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Non-Baire Proof of Uniform Boundedness Theorem and Its Applications in the Proof of Some Grand Theorems of Functional Analysis

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Abstract
The proving of three big theorems, known as the uniform boundedness theorem, the open mapping theorem, and the closed graph theorem, is the pinnacle of any first functional analysis course. All three rely on the completeness of some or all of the spaces involved, and their proofs are based on Baire's theorem (or, the Baire category theorem), a topological conclusion. The aim of this paper is to prove the Uniform Boundedness Theorem without using Baire's Theorem and to show the logical dependence of these above three results on each other.
Keywords: Uniform Boundedness; Open Mapping; Closed Graph; Norms Theorem; Sum Theorem; Closed Range.

1. Introduction

Many authors point out that the "main three" foundations of Functional Analysis (see for example [1], [2]) are the Uniform Boundedness Principle, the Open Mapping Theorem, and the Hahn–Banach Theorem ((HBT) for short). We simply present a few examples because there are so many references that support it: [3, Chapter 2], [4], [5, Chapter 4], [6, p. 97], [7]. The Closed Graph Theorem is included to the list in certain texts (see [8, p. 215] or [9], for instance). The first two theorems are proved (independently) from Baire's Category Theorem in most works on Functional Analysis, whereas the (HBT) is derived from Zorn's Lemma. Closed Graph Theorem is an application of the Open Mapping Theorem. The uniform boundedness theorem, the open mapping theorem, and the closed graph theorem- all the three theorem rely on the completeness of some or all of the spaces involved, and their proofs are based on Baire's theorem (or, the Baire category theorem), a topological conclusion. The open mapping theorem and the closed graph theorem are comparable in the sense that they may both be inferred from each other, available in most textbooks, one is proved first, starting with Baire's theorem, while the other is deduced afterwards [10], [11]. The uniform boundedness theorem is established on its own. We may also find publications in the mathematical literature that illustrate how to derive the uniform boundedness theorem from the closed graph theorem. With some limitations, the opposite is

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also true. The proof of Uniform Boundednes Theorem implies Open Mapping Theorem can be found in [12]. These findings appear to be less well-known. The purpose of this note is to lay out all of the possible equivalences and deductions in a clear and concise manner. In the case of Hilbert spaces, S. Kesavan [13] additionally proved that all three results are 'equivalent' to each other.

It is important to emphasize that all of these findings are distributed throughout the literature, and no claim to uniqueness in proof procedures is given. The proof of the uniform boundedness theorem from Baire's theorem is arguably the simplest of all of these proofs. However, multiple proofs of this result exist, in the sense that they do not utilize Baire's theorem, such as Hahn's using the "gliding hump" (also called "sliding hump") argument [8, Exercise 1.76]. In functional analysis, "gliding hump" proofs are still useful: see [14] for a full survey. In the context of Hilbert spaces, Halmos [15] also proves the uniform boundedness theorem without using Baire's theorem. We will show here a really simple demonstration of the uniform boundedness theorem that doesn't use Baire's theorem [due to Alan D. Sokal, 2011]. In slightly modifying an argument given by Alan D. Sokal [16], Adrian Fellhauer, [17] (March 23, 2018) is able to prove the uniform boundedness principle using nothing more than the Zermelo–Fraenkel system and the axiom of countable choice. Professor M. Victoria Velasco [18] recently (July, 2021) demonstrated that the Uniform Boundedness Theorem, the open mapping theorem, and five other theorems are fundamentally identical. Here, we don't include direct proofs of the open mapping or closed graph theorems because they can be found in any functional analysis textbook.

According to many of the most important theorems in analysis, point-wise hypotheses entail uniform conclusions. The result that "a continuous function on a compact set is uniformly continuous" is perhaps the simplest example. The uniform boundedness theorem is one of the most important results in functional analysis. It was first published in Banach's thesis in 1922. Lebesgue discovered the uniform boundedness principle in 1908 while working on the Fourier series, then Banach and Steinhaus isolated it as a general principle.

The uniform boundedness theorem (UBT) can be used to determine whether the norms of a given collection of bounded linear operators $\{T_\alpha\}$ have a finite least upper bound. As we know that the norm of each T_α must be finite and the norm was defined to be a real-valued (not an extended real-valued) function, but there is no guarantee that they will not form an increasing sequence. The uniform boundedness theorem provides a criterion for determining when such an increasing sequence is not formed. That is, it states that a point wise bounded sequence of bounded linear operators on Banach spaces is also uniformly bounded. The uniform boundedness theorem can be extended to relevant classes of non-normable and even non-metrizable topological vector spaces (see, for example, [19, pp. 82–87]).

