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# Modular Chromatic Number of Certain Cyclic Graphs

## P. Sumathi<sup>1</sup>, S.Tamilselvi<sup>2\*</sup>

<sup>1</sup> Associate professor, Department of Mathematics, C. Kandaswami Naidu College for Men, Anna Nagar, Chennai 600 102, India.

<sup>2\*</sup>Research scholar, Department of Mathematics, C. Kandaswami Naidu College for Men, Anna Nagar, Chennai 600 102, India.

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#### Abstract

A modular k- coloring,  $k \ge 2$  of a graph without isolated vertices is vertex coloring of G with the positive integers k, for all  $k \in \mathbb{Z}_k$ , where the adjacent vertices may be colored by the same integer and sums of the colors of their neighbors are different in  $\mathbb{Z}_k$ . The minimum k for which the G has a modular k- coloring is the modular chromatic number  $M_c(G)$  of G. In this paper, the modular chromatic number of generalized Jahangir graph, generalized Petersen graph, and generalized uniform theta graph are found.

*Keywords:* Modular K-coloring, Modular chromatic number, Generalized Jahangir graph, Generalized Petersen Graph, Generalized Uniform Theta Graph.

#### 1. Introduction

Graphs considered in this paper are simple, nontrivial, finite, connected, and undirected. The concept of modular coloring was first proposed by Okamoto, E.salehi, and P. Zhang in 2010. They executed the modular chromatic number of several well-known graphs and presented number of bounds in [1, 2, 3]. The modular coloring technique has been applied to many fields, such as scheduling, electrical circuits, networking, etc.

Let v be a vertex of graph G and let N(v) is the neighborhood of v it is denote the set of vertices adjacent to v in G. For a graph without isolated vertices, let  $C:V(G) \to \mathbb{Z}_k$ ,  $(k \ge 2)$  be a vertex coloring of G where adjacent vertices may be colored the same. The color sum S(v) of a vertex v of G is defined as the sum of the colors of the vertices in N(v), that is  $S(v) = \sum_{u \in N(v)} C(u)$ . The coloring C is called a modular sum k- coloring or simply a modular k- coloring of Gif  $S(x) \ne S(y)$  in  $\mathbb{Z}_k$  for all pairs x, y of adjacent vertices of G. A coloring C is a modular coloring if C is a modular k- coloring for some integer  $k \ge 2$ . The modular chromatic number  $m_c(G)$  of G is the minimum k for which G has a modular k- coloring.

In this paper we prove that the generalized Jahangir graph, generalized Petersen graph and generalized uniform theta graph admits modular coloring.

#### 2. Preliminaries

**Definition 2.1** The Generalized Jahangir graph  $J_{n,m}$  for  $m \ge 3$  is a graph on mn + 1 vertices, consisting of a cycle  $C_{nm}$  with one additional vertex that is adjacent to *n* vertices of  $C_{nm}$  at distance *m* to each other on  $C_{nm}$ .

Let  $J_{n,m}$  be a Generalized Jahangir graph, Let  $v_0$  be the centre vertex,  $v_i: i = 1,2,3,...,m$  be the join vertices and  $v_{ij}: i = 1,2,...,m$  and j = 1,2,...,n-1 be the petal vertices. Let  $E(J_{n,m}) = \{v_i v_{i+1}: i = 1,2,...,m(n-1)\} \cup \{u_{mn}, v_1\} \cup \{v_0 v_{1+(m(i-1))}: i = 1,2,3,...,m\}$  be the set of edges of  $J_{n,m}$ . Then |V| = mn + 1, |E| = (n + 1)m.

**Definition 2.2** The generalized Petersen graphs P(n, k) are defined to be a graph on 2n  $(n \ge 3)$  vertices with  $V(P(n, k)) = \{v_i, u_i: 0 \le i \le n-1\}$  and  $E(P(n, k)) = \{v_i v_{i+1}, v_i u_i, u_i u_{i+k}: 0 \le i \le n-1\}$ .

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**Definition 2.3** A generalized theta graph  $\theta(n, m)$  or simply a theta graph with *n* vertices has two vertices *N* and *S* of degree *m* such that every other vertices is of degree 2 and lies in one of the *m* paths joining the vertices *N* and *S*. A theta graph  $\theta(n, m)$  is said to be uniform if  $|L_1| + |L_2| + \dots + |L_i|$ , where  $L_i$  is a path between *N* and *S*.

