JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 2, 2022, p. 31-40
https://publishoa.com
ISSN: 1309-3452

Eulerian Polynomials and Ramanujan Summation

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ABSTRACT
Among several classes of numbers that occur in combinatorial problems, Eulerian Numbers play an significant role with its abundance occurrence. In this Paper, I will introduce Eulerian Numbers, Eulerian Polynomials and prove some of the interesting properties concerning them. I will finally relate these interesting classes of numbers with Ramanujan Summation, one of very interesting summation methods proposed by Srinivasa Ramanujan.

Keywords: Permutation Groups, Descents, Ascents, Eulerian Polynomials, Eulerian Numbers, Ramanujan Summation

1. Introduction

Leonhard Euler, the great Swiss mathematician was considered to be the greatest contributor in mathematics of all times. Among hundreds of mathematical concepts named after him, one of the interesting class of polynomials called Eulerian polynomials play significant role in several combinatorial problems. In this paper, I will discuss Eulerian Polynomials whose coefficients are Eulerian Numbers and connect these concepts with one of the beautiful ideas proposed by great Indian mathematician Srinivasa Ramanujan namely Ramanujan Summation.

2. Definitions and Examples

2.1 Let S_n denote the permutation group of n elements say $\{1,2,3,\dots,n\}$. Then we know that S_n represents the total number of bijections that can exist between $\{1,2,3,\dots,n\}$to itself. Also, $|S_n| = n!$. We define each element of S_n as permutation of n quantities.

2.2 Let $\sigma \in S_n$ be a permutation. We say that a permutation $\sigma \in S_n$ has descent at the position i if $\sigma(i) > \sigma(i+1)$. The set of all descents of a permutation $\sigma \in S_n$ is defined by $\operatorname{des}(\sigma) = \{ i / \sigma(i) > \sigma(i+1) \}$ (2.1)

2.3 Eulerian Numbers denoted by $\left\langle {n \atop k} \right\rangle$ is defined as the number of permutations in S_n with exactly k descents. That is,

$$\left\langle {n \atop k} \right\rangle = \left| \{ \sigma \in S_n / \operatorname{des}(\sigma) = k \} \right| \quad (2.2)$$

2.4 From the definition 2.3, we see that for the identity permutation $123...n \in S_n$ there is no descent since at any position i we have $\sigma(i) = i < \sigma(i+1) = i+1$ for all $i = 1,2,3,...,n-1$.

Similarly for the permutation $n...321 \in S_n$, there are $n - 1$ descents since, we have $\sigma(i) = n-i+1 > \sigma(i+1) = n-i$ for all $i = 1,2,3,...,n-1$. Hence, for any permutation in S_n, the minimum possible descent is 0 and maximum possible descent is $n - 1$.

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Volume 13, No. 2, 2022, p. 31-40
https://publishoa.com
ISSN: 1309-3452

Thus the Eulerian numbers $\left\langle {n \atop k} \right\rangle$ are defined for $k = 0, 1, 2, \ldots, n-1$. Moreover, since $\left\langle {n \atop k} \right\rangle$ accounts for all permutations in S_n, we have $\sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle = |S_n| = n! \quad (2.3)$

