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Asymptotic Behavior of Limiting Ratios of Generalized Recurrence Relations

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ABSTRACT

The most famous Fibonacci sequence possess a rather easy recurrence relation in which except the first two terms, each term is the sum of two previous terms. In this paper, I try to generalize the recurrence relation of Fibonacci sequence with respect to number of terms as well as considering not just the consecutive terms but terms whose index are in Arithmetic Progression. With this consideration, I try to obtain the limiting ratio of the new sequence produced through such generalization. The answers obtained cover more general class of Fibonacci type sequences and will provide new insights in the discussion of such generalizations.

Keywords: Fibonacci sequence, Recurrence Relations, Fibonacci type sequence of first order, Fibonacci type sequence of *r*th order, Limiting Ratio.

1. Introduction

Beginning with the most simple and elegant recurrence relation corresponding to the famous Fibonacci sequence, I try to generalize through modified recurrence relations in which terms are not necessarily successively defined but they are in Arithmetic Progression. With this generalization, I provide methods for computing the limiting ratios and provide some illustrations to justify the results obtained. The conclusion arrived will give new insights in to the understanding of the limiting ratios of more general class of Fibonacci type sequences.

2. Definitions

2.1 The recurrence relation of the Fibonacci sequence is defined by

$$P(n+2) = P(n+1) + P(n),\ n \geq 0, P(0) = 1, P(1) = 1 \quad (2.1)$$

From (2.1), we notice that except the first two terms, each term is sum of two previous terms.

2.2 If k is any positive real number then the generalized recurrence relation of Fibonacci type sequence of first order is defined by

$$P(n+mk) = P\left(n+(m-1)k\right) + P\left(n+(m-2)k\right) + \cdots + P(n+k) + P(n),\ n \geq 0, m \geq 2 \quad (2.2)$$

where

$$P(0) = P(k) = P(2k) = \cdots = P\left((m-2)k\right) = P\left((m-1)k\right) = 1.$$

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The (m+1)th term of the Arithmetic Progression (A.P.) whose first term is n and common difference k is $n+mk$. Thus from (2.2), we see that but for the first m terms, each term of the sequence is the sum of m previous A.P. index terms whose first term is n and common difference is k.

We notice that if $m = 2$, $k = 1$ then we get the standard recurrence relation corresponding to usual Fibonacci sequence defined in (2.1).

2.3 If k, r are positive real numbers, then the generalized recurrence relation of Fibonacci type sequence of rth order is defined by

$$P(n+mk)=r\left[P\left(n+(m-1)k\right)+P\left(n+(m-2)k\right)+\cdots+P(n+k)+P(n)\right],\ n\geq 0, m\geq 2 \quad (2.3)$$ where

$$P(0)=P(k)=P(2k)=\cdots=P\left((m-2)k\right)=P\left((m-1)k\right)=1.$$

We notice that if $m = 2$, $k = 1$, $r = 1$ we get the recurrence relation of standard Fibonacci sequence defined in (2.1) and if just $r = 1$, then we get the recurrence relation of Fibonacci type sequence of first order defined in (2.2).

2.4 The ratio of (n+1)th term to the nth term of a sequence as $n\to\infty$ is defined as the limiting ratio of the sequence. We denote the limiting ratio by λ . Thus $\lambda=\dfrac{P(n+1)}{P(n)}$ as $n\to\infty$ (2.4)

2.5 If λ is the limiting ratio, then for any integer $t\geq 1$ and as $n\to\infty$ we have

$$\frac{P(n+t)}{P(n)}=\frac{P(n+t)}{P(n+t-1)}\times\frac{P(n+t-1)}{P(n+t-2)}\times\cdots\times\frac{P(n+2)}{P(n+1)}\times\frac{P(n+1)}{P(n)}=\lambda\times\lambda\times\cdots\times\lambda\times\lambda=\lambda^{t} \quad (2.5)$$