**Theorem 2.1** [1]: For every non trivial connected graph  $m_c(G) \ge \chi(G)$ .

#### 3. Main results

In this section generalized Jahangir graph, generalized Petersen graph and generalized uniform theta graph are dealt and proved to admit modular coloring.

#### 3.1Generalized Jahangir Graph Theorem 3.1

Let  $J_{n,m}$  be a generalized Jahangir graph. Then  $m_c(J_{n,m}) = \begin{cases} 2 & \text{if } n \text{ is even} \\ 3 & \text{otherwise.} \end{cases}$ ,  $n \ge 3$ .

#### **Proof:**

Let  $J_{n,m}$  be a generalized Jahangir graph on  $n \ge 3$ , and  $V(J_{n,m}) = \{v_0\} \cup \{v_i: i = 1, 2, ..., m\} \cup \{v_{ij}: i = 1, 2, ..., m; j = 1, 2, ..., m, j = 1, 2, ..., m \} \cup \{v_{ij}: i = 1, 2, ..., m; j = 1, 2, ..., m \} \cup \{v_{ij}: i = 1, 2, ..., m\} \cup \{v_{ij}: i = 1, 2, ..., m\}$ 

## Case 1: $n \equiv 0 \pmod{4}$

When *n* is even and  $\equiv 0 \pmod{4}$ ,  $\chi(J_{n,m}) = 2$ .

By theorem 2.1,  $m_c(J_{n,m}) \ge 2$ .

Define an injective mapping  $f(v_i) : V(J_{n,m}) \to \mathbb{Z}_2$  as follows:  $(v_i) = \begin{cases} 0 & \{v_i : i = 0, 1, 2, ..., m\} \text{ and } \{v_{ij} : i = 1, 2, ..., m; j = 2, 3, ..., n - 1\} \\ 1 & \{v_{ij} : i = 1, 2, ..., m; j = 1\} \end{cases}$ 

Clearly it yields the modular coloring of  $J_{n,m}$ .

Let  $S(v_i) = \begin{cases} 0 & \{v_i: i = 0\} \text{ and } \{v_{ij}: i = 1, 2, ..., m; j \text{ is odd} \} \\ 1 & \{v_i: i = 1, 2, ..., m\} \text{ and } \{v_{ij}: i = 1, 2, ..., m; j \text{ is even} \}, \end{cases}$  which is a modular 2-coloring, since  $S(v_i) \neq S(v_j)$  for all pairs of  $v_i, v_j$  of all adjacent vertices of  $J_{n,m}$ . Therefore  $m_c(J_{n,m}) \leq 2$ . Hence  $m_c(J_{n,m}) = 2$ .

# Case 2: $n \equiv 2 \pmod{4}$

When *n* is even and  $n \equiv 2 \pmod{4}$ ,  $\chi(J_{n,m}) = 2$ .

By theorem 2.1,  $m_c(J_{n,m}) \ge 2$ .

Define an injective mapping  $f(v_i): V(J_{n,m}) \to \mathbb{Z}_2$  as follows:

$$f(v_i) = \begin{cases} 0 & \{v_i : i = 1, 2, \dots, m\} \text{ and} \\ & \{v_{ij} - v_{i,3+4k} : i = 1, 2, \dots, m; j = 1, 2, 3, \dots, n-1; k = 0, 1, \dots, \frac{n-6}{4} \} \\ & 1 & \{v_i : i = 0\} \text{ and } \{v_{i,3+4k} : i = 1, 2, \dots, m; k = 0, 1, \dots, \frac{n-6}{4} \} \end{cases}.$$

Clearly it yields the modular coloring of  $J_{n,m}$ .

Let 
$$S(v_i) = \begin{cases} 0 & \{v_i: i = 0\} \text{ and } \{v_{ij}: i = 1, 2, ..., m; j \text{ is odd} \} \\ 1 & \{v_i: i = 1, 2, ..., m\} \text{ and } \{v_{ij}: i = 1, 2, ..., m; j \text{ is even} \} \end{cases}$$

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which is a modular 2-coloring, since  $\mathcal{S}(v_i) \neq \mathcal{S}(v_j)$  for all pairs of  $v_i, v_j$  of all adjacent vertices of  $J_{n,m}$ . Therefore  $m_c(J_{n,m}) \leq 2$ . Hence  $m_c(J_{n,m}) = 2$ .

# Case 3: When *n* is odd

When *n* is odd,  $\chi(J_{n,m}) = 3$ .

By theorem 2.1,  $m_c(J_{n,m}) \ge 3$ .