2.5 In S_3 we have 3! possible permutations given by $\{123, 132, 213, 231, 312, 321\}$. For these permutations we find that des(123) = 0, des(132) = 1 since $\sigma(2) = 3 > \sigma(3) = 2$. Similarly, des(213) = 1, des(231) = 1, des(312) = 1, des(321) = 2. Thus, there is only one permutation in S_3 with 0 descent, four permutations with 1 descent and one permutation with 2 descents. According to (2.2), we can record this information as $\left\langle {3 \atop 0} \right\rangle = 1, \left\langle {3 \atop 1} \right\rangle = 4, \left\langle {3 \atop 2} \right\rangle = 1$ and we notice that $\sum_{k=0}^{2} \left\langle {3 \atop k} \right\rangle = \left\langle {3 \atop 0} \right\rangle + \left\langle {3 \atop 1} \right\rangle + \left\langle {3 \atop 2} \right\rangle = 1 + 4 + 1 = 6 = 3! = |S_3|$ verifying (2.3) . Similarly, in 1234 is the only permutation with 0 descent and there will be 11 permutations: 1243, 1324, 1342, 1423, 2134, 2314, 2341, 2413, 3124, 3412, 4123 with 1 descent, there will be 11 permutations: 3421, 4231, 2431, 3241, 4312, 4132, 1432, 3142, 4213, 2143, 3214 with 2 descents and there is only one permutation 4321 with 3 descents. Thus we have $\left\langle {4 \atop 0} \right\rangle = 1, \left\langle {4 \atop 1} \right\rangle = 11, \left\langle {4 \atop 2} \right\rangle = 11, \left\langle {4 \atop 3} \right\rangle = 1$ and we observe that $\sum_{k=0}^{3} \left\langle {4 \atop k} \right\rangle = \left\langle {4 \atop 0} \right\rangle + \left\langle {4 \atop 1} \right\rangle + \left\langle {4 \atop 2} \right\rangle + \left\langle {4 \atop 3} \right\rangle = 1 + 11 + 11 + 1 = 24 = 4! = |S_4|$ verifying (2.3).

2.6 Let $\sigma \in S_n$ be a permutation. We say that a permutation $\sigma \in S_n$ has ascent at the position *i* if $\sigma(i) < \sigma(i+1)$. The set of all ascents of a permutation $\sigma \in S_n$ is defined by $\mathrm{asc}(\sigma) = \{ i / \sigma(i) < \sigma(i+1) \}$ (2.4)

Since there are maximum $n - 1$ descents that can occur for permutations in S_n, the number of ascents of a permutation can be expressed in terms of number of its descents by the equation $|asc\,\sigma| = |\{ i / \sigma(i) < \sigma(i+1) \}| = |(n-1) - |des\sigma||$ (2.5). That is, the number of ascents of a permutation in S_n is equal to $(n - 1)$ minus number of its descents. See Figure 1 for values of Eulerian Numbers provided for the first ten rows.

$n \backslash k$	0	1	2	3	4	5	6	7	8	9
1	1									
2	1	1								
3	1	4	1							
4	1	11	11	1						
5	1	26	66	26	1					
6	1	57	302	302	57	1				
7	1	120	1191	2416	1191	120	1			
8	1	247	4293	15619	15619	4293	247	1		
9	1	502	14608	88234	156190	88234	14608	502	1	
10	1	1013	47840	455192	1310354	1310354	455192	47840	1013	1

Figure 1: Eulerian Numbers $\left\langle {n \atop k} \right\rangle, 0 \le k \le n-1$

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We can actually compute the Eulerian Numbers by using the following well known recurrence relation (for proof, see **[3]**)

$$\left\langle {n \atop k} \right\rangle = (n-k)\left\langle {n-1 \atop k-1} \right\rangle + (k+1)\left\langle {n-1 \atop k} \right\rangle \quad (2.6)$$

3. Theorem 1

The Eulerian Numbers are symmetric with respect to number of its descents. That is,

$$\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-k-1} \right\rangle, \; 0 \le k \le n-1 \quad (3.1)$$

Proof: First we notice that if $\sigma \in S_n$ is a permutation with *k* descents then reversal of entries of σ swaps *k* descents of σ with *k* ascents of reversal of σ. For example, if we consider the permutation $\sigma = 13524 \in S_5$ then σ is a permutation with 1 descent and 3 ascents. Now reversing σ we get 42531, a permutation in S_5 with 3 descents and 1 ascent.

Thus reversing the permutation operation provides a bijection between the set of permutations with *k* descents and *k* ascents. But by (2.5), we know that the number of *k* ascents will be equal to $(n-1)-k$ descents. Thus in S_n through reversal operation, there exists a bijection between the set of all permutations with *k* descents and $n-k-1$ descents.