3. Limiting Ratio of Fibonacci type sequence of first order

3.1 Theorem 1

If $k\geq 1$ is any real number, then the limiting ratio of Fibonacci type sequence of first order converges to $\sqrt[k]{2}$ (3.1)

Proof: The recurrence relation corresponding to Fibonacci type sequence of first order according to (2.2) is given by

$$P(n+mk)=P\left(n+(m-1)k\right)+P\left(n+(m-2)k\right)+\cdots+P(n+k)+P(n),\ n\geq 0, m\geq 2 \quad (3.2)$$ where

$$P(0)=P(k)=P(2k)=\cdots=P\left((m-2)k\right)=P\left((m-1)k\right)=1$$

From (3.2), we get $\dfrac{P(n+mk)}{P(n)}=\dfrac{P(n+(m-1)k)}{P(n)}+\dfrac{P(n+(m-2)k)}{P(n)}+\cdots+\dfrac{P(n+k)}{P(n)}+1$ (3.3)

Now using (2.5), we can write (3.3) as $\lambda^{mk}=\lambda^{(m-1)k}+\lambda^{(m-2)k}+\cdots+\lambda^{k}+1$ (3.4)

Using the Geometric series identity $1+\lambda^{k}+\cdots+\lambda^{(m-2)k}+\lambda^{(m-1)k}=\dfrac{\lambda^{mk}-1}{\lambda^{k}-1}$ (3.5), (3.4) becomes $\lambda^{mk}=\dfrac{\lambda^{mk}-1}{\lambda^{k}-1}$.

This simplifies to $\lambda^{(m+1)k}+1=2\lambda^{mk}$ (3.6)

From (3.6), we get $\lambda^{k}+\dfrac{1}{\lambda^{mk}}=2$. By definition (2.2), we observe that $\lambda>1$. Hence, $\dfrac{1}{\lambda^{mk}}\to 0$ as $m\to\infty$. Thus when $m\to\infty$, we get $\lambda^{k}=2$ from which $\lambda=\sqrt[k]{2}$ as we wished.

Thus the limiting ratio of Fibonacci type sequence of first order converges to $\sqrt[k]{2}$. This completes the proof.

3.2 Illustrations

If we consider *k* = 3, *m* = 8 then from (3.4), we get

$$\lambda^{24}-\lambda^{21}-\lambda^{18}-\lambda^{15}-\lambda^{12}-\lambda^{9}-\lambda^{6}-\lambda^{3}-1=0 \quad (3.7)$$

The positive real root of (3.7) by Newton – Raphson method is found to be $\sqrt[3]{1.99603}$ which is very close to $\sqrt[3]{2}$ verifying (3.1) of theorem 1. Figure 1 confirms this fact.

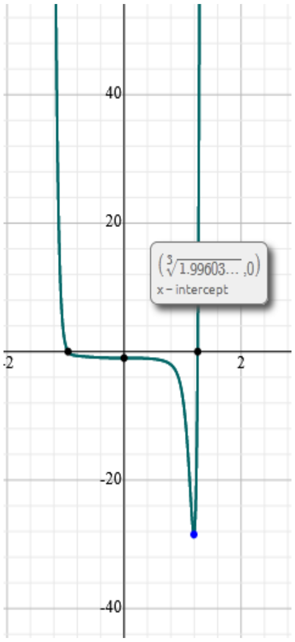

Figure 1: Graph of $y=x^{24}-x^{21}-x^{18}-x^{15}-x^{12}-x^{9}-x^{6}-x^{3}-1$

If we consider *k* = 5, *m* = 7 then from (3.4), we get

$$\lambda^{35}-\lambda^{30}-\lambda^{25}-\lambda^{20}-\lambda^{15}-\lambda^{10}-\lambda^{5}-1=0 \quad (3.8)$$

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The positive real root of (3.8) by Newton – Raphson method is found to be 1.148 approximately which is very close to $\sqrt[5]{2} = 1.148698$ verifying (3.1) of theorem 1. Figure 2 confirms this fact.