Since the injective mapping  $f(v_i): V(J_{n,m}) \to \mathbb{Z}_3$  is defined by

$$f(v_i) = \begin{cases} 0 & \{v_i: i = 1, 2, \dots, m\} \text{ and } \{v_{ij}: i = 1, 2, \dots, m; j \text{ is even} \} \\ 1 & \{v_i: i = 0\} \text{ and } \{v_{ij}: i = 1, 2, \dots, m; j \text{ is odd} \} \end{cases}.$$

Clearly it yields the modular coloring of  $J_{n,m}$ .

$$\text{Let } \mathcal{S}(v_i) = \begin{cases} 0 & \{v_i: i = 0\} \text{ and } \{v_{ij}: i = 1, 2, \dots, m; j \text{ is odd} \} \\ 1 & \{v_{ij}: i = 1, 2, \dots, m; j = n - 1 \\ 2 & \{v_i: i = 1, 2, \dots, m\} \text{ and } \{v_{ij}: i = 1, 2, \dots, m; 1 \le j \le n - 2, \text{where } j \text{ is even} \} \end{cases}$$

which is a modular 3-coloring, since  $S(v_i) \neq S(v_j)$  for all pairs of  $v_i, v_j$  of all adjacent vertices of  $J_{n,m}$ . Therefore  $m_c(J_{n,m}) \leq 3$ . Hence  $m_c(J_{n,m}) = 3$ . Refer figure 3.1.



Figure 3.1 [J<sub>5,4</sub>]

#### **3.2Generalized Petersen Graph**

Here we prove the following theorem for the Petersen graph P(n, m) where m = 1. Theorem 3.2

For each integer n > 3,  $m_c(P(n, 1)) = \begin{cases} 2 & if n is even \\ 3 & otherwise. \end{cases}$ 

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## **Proof:**

Let P(n, 1) be Petersen graph. Let  $V = \{v_i : i = 1, 2, ..., n\} \cup \{u_i : i = 1, 2, ..., n\}$  be the vertices of P(n, 1).

We consider two cases,

## Case 1: *n* is even

When *n* is even,  $\chi(P(n, 1)) = 2$ .

By theorem 2.1,  $m_c(P(n, 1)) \ge 2$ .

Let  $f(v_i): V(P(n, 1)) \to \mathbb{Z}_2$  be an injective mapping such that

 $f(P(n,1)) = \begin{cases} 0 & \{v_i: i \text{ is even}\} \text{ and } \{u_i: i \text{ is odd}\}\\ 1 & \{v_i: i \text{ is odd}\} \text{ and } \{u_i: i \text{ is even}\} \end{cases}$ 

Clearly it gives modular coloring of P(n, 1).

Let  $S(P(n, 1)) = \begin{cases} 0 & \{v_i: i \text{ is odd}\} \text{ and } \{u_i: i \text{ is even}\} \\ 1 & \{v_i: i \text{ is even}\} \text{ and } \{u_i: i \text{ is odd}\} \end{cases}$ 

which is a modular 2-coloring, since  $S(v_i) \neq S(v_j)$  for all pairs of  $v_i, v_j$  of all adjacent vertices of P(n, 1). Therefore  $m_c(P(n, 1)) \leq 2$  and so  $m_c(P(n, 1)) = 2$ . Refer figure 3.2.



Figure 3.2 [*P*(6,1)]

#### Case 2: n is odd

When *n* is odd,  $\chi(P(n, 1)) = 3$ .

By theorem 2.1,  $m_c(P(n, 1)) \ge 3$ .

Let  $f(v_i): V(P(n, 1)) \to \mathbb{Z}_2$  be an injective mapping such that

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$$f(P(n,1)) = \begin{cases} 0 & \{v_i: i = n, n-2, 1 \le i \le n, where \ i \ is \ even\} \ and \ \{u_i - u_{n-2}: 1 \le i \le n\} \\ 1 & \{u_i: i = n-2\} \\ 2 & \{v_i: 1 \le i \le n-4, where \ i \ is \ odd\} \end{cases}$$

Clearly it gives modular coloring of P(n, 1).

$$S(P(n,1)) = \begin{cases} 0 & \{v_i: i = n - 1, 1 \le i \le n - 4, where \ i \ is \ odd\} \ and \\ \{u_i: i = n, n - 2, 1 \le i \le n - 5, where \ i \ is \ even\} \\ 1 & \{v_i: i = n - 2, 1 \le i \le n - 5, where \ i \ is \ even\} \ and \ \{u_i: i = n - 1, n - 3\} \\ 2 & \{v_i: i = n, n - 3\} \ and \ \{u_i: 1 \le i \le n - 4, where \ i \ is \ odd\} \end{cases}$$

which is a modular 3-coloring, since  $S(v_i) \neq S(v_j)$  for all pairs of  $v_i, v_j$  of all adjacent vertices of  $J_{n,m}$ . Therefore  $m_c(P(n, 1)) \leq 3$ . Hence  $m_c(P(n, 1)) = 3$ .

