3}h$

$MLH_n = ML_n + ML_{n+1}i + ML_{n+2}\varepsilon + ML_{n+3}h$

where i, ε, h are hybrid units, with $i^2 = -1, \varepsilon^2 = 0, h^2 = 1, ih = -hi = \varepsilon + i$.

The Gaussian Mersenne hybrid sequence and Gaussian Mersenne-Lucas hybrid sequence are defined as

$GMH_n = GM_n + iGM_{n+1} + \varepsilon GM_{n+2} + hGM_{n+3}$

and

$GMLH_n = GML_n + iGML_{n+1} + \varepsilon GML_{n+2} + hGML_{n+3}$

Theorem 9

The Binet formula for the Gaussian Mersenne hybrid sequence and Gaussian Mersenne-Lucas hybrid sequence are

$GMH_n = 2^{n-1}\alpha^*\mathscr{a} - \beta^*\mathscr{b}$ and $GMLH_n = 2^{n-1}\alpha^*\mathscr{a} + \beta^*\mathscr{b}$

where $\alpha^* = 1 + 2i + 2^2\varepsilon + 2^3h,\ \beta^* = 1 + i + \varepsilon + h,\ \mathscr{a} = (2 + i),\ \mathscr{b} = (1 + i)$.

Proof

$GMH_n = (M_n + M_{n+1}i + M_{n+2}\varepsilon + M_{n+3}h) + i(M_{n-1} + M_ni + M_{n+1}\varepsilon + M_{n+2}h)$

$= (M_n + iM_{n-1}) + i(M_{n+1} + iM_n) + \varepsilon(M_{n+2} + iM_{n+1}) + h(M_{n+3} + iM_{n+2})$

$= GM_n + iGM_{n+1} + \varepsilon GM_{n+2} + hGM_{n+3}$

$= (2^{n-1}\mathscr{a} - \mathscr{b}) + i(2^n\mathscr{a} - \mathscr{b}) + \varepsilon(2^{n+1}\mathscr{a} - \mathscr{b}) + h(2^{n+2}\mathscr{a} - \mathscr{b})$

$= 2^{n-1}(1 + 2i + 2^2\varepsilon + 2^3h)\mathscr{a} - (1 + i + \varepsilon + h)\mathscr{b}$

$= 2^{n-1}\alpha^*\mathscr{a} - \beta^*\mathscr{b}$

$GMLH_n = (ML_n + ML_{n+1}i + ML_{n+2}\varepsilon + ML_{n+3}h) + i(ML_{n-1} + ML_ni + ML_{n+1}\varepsilon + ML_{n+2}h)$

$= (ML_n + iML_{n-1}) + i(ML_{n+1} + iML_n) + \varepsilon(ML_{n+2} + iML_{n+1}) + h(ML_{n+3} + iML_{n+2})$

$= GML_n + iGML_{n+1} + \varepsilon GML_{n+2} + hGML_{n+3}$

$= (2^{n-1}\mathscr{a} + \mathscr{b}) + i(2^n\mathscr{a} + \mathscr{b}) + \varepsilon(2^{n+1}\mathscr{a} + \mathscr{b}) + h(2^{n+2}\mathscr{a} + \mathscr{b})$

$= 2^{n-1}(1 + 2i + 2^2\varepsilon + 2^3h)\mathscr{a} + (1 + i + \varepsilon + h)\mathscr{b}$

$= 2^{n-1}\alpha^*\mathscr{a} + \beta^*\mathscr{b}$

Theorem 10

For any positive integers m, n we have

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https://publishoa.com
ISSN: 1309-3452

$GMH_mGMH_n + GMH_{m+1}GMH_{n+1} = 2^{m+n-2}\alpha^{*2}\mathscr{a}^2ML_2 - 2^{n-1}\beta^*\alpha^*\mathscr{b}\mathscr{a}ML_1 - 2^{m-1}\alpha^*\beta^*\mathscr{a}\mathscr{b}ML_1 + 2\beta^{*2}\mathscr{b}^2$

Proof

$GMH_mGMH_n + GMH_{m+1}GMH_{n+1}$

$= (2^{m-1}\alpha^*\mathscr{a} - \beta^*\mathscr{b})(2^{n-1}\alpha^*\mathscr{a} - \beta^*\mathscr{b}) + (2^m\alpha^*\mathscr{a} - \beta^*\mathscr{b})(2^n\alpha^*\mathscr{a} - \beta^*\mathscr{b})$