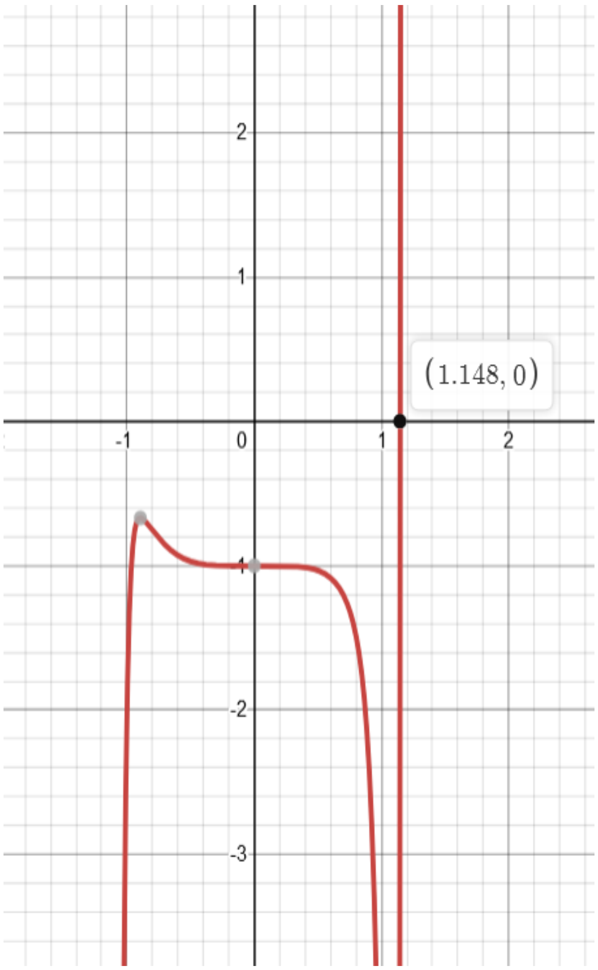

Figure 2: Graph of $y = x^{35} - x^{30} - x^{25} - x^{20} - x^{15} - x^{10} - x^5 - 1$

If we consider k = 3.5, m = 8 then from (3.4) we get

$$\lambda^{28} - \lambda^{24.5} - \lambda^{21} - \lambda^{17.5} - \lambda^{14} - \lambda^{10.5} - \lambda^7 - \lambda^{3.5} - 1 = 0 \quad (3.9)$$

The positive real root of (3.9) by Newton – Raphson method is found to be 1.21832 approximately which is very close to $\sqrt[3.5]{2} = 1.21901$ again verifying (3.1) of theorem 1. Figure 3 confirms this fact.

We also notice that if k = 1 then from (3.1) the limiting ratio converges to $\sqrt{2}$ and if k < 1, then the limiting ratio corresponding to (3.4) does not exist.

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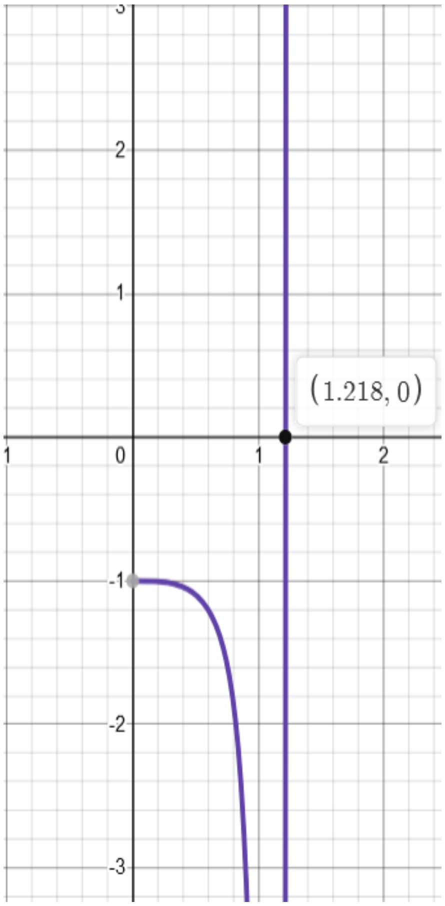