The Baire category theorem is used in the traditional textbook proof of the Uniform Boundedness Principle (e.g., [20, p. 81]), which dates back to Stefan Banach, Hugo Steinhaus, and Stanislaw Saks in 1927 [21]. This proof is simple, although it is not totally

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elementary due to its dependence on the Baire category theorem. The initial proofs offered by Hans Hahn [22] and Stefan Banach in 1922 were somewhat different: they began with the premise that $sup_{T\in F}\|T\| = \infty$ and used a "gliding hump" (also known as "sliding hump") technique to construct a sequence (T_n) in F and a point $x \in X$ such that $lim_{n\to\infty}\|T_n x\| = \infty$. Variants of this proof were later given by T. H. Hildebrandt [23] and Felix Hausdorff [24]. These proofs are simple, but the arguments are a little tricky. In this paper, we include a really simple proof along similar lines:

Again it is important to note that the purpose of this article is not to minimize the significance of the Baire category theorem. Indeed, proofs of these statements using the Baire category theorem, which can be found in mainstream textbooks, are easier and more intuitive. The conventional Baire category technique produces a slightly stronger version of the uniform boundedness theorem than the one presented here: if $sup_{T\in F}\|Tx\| < \infty$ for a nonmeager (i.e., second category) set of $x \in X$, then F is norm-bounded. The fact that they can be proved without using Baire's theorem, on the other hand, indicates that the completeness of the spaces involved is the foundation for these theorems.

2. Discussions and Result

We shall offer all the statement of the theorems for the sake of thoroughness of the exposition and to show the logical dependence of these results on each other. (For proof of the following two theorems, see S. Kesavan [13], S. Kesavan [25] respectively).

Theorem 2.1: Each of the following statements implies the others.

(i) **The Closed Graph Theorem**: Let V and W be Banach spaces and let $T: V \to W$ be a linear map. If the graph of T is defined by

$$G(T) = \{(x, Tx): x \in V\} \subset V \times W$$

is closed in $V \times W$, then T is continuous.

(ii) **The Open Mapping Theorem**: Let V and W be Banach spaces and let $T: V \to W$ be a continuous linear map which is surjective. Then T is an open map, i.e. T maps open sets of V onto open sets of W.

(iii) **The Bounded Inverse Theorem**: Let V and W be Banach spaces and let $T: V \to W$ be a continuous linear bijection. Then T is an isomorphism, i.e. T^{-1} is also continuous.

(iv) **The Two Norms Theorem**: Let V be a vector space and let $\|.\|_1$ and $\|.\|_2$ be two norms on V. If V is a Banach space with respect to either norm and if there exists a constant $C > 0$ such that $\|x\|_1 \leq C\|x\|_2$ for every $x \in V$ then the two norms are equivalent.

(v) **Uniform Boundedness Theorem**: Let V be a Banach space and W be a normed linear space. Let $T_i: V \to W$ be a continuous linear map for each $i \in I$. If $sup_{i\in I}\|T_i x\| < \infty$) for each $x \in V$ then there exists a constant $C > 0$ such that $\|T_i\| \le C$ for each $i \in I$.

The closed graph theorem was the beginning point in the first article of S. Kesavan [13], and in order to finish the loop of the different implications, the argument that the uniform boundedness principle entails the closed graph theorem required the reflexivity of the target space. This consequence was demonstrated in the context of Hilbert spaces in the article referenced above. Ramaswamy and Ramasamy deal with the scenario where W is a reflexive Banach space.

But in later, S. Kesavan [25] began with the Uniform Boundedness theorem and were able to conclude the loop without the use of additional hypotheses. S. Kesavan [25] also proved that the uniform boundedness theorem implies the closed graph theorem without any further hypotheses. [For observing the order, we state the theorem 2.2]

Theorem 2.2: Each of the following statements implies the others.

(i) **Uniform Boundedness Theorem**: Let V be a Banach space and W be a normed linear space. Let $T_i: V \to W$ be a continuous linear map for each $i \in I$. If $sup_{i\in I}\|T_i x\| < \infty)$ for each $x \in V$ then there exists a constant $C > 0$ such that $\|T_i\| \le C$ for each $i \in I$.

(ii) **The Open Mapping Theorem**: Let V and W be Banach spaces and let $T: V \to W$ be a continuous linear map which is surjective. Then T is an open map, i.e. T maps open sets of V onto open sets of W.