$= 2^{m+n-2}\alpha^{*2}\mathscr{a}^2 - 2^{n-1}\beta^*\alpha^*\mathscr{b}\mathscr{a} - 2^{m-1}\alpha^*\beta^*\mathscr{a}\mathscr{b} + \beta^{*2}\mathscr{b}^2 + 2^{m+n}\alpha^{*2}\mathscr{a}^2 - 2^n\beta^*\alpha^*\mathscr{b}\mathscr{a} - 2^m\alpha^*\beta^*\mathscr{a}\mathscr{b} + \beta^{*2}\mathscr{b}^2$

$= 2^{m+n-2}\alpha^{*2}\mathscr{a}^2(2^2 + 1) - 2^{n-1}\beta^*\alpha^*\mathscr{b}\mathscr{a}(2 + 1) - 2^{m-1}\alpha^*\beta^*\mathscr{a}\mathscr{b}(2 + 1) + 2\beta^{*2}\mathscr{b}^2$

$= 2^{m+n-2}\alpha^{*2}\mathscr{a}^2ML_2 - 2^{n-1}\beta^*\alpha^*\mathscr{b}\mathscr{a}ML_1 - 2^{m-1}\alpha^*\beta^*\mathscr{a}\mathscr{b}ML_1 + 2\beta^{*2}\mathscr{b}^2$

Theorem 11

For any positive integer n,

i. $GMH_{n+1} + 2GMH_{n-1} = 3GMH_n$
ii. $GMLH_{n+1} + 2GMLH_{n-1} = 3GMLH_n$

Proof

i. $GMH_{n+1} + 2GMH_{n-1}$
$= (GM_{n+1} + iGM_{n+2} + \varepsilon GM_{n+3} + hGM_{n+4}) + 2(GM_{n-1} + iGM_n + \varepsilon GM_{n+1} + hGM_{n+2})$
$= (GM_{n+1} + 2GM_{n-1}) + i(GM_{n+2} + 2GM_n) + \varepsilon(GM_{n+3} + 2GM_{n+1}) + h(GM_{n+4} + 2GM_{n+2})$
$= 3GM_n + i3GM_{n+1} + \varepsilon 3GM_{n+2} + h3GM_{n+3}$
$= 3GMH_n$

ii. $GMLH_{n+1} + 2GMLH_{n-1}$
$= (GML_{n+1} + iGML_{n+2} + \varepsilon GML_{n+3} + hGML_{n+4}) + 2(GML_{n-1} + iGML_n + \varepsilon GML_{n+1} + hGML_{n+2})$
$= (GML_{n+1} + 2GML_{n-1}) + i(GML_{n+2} + 2GML_n) + \varepsilon(GML_{n+3} + 2GML_{n+1}) + h(GML_{n+4} + 2GML_{n+2})$
$= 3GML_n + i3GML_{n+1} + \varepsilon 3GML_{n+2} + h3GML_{n+3}$
$= 3GMLH_n$

Theorem 12

For any positive integer n, m, k we have

i. $GMH_mGMH_n - GMH_{m+k}GMH_{n-k} = 2^{m-1}\alpha^*\beta^*\mathscr{a}\mathscr{b}M_k - 2^{n-k-1}\beta^*\alpha^*\mathscr{b}\mathscr{a}M_k$
ii. $GMLH_mGMLH_n - GMLH_{m+k}GMLH_{n-k} = 2^{n-k-1}\beta^*\alpha^*\mathscr{b}\mathscr{a}M_k - 2^{m-1}\alpha^*\beta^*\mathscr{a}\mathscr{b}M_k$

Proof

i. $GMH_mGMH_n - GMH_{m+k}GMH_{n-k}$

$= (2^{m-1}\alpha^*\mathscr{a} - \beta^*\mathscr{b})(2^{n-1}\alpha^*\mathscr{a} - \beta^*\mathscr{b}) - (2^{m+k-1}\alpha^*\mathscr{a} - \beta^*\mathscr{b})(2^{n-k-1}\alpha^*\mathscr{a} - \beta^*\mathscr{b})$

$= 2^{m+n-2}\alpha^{*2}\mathscr{a}^2 - 2^{n-1}\beta^*\alpha^*\mathscr{b}\mathscr{a} - 2^{m-1}\alpha^*\beta^*\mathscr{a}\mathscr{b} + \beta^{*2}\mathscr{b}^2 - 2^{m+n-2}\alpha^{*2}\mathscr{a}^2 + 2^{n-k-1}\beta^*\alpha^*\mathscr{b}\mathscr{a}$
$+ 2^{m+k-1}\alpha^*\beta^*\mathscr{a}\mathscr{b} - \beta^{*2}\mathscr{b}^2$

$= 2^{n-k-1}\beta^*\alpha^*\mathscr{b}\mathscr{a}(1 - 2^k) + 2^{m-1}\alpha^*\beta^*\mathscr{a}\mathscr{b}(2^k - 1)$