Figure 3:

Graph of $y = x^{28} - x^{24.5} - x^{21} - x^{17.5} - x^{14} - x^{10.5} - x^{7} - x^{3.5} - 1$

4. Limiting Ratio of Fibonacci type sequence of *r*th order

4.1 Theorem 2

If *k*, *r* are positive real numbers, then the limiting ratio of Fibonacci type sequence of *r*th order converges to $\sqrt[k]{r+1}$ (4.1)

Proof: The recurrence relation corresponding to Fibonacci type sequence of *r*th order according to (2.3) is given by

$P(n+mk) = r\left[P\left(n+(m-1)k\right)+P\left(n+(m-2)k\right)+\cdots+P(n+k)+P(n)\right],\ n\geq 0, m\geq 2$ (4.2) where
$P(0)=P(k)=P(2k)=\cdots=P\left((m-2)k\right)=P\left((m-1)k\right)=1$.

From (4.2), we get $\dfrac{P(n+mk)}{P(n)} = r\left[\dfrac{P(n+(m-1)k)}{P(n)}+\dfrac{P(n+(m-2)k)}{P(n)}+\cdots+\dfrac{P(n+k)}{P(n)}+1\right]$ (4.3)

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Using (2.5), we can write (4.3) as $\lambda^{mk} = r\left(\lambda^{(m-1)k} + \lambda^{(m-2)k} + \cdots + \lambda^{k} + 1\right)$ (4.4)

Using the Geometric series identity $1+\lambda^{k}+\cdots+\lambda^{(m-2)k}+\lambda^{(m-1)k} = \frac{\lambda^{mk}-1}{\lambda^{k}-1}$ (4.5), (4.4) becomes $\lambda^{mk} = r\left(\frac{\lambda^{mk}-1}{\lambda^{k}-1}\right)$. This simplifies to $\lambda^{(m+1)k} + r = (r+1)\lambda^{mk}$ (4.6).

From (4.6), we get $\lambda^{k} + \frac{r}{\lambda^{mk}} = r+1$.

By definition (2.3), we observe that $\lambda > 1$. Hence, $\frac{1}{\lambda^{mk}} \to 0$ as $m \to \infty$. Thus when $m \to \infty$, and r is finite, we get $\lambda^{k} = r+1$ from which $\lambda = \sqrt[k]{r+1}$ as we wished.

Thus the limiting ratio of Fibonacci type sequence of rth order converges to $\sqrt[k]{r+1}$. This completes the proof.

4.2 Illustrations

In view of (4.1) of theorem 2, we find that if r = 1, then the limiting ratio is $\sqrt[k]{2}$ the result we obtained in (3.1) of theorem 1. We will now consider some illustrations to verify the result obtained in theorem 2.

If r = 2, k = 3 and m = 5 then from (4.4), we get $\lambda^{15} - 2\left(\lambda^{12} + \lambda^{9} + \lambda^{6} + \lambda^{3} + 1\right) = 0$ (4.7)

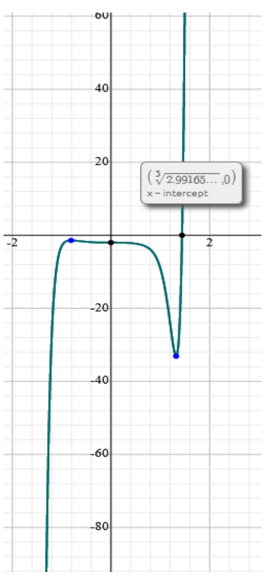

Figure 4: Graph of $y = x^{15} - 2\left(x^{12} + x^{9} + x^{6} + x^{3} + 1\right)$

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The positive real root of (4.7) by Newton – Raphson method is found to be $\sqrt[3]{2.99165}$ which is very close to $\sqrt[k]{r+1}=\sqrt[3]{3}$ verifying (4.1) of theorem 2. Figure 4 confirms this fact.