#### **3.3Generalized Uniform Theta Graph**

#### Theorem 3.3

Let  $\theta(n,m)$  is Generalized Uniform Theta Graph for  $n,m \ge 3$ , then  $m_c(\theta(n,m)) = \begin{cases} 2 & if n is odd and n is even, m is odd \\ 3 & n is even, m is even \end{cases}$ 

#### **Proof** :

Let  $G \cong \theta(n,m)$  for any integer  $n, m \ge 3$ . Let  $v_m = N$  and  $v_0 = S$  is North Pole and South Pole of *G* respectively.  $\theta(n,m)$  is a graph containing *m* disjoint paths of *n* vertices joining the poles *N* and *S*.

We prove the theorem using the following two cases.

#### Case 1: *n* is odd

When *n* is odd,  $\chi(\theta(n, m)) = 2$ .

By theorem 2.1,  $m_c(\theta(n, m)) \ge 2$ .

We define the injective mapping as  $C(v_{ij}): V(\theta(n,m)) \to \mathbb{Z}_2$  Such that, for  $n \equiv 1 \pmod{4}$ 

$$C(v_{ij}) = \begin{cases} 0 & \{v_{n+1}\} \cup \{v_{ij}: 1 \le i \le l, 1 \le j \le m\} - \{v_{ij}: i = 4k, k = 0, 1, \dots, \frac{(n-1)}{4}, 1 \le j \le m\} \\ 1 & \{v_0\} \cup \{v_{ij}: i = 2 + 4k, k = 0, 1, \dots, \frac{(n-1)}{4}, j = 1, 2, \dots, m\} \end{cases}$$

and for  $n \equiv 3(mod4)$ ,

$$C(v_{ij}) = \begin{cases} 0 & \{v_0, v_{n+1}\} \cup \{v_{ij} : 1 \le i \le l, \ 1 \le j \le m\} - \{v_{ij} : i = 4k, k = 0, 1, \dots, \frac{(n-3)}{4}, 1 \le j \le m\} \\ 1 & \{v_{ij} : i = 2 + 4k, k = 0, 1, \dots, \frac{(n-3)}{4}, 1 \le j \le m\} \end{cases}$$

Let the modular coloring be

$$\mathcal{S}(v_{ij}) = \begin{cases} 0 & \{v_0, v_{n+1}\} \cup \{v_{ij} : 0 \le i \le n \text{ and } i \text{ is even, } 1 \le j \le m \} \\ 1 & \{v_{ij} : 0 \le i \le n \text{ and } i \text{ is odd, } 1 \le j \le m \} \end{cases}.$$

It achieves the modular 2- coloring. Therefore  $m_c(\theta(n,m)) \leq 2$ . Hence  $m_c(\theta(n,m)) = 2$ .

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Refer figure 3.3



figure3.3 [θ(3,4)]

Case 2: *n* is even

## Subcase 1: *m* is odd

When *n* is even and *m* is odd,  $\chi(\theta(n, m)) = 2$ .

By theorem 2.1,  $m_c(\theta(n, m)) \ge 2$ .

We define the injective mapping as  $C(v_{ij}): V(\theta(n,m)) \to \mathbb{Z}_2$  Such that, for  $n \equiv 2 \pmod{4}$ .

and for  $n \equiv 0 \pmod{4}$ ,

$$C(v_{ij}) = \begin{cases} 0 & \{v_0\} \cup \{v_{ij}: 1 \le i \le l, 1 \le j \le m\} - \{v_{ij}: i = 1 + 4k, k = 0, 1, \dots, \frac{(n-4)}{4}, 1 \le j \le m\} \\ 1 & \{v_{n+1}\} \cup \{v_{ij}: i = 1 + 4k, k = 0, 1, \dots, \frac{(n-4)}{4}, 1 \le j \le m\} \end{cases}$$

Let the modular coloring be

$$\boldsymbol{\mathcal{S}}(v_{ij}) = \begin{cases} 0 & \{v_{n+1}\} \cup \{v_{ij}: 0 \le i \le n \text{ and } i \text{ is odd}, 1 \le j \le m\} \\ 1 & \{v_0\} \cup \{v_{ij}: 0 \le i \le n \text{ and } i \text{ is evev}, 1 \le j \le m\} \end{cases},$$

It achieves the modular 2- coloring. Therefore  $m_c(\theta(n,m)) \leq 2$ . Hence  $m_c(\theta(n,m)) = 2$ .