$= 2^{m-1}\alpha^*\beta^*\mathscr{a}\mathscr{b}M_k - 2^{n-k-1}\beta^*\alpha^*\mathscr{b}\mathscr{a}M_k$

ii. $GMLH_mGMLH_n - GMLH_{m+k}GMLH_{n-k}$
$= (2^{m-1}\alpha^*\mathscr{a} + \beta^*\mathscr{b})(2^{n-1}\alpha^*\mathscr{a} + \beta^*\mathscr{b}) - (2^{m+k-1}\alpha^*\mathscr{a} + \beta^*\mathscr{b})(2^{n-k-1}\alpha^*\mathscr{a} + \beta^*\mathscr{b})$
$= 2^{m+n-2}\alpha^{*2}\mathscr{a}^2 + 2^{n-1}\beta^*\alpha^*\mathscr{b}\mathscr{a} + 2^{m-1}\alpha^*\beta^*\mathscr{a}\mathscr{b} + \beta^{*2}\mathscr{b}^2 - 2^{m+n-2}\alpha^{*2}\mathscr{a}^2 - 2^{n-k-1}\beta^*\alpha^*\mathscr{b}\mathscr{a}$
$- 2^{m+k-1}\alpha^*\beta^*\mathscr{a}\mathscr{b} - \beta^{*2}\mathscr{b}^2$
$= 2^{n-k-1}\beta^*\alpha^*\mathscr{b}\mathscr{a}(2^k - 1) + 2^{m-1}\alpha^*\beta^*\mathscr{a}\mathscr{b}(1 - 2^k)$
$= 2^{n-k-1}\beta^*\alpha^*\mathscr{b}\mathscr{a}M_k - 2^{m-1}\alpha^*\beta^*\mathscr{a}\mathscr{b}M_k$

Theorem 13

For any positive integer n,

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https://publishoa.com
ISSN: 1309-3452

i. $GMH_n + GMLH_n = 2^n \alpha^* a$
ii. $GMH_n - GMLH_n = -2\beta^* b$

Proof

i. $GMH_n + GMLH_n = 2^{n-1}\alpha^* a - \beta^* b + 2^{n-1}\alpha^* a + \beta^* b$

$= 2(2^{n-1}\alpha^* a)$

$= 2^n \alpha^* a$

ii. $GMH_n - GMLH_n = 2^{n-1}\alpha^* a - \beta^* b - 2^{n-1}\alpha^* a - \beta^* b$
$= -2\beta^* b$

Theorem 14 (Catalan Identity for Hybrid Gaussian Mersenne and Gaussian Mersenne-Lucas Sequences)

For any positive integers n, k we have

$$GMH_{n+k}GMH_{n-k} - {GMH_n}^2 = 2^{n-k-1}M_k[\beta^*\alpha^* b a - 2^k \alpha^*\beta^* a b]$$

$$GMLH_{n+k}GMLH_{n-k} - {GMLH_n}^2 = 2^{n-k-1}M_k[2^k \alpha^*\beta^* a b - \beta^*\alpha^* b a]$$

Proof

$GMH_{n+k}GMH_{n-k} - {GMH_n}^2$

$= (2^{n+k-1}\alpha^* a - \beta^* b)(2^{n-k-1}\alpha^* a - \beta^* b) - (2^{n-1}\alpha^* a - \beta^* b)(2^{n-1}\alpha^* a - \beta^* b)$

$= 2^{n-k-1}\beta^*\alpha^* b a(2^k - 1) - 2^{n-1}\alpha^*\beta^* a b(2^k - 1)$

$= 2^{n-k-1}M_k[\beta^*\alpha^* b a - 2^k \alpha^*\beta^* a b]$

$GMLH_{n+k}GMLH_{n-k} - {GMLH_n}^2$

$= (2^{n+k-1}\alpha^* a + \beta^* b)(2^{n-k-1}\alpha^* a + \beta^* b) - (2^{n-1}\alpha^* a + \beta^* b)(2^{n-1}\alpha^* a + \beta^* b)$

$= -2^{n-k-1}\beta^*\alpha^* b a(2^k - 1) + 2^{n-1}\alpha^*\beta^* a b(2^k - 1)$

$= 2^{n-k-1}M_k[2^k \alpha^*\beta^* a b - \beta^*\alpha^* b a]$

Theorem 15 (Cassini Identity for Hybrid Gaussian Mersenne and Gaussian Mersenne-Lucas Sequences)

For any positive integer n, we have

$$GMH_{n+1}GMH_{n-1} - {GMH_n}^2 = 2^{n-2}[\beta^*\alpha^* b a - 2\alpha^*\beta^* a b]$$

$$GMLH_{n+1}GMLH_{n-1} - {GMLH_n}^2 = 2^{n-2}[2\alpha^*\beta^* a b - \beta^*\alpha^* b a]$$