(iii) **The Bounded Inverse Theorem**: Let V and W be Banach spaces and let $T: V \to W$ be a continuous linear bijection. Then T is an isomorphism, i.e. T^{-1} is also continuous.

(iv) **The Closed Graph Theorem**: Let V and W be Banach spaces and let $T: V \to W$ be a linear map. If the graph of T is defined by

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is closed in $V \times W$, then T is continuous.

(v) **The Two Norms Theorem**: Let V be a vector space and let $\|.\|_1$ and $\|.\|_2$ be two norms on V. If V is a Banach space with respect to either norm and if there exists a constant $C > 0$ such that $\|x\|_1 \le C\|x\|_2$ for every $x \in V$ then the two norms are equivalent.

In the year 2021, M. Victoria Velasco [18] demonstrated the equivalence of the eight theorems that make up Theorem 2.3 below. The proof of Theorem 2.3 given by M. Victoria Velasco is relatively brief and straightforward. He concludes that because a straightforward and elementary proof of the Uniform Boundedness Principle that does not require Baire's Theorem can be given, this is also true for all of the outcomes involved in Theorem 2.3. Thus, he does not present a brief and simultaneous proof of all of them; he only demonstrates that they are

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all equally relevant since they are logically equivalent, as Theorem 2.3 establishes. In fact, the derivation of this result demonstrates how closely the eight theorems involved are related. (For details, see [18])

Theorem 2.3: The following statements are equivalent:

(i) **Uniform Boundedness Theorem (UBP)**: Let $\{T_i\}, i \in I$ be a family of bonded linear maps from Banach space X into a normed linear space Y_i. If $\{T_i\}, i \in I$ is pointwise bounded then $sup_{\ i \in I} \|T_i\| \leq \infty$

(ii) **The Open Mapping Theorem (OMT)**: Let X and Y be Banach spaces and let $T: X \to Y$ be a continuous linear map which is surjective. Then T is an open map.

(iii) **The Open Mapping Theorem (bis) (OMTbis)**: Let X be a Banach space, Y a normed space, and $T \in L(X, Y)$ a surjective map. Then T is open if and only if Y is complete.

(iv) **Banach Isomorphism Theorem (BIT)**: Let X and Y be Banach spaces. If $T \in L(X, Y)$ is bijective, then T^{-1} is continuous.

(v) **Norms Theorem (NT):** Let $\|.\|$ and $|.|$ be complete norms on a linear space X such that they are comparable. Then $\|.\|$ and $|.|$ are equivalent.

(vi) **Closed Graph Theorem (CGT):** If X and Y are Banach spaces then a linear operator $T: X \to Y$ is continuous if and only if its graph is closed (i.e. the separating subspace of T is zero.)

(vii) **Sum Theorem (ST):** Let M and N be closed subspace of a Banach space X. Then $M + N$ is closed if and only if the map $(m, n) \to m + n$ from $M \times N$ into $M + N$ is open.

(viii) **Closed Range Theorem (CRT):):** Let X and Y be Banach spaces and $T: D(T) \subseteq X \to Y$ a closed linear operator whose domain $D(T)$ is dense in X. Let $R(T)$ be the range of T and $T^*: D(T^*) \subseteq Y^* \to R(T^*) \subseteq X^*$ the transpose of T. Then, the following assertions are equivalent:

(a) $R(T)$ is closed in Y.

(b) $R(T^*)$ is closed in X^*

(c) $R(T^*) = (\ker T)^{\perp}$

(d) $R(T) = (\ker T^*)^{T}$

(e) $T: D(T) \to R(T)$ is open.

(f) $T^*: D(T^*) \to R(T^*)$ is open.

3.1 *Gliding hump argument*

As mentioned earlier, we now see a variant derivation of the uniform boundedness principle that does not use any version of the Baire category theorem. The argument is a "gliding hump" (also known as "sliding hump") argument, which is mostly attributable to Hahn. The only use of completeness in the argument is to assure that a certain absolutely convergent series converges. The proof of uniform boundedness principle without using any other form of the Baire category theorem is essentially from Hahn's 1922 paper; though he stated the result only for sequences of linear functionals. This is called a gliding hump argument.