## Subcase 2: *m* is even

When *n* is even and *m* is even,  $\chi(\theta(n, m) = 2)$ .

By theorem 2.1,  $m_c(\theta(n, m)) \ge 2$ .

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We define an injective mapping as  $C(v_{ij}): V(\theta(n,m)) \to \mathbb{Z}_3$  Such that, for  $n \equiv 2 \pmod{4}$ 

and for  $n \equiv 0 \pmod{4}$ ,

$$C(v_{ij}) = \begin{cases} 0 \quad \{v_0\} \cup \{v_{ij}: 1 \le i \le l, 1 \le j \le m\} - \{v_{ij}: i = 1 + 4k, k = 0, 1, \dots, \frac{(n-4)}{4}, 1 \le j \le m\} \\ 1 \qquad \qquad \{v_{n+1}\} \cup \{v_{ij}: i = 1 + 4k, k = 0, 1, \dots, \frac{(n-4)}{4}, 1 \le j \le m\} \end{cases}$$

Let the modular coloring be

$$\mathcal{S}(v_{ij}) = \begin{cases} 0 & \{v_0, v_{n+1}\} \cup \{v_{ij} : 0 \le i \le n \text{ and } i \text{ is odd}, 1 \le j \le m\} \\ 1 & \{v_{ij} : 0 \le i \le n \text{ and } i \text{ is even}, 1 \le j \le m\} \end{cases}.$$

It achieves the modular 3- coloring, it follows that  $m_c(\theta(n,m)) \leq 3$ . We show that  $m_c(\theta(n,m)) \neq 2$ . Assume, to the contrary, that there exist modular 2- colorings  $C_1(v_{ij}): V(\theta(n,m)) \rightarrow \mathbb{Z}_2$  of  $\theta(n,m)$ . such that, for  $n \equiv 2 \pmod{4}$ 

$$C(v_{ij}) = \begin{cases} 0 \quad \{v_0, v_{n+1}\} \cup \{v_{ij}: 1 \le i \le l, 1 \le j \le m\} - \{v_{ij}: i = 1 + 4k, k = 0, 1, \dots, \frac{(n-2)}{4}, 1 \le j \le m\} \\ 1 \quad \{v_{ij}: i = 1 + 4k, k = 0, 1, \dots, \frac{(n-2)}{4}, 1 \le j \le m\} \end{cases}$$

and for  $n \equiv 0 \pmod{4}$ 

$$C(v_{ij}) = \begin{cases} 0 & \{v_0\} \cup \{v_{ij}: 1 \le i \le l, 1 \le j \le m\} - \{v_{ij}: i = 1 + 4k, k = 0, 1, \dots, \frac{(n-4)}{4}, 1 \le j \le m\} \\ 1 & \{v_{n+1}\} \cup \{v_{ij}: i = 1 + 4k, k = 0, 1, \dots, \frac{(n-4)}{4}, 1 \le j \le m\} \end{cases}$$
Clearly it gives

modular coloring of  $\theta(n, m)$ .

Then we may assume that

$$\operatorname{Let} \mathcal{S}(v_{ij}) = \begin{cases} 0 & \{v_{n+1}\} \cup \{v_{ij}: 0 \le i \le n \text{ and } i \text{ is odd}, 1 \le j \le m\} \\ 1 & \{v_0\} \cup \{v_{ij}: 0 \le i \le n \text{ and } i \text{ is even}, 1 \le j \le m\} \end{cases}$$

It is clear that  $S(v_0) = 0$  and  $S(v_{1j}) = 0$  which is a contradiction that adjacent vertices must receive different coloring. Therefore our assumption  $m_c(\theta(n, m)) = 2$  is wrong. It follows that  $m_c(\theta(n, m)) \ge 3$  and so  $m_c(\theta(n, m)) = 3$ .

## 4. Conclusion:

Based on the significance of the theorems we determine the specific graphs such as generalized Jahangir graph, generalized Petersen graph, and generalized uniform theta graph are modular coloring.

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