If r = 15, k = 5, m = 6 then from (4.4), we get $\lambda^{30}-15\left(\lambda^{25}+\lambda^{20}+\lambda^{15}+\lambda^{10}+\lambda^{5}+1\right)=0$ (4.8)

The positive real root of (4.8) by Newton – Raphson method is found to be $\sqrt[5]{15.99999}$ which is very close to $\sqrt[k]{r+1}=\sqrt[5]{16}$ verifying (4.1) of theorem 2. Figure 5 confirms this fact.

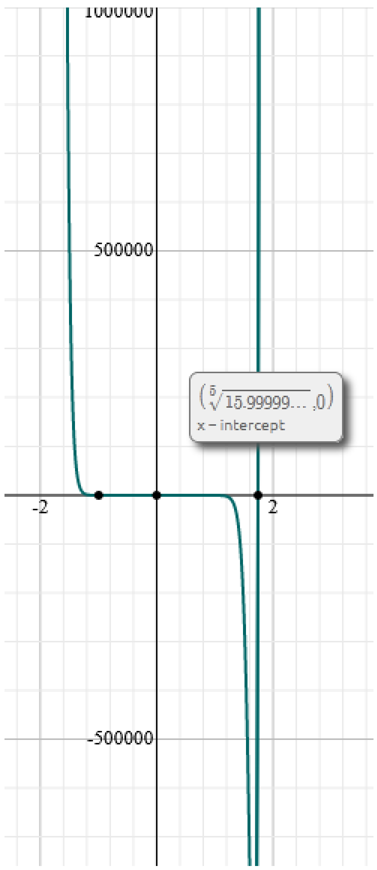

Figure 5: Graph of $y=x^{30}-15\left(x^{25}+x^{20}+x^{15}+x^{10}+x^{5}+1\right)$

If r = 10.25, m = 8, k = 0.9 then from (4.4) we get $\lambda^{7.2}-10.25\left(\lambda^{6.3}+\lambda^{5.4}+\lambda^{4.5}+\lambda^{3.6}+\lambda^{2.7}+\lambda^{1.8}+\lambda^{0.9}+1\right)=0$ (4.9)

The positive root of (4.9) by Newton – Raphson method is found to be $\sqrt[0.9]{11.24999}=14.721322$ which is very close to $\sqrt[k]{r+1}=\sqrt[0.9]{11.25}=14.721336$ verifying (4.1) of theorem 2.

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By choosing sufficient values of r, m, k we see that the limiting ratio of the recurrence relation of Fibonacci type sequence of *r*th order is always $\sqrt[k]{r+1}$ approximately.

5. Conclusion

By considering the generalization of standard Fibonacci sequence in two ways namely, Fibonacci sequence of first order and Fibonacci sequence of *r*th order as defined in (2.2) and (2.3), I have proved that the limiting ratios of these two generalized sequences converges to $\sqrt[k]{2}$ and $\sqrt[k]{r+1}$ through theorems 1 and 2 respectively. Three illustrations were provided in each case to justify the results obtained regarding the limiting ratios obtained in theorems 1 and 2.

These six illustrations were aided with six appropriate figures to aid better understanding of the limiting ratios obtained in each case. It is clear that in all the six examples, the results obtained agree to greater degree of accuracy. Thus modifying the usual recurrence relation of standard Fibonacci sequence, we could generate completely new type of sequences whose limiting ratios turns out be very interesting numbers. By further modification like considering the terms of the recurrence relation whose index is in Geometric Progression or Harmonic Progression instead of Arithmetic Progression, we can try to explore more in determining the limiting ratios of such newly defined sequences. Since, the limiting ratios provides the asymptotic behavior of the terms of a sequence, these values provide a clear thought of how the newly defined sequences behave in general. Literally there is no end to these kind of explorations.

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