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Gliding hump arguments probably first appeared in work by Henri Lebesgue from 1905. Hahn specifically stated in his paper that the basic method for his proof was taken from a 1909 paper by Lebesgue. The original proofs given by Hans Hahn and Stefan Banach in 1922 were quite different: they began from the assumption that $\sup_{T\in F}\|T\| = \infty$ and used a "gliding hump" (also called "sliding hump") technique to construct a sequence (T_n) in F and a point $x \in X$ such that $\lim_{n\to\infty}\|T_n x\| = \infty$

3.2 *A non-Baire Proof of Banach Steinhaus Theorem*

We now have a proof for the uniform boundedness theorem that can be understood easily. In Halmos's sense of the work, the proof is elementary because it does not use the Baire category theorem or any associated lemmas. It employs a technique known as "gliding-hump." It is weaker than the Baire-based proof in that the latter establishes that an unbounded family of operators can only be pointwise bounded on a meager set of points, whereas this proof just shows only that some sequence may be constructed on which an unbounded family of operators is unbounded at some point.

3.3 Banach-Steinhaus Theorem: [26] Let X be Banach Space and Y be a normed space and $F \in B(X, Y)$. Then if $\sup\{\|Tx\|: T \in F\} < \infty$ for all $x \in X$ we must have that

$$\sup\{\|T\|: T \in F\} < \infty$$

Proof: Suppose that F is uniformly bounded, ie. $\sup\|T\| = \infty \quad where \quad T \in F$. We wish to establish the existence of a point at which F is not bounded.

Fix $0 < \delta < \frac{1}{2}$ Select T_1 from F. Let x_1 in X so $\|x_1\| = \delta$ and $\|T_1 x_1\| > (1-\delta)\|T_1\|\|x_1\|$

We now conduct an induction. Having selected $T_1, \dots\dots T_{n-1}$ and $x_1, \dots\dots x_{n-1}$ select T_n from F for which $\|T_n\| > \frac{M_{n-1}+n}{(1-2\delta)\delta^n}$ where $M_{n-1} = \sup_{T\in F}\|T(x_1 + \cdots . + x_{n-1})\|$ and then choose x_n in X_0 with $\|x_n\| = \delta^n$ and $\|T_n x_n\| > (1-\delta)\|T_n\|\|x_n\| = (1-\delta)\delta^n\|T_n\|$

Notice that the series $\sum_{k=1}^{\infty} x_k$ has Cauchy sequence of partial sums, hence converges in the Banach space X. Observe that the choices of T_n and x_n entail that

$$\left(1 - \frac{\delta}{1-\delta}\right)\|T_n x_n\| = \frac{1-2\delta}{1-\delta}\|T_n x_n\|$$
$$> (1-2\delta)\delta^n\|T_n\|$$
$$> M_{n-1} + n$$

while $\|T_n \sum_{k=n+1}^{\infty} x_k\| \le \|T_n\| \sum_{k=n+1}^{\infty} \delta^k = \|T_n\| \frac{\delta^{n+1}}{1-\delta} < \frac{\delta}{1-\delta}\|T_n x_n\|$

We put this together to compute for $x = \sum_{k=1}^{\infty} x_k$ that

$$\|T_n x\| \ge \|T_n x_n\| - \left\|T_n \sum_{k=1}^{n-1} x_k\right\| - \|T_n \sum_{k=n+1}^{\infty} x_k\| > \left(1 - \frac{\delta}{1-\delta}\right)\|T_n x_n\| - M_{n-1} > n$$

Hence F is not pointwise bounded on all of X which contradicts the assumption. Thus the proof is completed.

Now we want to present the proof of Uniform Boundedness theorem given by the Alan D.

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Sokal [16]. To prove the Uniform Boundedness theorem, we need the following trivial result.

Lemma 3.4: Let T be a bounded linear operator from a normed linear space X to a normed linear space Y. Then for any $x \in X$ and $r > 0$, we have

$$\sup_{x' \in B(x,r)} \|Tx'\| \geq \|T\| r \text{ ....(1)}$$

Where $B(x,r) = \{x' \in X: \|x' - x\| < r\}$

Proof: For $\xi \in X$ we have

max{$\|T(x+\xi)\|, \|(x-\xi)\|\} \geq \frac{1}{2}[\|T(x+$ □)+□(□−□)]≥□□……(2)

where the ≥ uses the triangle inequality in the form $\|\alpha - \beta\| \leq \|\alpha\| + \|\beta\|$.

Now take the supremum over $\xi \in B(0,r)$.