Theorem 16 (d'Ocagne's Identity for Hybrid Gaussian Mersenne and Gaussian Mersenne-Lucas Sequences)

For any positive integers n, m we have

$$GMH_m GMH_{n+1} - GMH_{m+1}GMH_n = 2^{m-1}\alpha^*\beta^* a b - 2^{n-1}\beta^*\alpha^* b a$$

$$GMLH_m GMLH_{n+1} - GMLH_{m+1}GMLH_n = 2^{n-1}\beta^*\alpha^* b a - 2^{m-1}\alpha^*\beta^* a b$$

Proof

$GMH_m GMH_{n+1} - GMH_{m+1}GMH_n$

$= (2^{m-1}\alpha^* a - \beta^* b)(2^n \alpha^* a - \beta^* b) - (2^m \alpha^* a - \beta^* b)(2^{n-1}\alpha^* a - \beta^* b)$

$= 2^{n-1}\beta^*\alpha^* b a(1 - 2) + 2^{m-1}\alpha^*\beta^* a b(2 - 1)$

$= 2^{m-1}\alpha^*\beta^* a b - 2^{n-1}\beta^*\alpha^* b a$

$GMLH_m GMLH_{n+1} - GMLH_{m+1}GMLH_n$

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 4194-4201
https://publishoa.com
ISSN: 1309-3452

$= (2^{m-1}\alpha^* a + \beta^* b)(2^n\alpha^* a + \beta^* b) - (2^m\alpha^* a + \beta^* b)(2^{n-1}\alpha^* a + \beta^* b)$

$= 2^{n-1}\beta^*\alpha^* b a(2-1) - 2^{m-1}\alpha^*\beta^* a b(2-1)$

$= 2^{n-1}\beta^*\alpha^* b a - 2^{m-1}\alpha^*\beta^* a b$

Theorem 17 (Honsberger Identity for Hybrid Gaussian Mersenne and Gaussian Mersenne-Lucas Sequences)

For any positive integers k, n we have

$GMH_{k-1}GMH_n + GMH_kGMH_{n+1} = 2^{k+n-3}\alpha^{*2}a^2ML_2 - 2^{k-2}\alpha^*\beta^* a b ML_1 - 2^{n-1}\beta^*\alpha^* b a ML_1 + 2\beta^{*2} b^2$

$GMLH_{k-1}GMLH_n + GMLH_kGMLH_{n+1} = 2^{k+n-3}\alpha^{*2}a^2ML_2 + 2^{k-2}\alpha^*\beta^* a b ML_1 + 2^{n-1}\beta^*\alpha^* b a ML_1 + 2\beta^{*2} b^2$

Proof

$GMH_{k-1}GMH_n + GMH_kGMH_{n+1}$

$= (2^{k-2}\alpha^* a - \beta^* b)(2^{n-1}\alpha^* a - \beta^* b) + (2^{k-1}\alpha^* a - \beta^* b)(2^n\alpha^* a - \beta^* b)$

$= 2^{k+n-3}\alpha^{*2}a^2(2^2+1) - 2^{n-1}\beta^*\alpha^* b a(2+1) - 2^{k-2}\alpha^*\beta^* a b(2+1) + 2\beta^{*2} b^2$

$= 2^{k+n-3}\alpha^{*2}a^2ML_2 - 2^{k-2}\alpha^*\beta^* a b ML_1 - 2^{n-1}\beta^*\alpha^* b a ML_1 + 2\beta^{*2} b^2$

$GMLH_{k-1}GMLH_n + GMLH_kGMLH_{n+1}$

$= (2^{k-2}\alpha^* a + \beta^* b)(2^{n-1}\alpha^* a + \beta^* b) + (2^{k-1}\alpha^* a + \beta^* b)(2^n\alpha^* a + \beta^* b)$

$= 2^{k+n-3}\alpha^{*2}a^2(2^2+1) + 2^{n-1}\beta^*\alpha^* b a(2+1) + 2^{k-2}\alpha^*\beta^* a b(2+1) + 2\beta^{*2} b^2$

$= 2^{k+n-3}\alpha^{*2}a^2ML_2 + 2^{k-2}\alpha^*\beta^* a b ML_1 + 2^{n-1}\beta^*\alpha^* b a ML_1 + 2\beta^{*2} b^2$

Conclusion

In this paper, we defined new sequences named by Hybrid Gaussian Mersenne and Hybrid Gaussian Mersenne-Lucas sequences. Further, we obtained some relations among them. Moreover, we verified some identities through Binet's formula.

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