Theorem 3.5 (Uniform Boundedness Theorem (Alan D Sokal)): Let $\mathcal{F}$ be a family of bounded linear operator form a Banach Space X to a normed linear space Y. If $\mathcal{F}$ is point wise bounded (i.e. $\sup_{T \in \mathcal{F}} \|Tx\| < \infty$ for all $x \in X$, then $\mathcal{F}$ is norm bounded. (i.e. $\sup_{T \in \mathcal{F}} \|T\| < \infty$)

Proof of theorem 3.5 : Suppose $\sup_{T \in \mathcal{F}} \|Tx\| = \infty$ and choose $(T_n)_{n=1}^{\infty}$ in $\mathcal{F}$ such that $\|T_n\| \geq 4^n$. Then set $x_0 = 0$ and for $n \geq 1$ use the lemma to choose inductively $x_n \in X$ such that $\|x_n - x_{n-1}\| \leq 3^{-n}$ and $\|T_n x_n\| \geq \frac{2}{3} 3^{-n} \|T_n\|$. The sequence (x_n) is a Cauchy, hence convergent to some $x \in X$; and it is easy to see that $\|x - x_n\| \leq \frac{1}{2} 3^{-n}$ and hence $\|T_n x\| \geq \frac{1}{6} 3^{-n} \|T_n\| \geq \frac{1}{6} (\frac{4}{3})^n \to \infty$.

Now, we present here, another proof of the Uniform Boundedness Theorem that does not require the Baire's Theorem in a similar fashion as proved by Alan D Sokal but in a slightly different way.

The new proof of theorem 3.5 (UBT): Assume that $\sup_{T \in \mathcal{F}} \|Tx\| = \infty$ and choose $(T_n)_{n=1}^{\infty}$ in $\mathcal{F}$ such that $\|T_n\| \geq 6^n$. Then set $x_0 = 0$ and for $n \geq 1$ use the lemma 3.4 to choose inductively $x_n \in X$ such that $\|x_n - x_{n-1}\| \leq 5^{-n}$ and $\|T_n x_n\| \geq \frac{4}{5} 5^{-n} \|T_n\|$. The sequence (x_n) is a Cauchy, hence convergent to some $x \in X$;

Now if $m > n$ we have $\|x_n - x_m\| \leq 5^{-(n+1)} + 5^{-(n+2)} + \cdots + 5^{-m}$

Keeping n fixed and letting $m \to \infty$ we deduce that

$$\|x_n - x\| \leq \frac{5^{-n-1}}{1 - \frac{1}{5}} = \frac{1}{4} 5^{-n}$$

Then by the triangle inequality, we get

$\|T_n x\| = \|T_n x_n - T_n x_n + T_n x\| \geq \|T_n(x_n)\| - \|T_n(x - x_n)\| > \left(\frac{4}{5} 5^{-n} -\right.$ 145−□□□=11205−□□□=112065□→∞, a contradiction.

As stated in [16], the above proof is most easily represented in terms of a sequence (x_n) that converges to x, as we have seen. This differs from previous "gliding hump" proofs, which employed a sequence that added up to x. Of course, because sequences and series are equivalent, each proof can be expressed in either language; it is a matter of personal preference. Moreover from a quantitative standpoint, this proof is incredibly wasteful. Ball's "plank theorem" [27] leads to a quantitatively sharp version of the uniform boundedness theorem: namely, if

$$\sum_{n=1}^{\infty} \|T_n\|^{-1} < \infty$$

then there exists $x \in X$ such that $\lim_{n\to\infty} \|T_n x\| = \infty$ (see also [28]).

A similar (but slightly more complicated) elementary proof of the uniform boundedness theorem can be found in [29, p. 83]

Apart from the Uniform Boundedness Theorem and other great theorems of Functional Analysis, we will look at two more conclusions included in most functional analysis textbooks that may be proved without utilising Baire's Theorem in this article. These are, without a doubt, consequence of UBT.

Theorem 3.6: Let $y = (\eta_i)$, $\eta_i \in C$ be such that $\sum \xi_i \eta_i$ converges for every $x = (\xi_i) \in c_0$ where $c_0 \subset l^{\infty}$is a subspace of all complex sequences converging to zero. Then $\sum|\eta_i| < \infty$.

Theorem 3.7: Let X be a Banach space, Y a normed space and $T_n \in B(X, Y)$ such that $(T_n x)$ is Cauchy in Y for every $x \in X$. Then $(\|T_n\|)$ is bounded

To prove these results we have to use UBT and as proof of UBT is Baire-free so these results are also Baire-free.

4. Conclusion

As these results (involved in the above theorems 2.1, 2.2, 2.3) are interdependent and the Uniform Boundedness Theorem can be proved without appealing to Baire's Theorem, so these results are also Baire Free and we can use the Uniform Boundedness Theorem as a tool to prove these results.

References:

1. S. G. Krantz, A Guide to Functional Analysis (The Mathematical Association of America, 2013).
2. B. D. MacCluer, Elementary functional analysis, Graduate Texts in Mathematics, vol. 253 (Springer, New York, 2009).
3. T. Bühler, and D. A. Solomon, Functional analysis. Graduate Studies in Mathematics, 191 (American Mathematical Society, Providence, RI, 2018).
4. K. Ciesielski and Moslehian, Some remarks on the history of functional analysis, Ann. Funct. Anal. **1** (2010) pp 1–12.
5. E. Kreyszig, Introductory functional analysis with applications (JohnWiley & Sons, New York, London, Sydney, 1978).
6. W. Rudin, Real and Complex Analysis, 3rd edn. International (Mc Graw-Hill, New York, 1987).
7. C. Swartz, An introduction to functional analysis, Monographs and Textbooks in Pure and Applied Mathematics, vol. 157 (Marcel Dekker, Inc., New York, 1992).
8. E. Hewitt and K. Stromberg, Real and abstract analysis. A modern treatment of the theory of functions of a real variable (Springer, New York , 1965).

9. R. E. Megginson, An Introduction to Banach Space Theory, Graduate Texts in Mathematics, vol. 183. (Springer, New York, 1998).
10. Henri Bourles , On the Closed Graph Theorem and the Open Mapping Theorem (Nov, 2014). https://doi.org/10.48550/arXiv.1411.5500
11. R.S. Monahan and P.L. Robinson, The Closed Graph Theorem is the Open mapping Theorem, (December 2019). arXiv 1912.02626 (2019).
12. F. Liebaug and K. Spindler, Logical equivalence of the fundamental theorems on operators between Banach spaces, Elem. Math. **75** (2020) pp 15–22.
13. S. Kesavan, A note on the grand theorems of functional analysis, *Mathematics Newsletter*, vol. 27, no. 3, (2017) pp 188-191.
14. C. Swartz, Infinite Matrices and the Gliding Hump, World Scientific, River Edge, NJ (1996).
15. P.R. Halmos, A Hillbert Space Problem Book (Springer-Verlag, New York, 1974).
16. A.D. Sokal, A really simple elementary proof of the uniform boundedness theorem, Amer. Math. Monthly, vol. 118, no. 5, (2011) pp. 450-452.
17. A. F. D. Fellhauer, On the relation of three theorems of analysis to the axiom of choice, Journal of Logic & Analysis 9:1 (2017) pp 1–23. . https://doi.org/10.4115/jla.2017.9.1
18. M. Victoria Velasco, On the pillars of Functional Analysis, RACSAM (2021) https://doi.org/10.1007/s13398-021-01108-1
19. H.H. Schaefer and M.P. Wolf, Topological Vector Spaces, 2nd ed., Springer-Verlag, New York,1999
20. M. Reed and B. Simon, Methods of Modern Mathematical Physics, Vol I, Functional Analysis, Academic Press, New York, 1972.
21. S. Banach and H. Steinhaus, Sur le principle de la condensation des singularities, Fund, Math. 9 (1927) 50-61
22. H. Hahn, Uber Folgen linearer Operationen, Monatsh. Math. Phys. 32 (1922) 3-88.
23. T.H. Hildebrandt, On uniform limitedness of sets of functional operations, Bull Amer. Math. Soc. 29 (1923) 309-315
24. J. Hennefeld, A nontopological proof of the uniform boundedness theorem, Amer. Math. Monthly 87 (1980) 217
25. S. Kesavan, The grand theorems of Functional Analysis Revisited: A Baire-free Approach, *Mathematics Newsletter*, 31 no. 3, (2020-2021) 89-93.
26. N. L. Carothers, A Short Course on Banach Space Theory, Cambridge University Press, 2010.
27. K. Ball, The plank problem for symmetric bodies, Invent. Math. 104 (1991) 535–543.
28. V. Muller and J. Vrsovsky, Orbits of linear operators tending to infinity, Rocky Mountain J. Math. 39 (2009) 219–230.
29. I. Gohberg, S. Goldberg and M. A. Kaashoek, Basic Classes of Linear Operators, Birkhauser, Basel, 2003.
30. S. Kesavan, Functional analysis, TRIM, Hindustan Book Agency, 52 (2009).
31. R. Mennicken and B. Sagraloff, On Banach's closed range theorem. Arch. Math **33**, 461–465 (1979)