Hence by the definition of Eulerian Numbers, we get $\left\langle {n \atop k} \right\rangle = \left\langle {n \atop n-k-1} \right\rangle$ proving (3.1)

4. Eulerian Polynomials

4.1 Definition

The polynomials $E_n(x)$ for all $x \in R$ defined by $E_n(x) = \sum_{k=0}^{n-1} \left\langle {n \atop k} \right\rangle x^k$ (4.1) are called as Eulerian Polynomials. Through entries of Figure and (4.1) we can list the first seven Eulerian Polynomials

$$E_1(x) = \left\langle {1 \atop 0} \right\rangle x^0 = 1 \quad (4.2)$$

$$E_2(x) = \sum_{k=0}^{1} \left\langle {2 \atop k} \right\rangle x^k = \left\langle {2 \atop 0} \right\rangle x^0 + \left\langle {2 \atop 1} \right\rangle x^1 = 1 + x \quad (4.3)$$

$$E_3(x) = \sum_{k=0}^{2} \left\langle {3 \atop k} \right\rangle x^k = \left\langle {3 \atop 0} \right\rangle x^0 + \left\langle {3 \atop 1} \right\rangle x^1 + \left\langle {3 \atop 2} \right\rangle x^2 = 1 + 4x + x^2 \quad (4.4)$$

$$E_4(x) = \sum_{k=0}^{3} \left\langle {4 \atop k} \right\rangle x^k = \left\langle {4 \atop 0} \right\rangle x^0 + \left\langle {4 \atop 1} \right\rangle x^1 + \left\langle {4 \atop 2} \right\rangle x^2 + \left\langle {4 \atop 3} \right\rangle x^3 = 1 + 11x + 11x^2 + x^3 \quad (4.5)$$

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ISSN: 1309-3452

$$E_5(x)=\sum_{k=0}^{4}\left\langle {5 \atop k} \right\rangle x^k=\left\langle {5 \atop 0} \right\rangle x^0+\left\langle {5 \atop 1} \right\rangle x^1+\left\langle {5 \atop 2} \right\rangle x^2+\left\langle {5 \atop 3} \right\rangle x^3+\left\langle {5 \atop 4} \right\rangle x^4=1+26x+66x^2+26x^3+x^4 \quad (4.6)$$

$$E_6(x)=\sum_{k=0}^{5}\left\langle {6 \atop k} \right\rangle x^k=\left\langle {6 \atop 0} \right\rangle x^0+\left\langle {6 \atop 1} \right\rangle x^1+\left\langle {6 \atop 2} \right\rangle x^2+\left\langle {6 \atop 3} \right\rangle x^3+\left\langle {6 \atop 4} \right\rangle x^4+\left\langle {6 \atop 5} \right\rangle x^5$$
$$=1+57x+302x^2+302x^3+57x^4+x^5 \quad (4.7)$$

$$E_7(x)=\sum_{k=0}^{6}\left\langle {7 \atop k} \right\rangle x^k=\left\langle {7 \atop 0} \right\rangle x^0+\left\langle {7 \atop 1} \right\rangle x^1+\left\langle {7 \atop 2} \right\rangle x^2+\left\langle {7 \atop 3} \right\rangle x^3+\left\langle {7 \atop 4} \right\rangle x^4+\left\langle {7 \atop 5} \right\rangle x^5+\left\langle {7 \atop 6} \right\rangle x^6$$
$$=1+120x+1191x^2+2416x^3+1191x^4+120x^5+x^6 \quad (4.8)$$

In view of above equations we notice that the Eulerian Polynomial $E_n(x)$ for any natural number n is a polynomial is a polynomial of degree n – 1 containing n terms. Moreover by symmetry of Eulerian Numbers we see that the Eulerian Polynomials $E_n(x)$ exhibit mirror symmetry with respect to their coefficients.

4.2 Theorem 2

If n is an even positive integer, then $E_n(-1)=0 \quad (4.9)$

Proof: By definition (4.1), we have

$$E_n(-1)=\sum_{k=0}^{n-1}\left\langle {n \atop k} \right\rangle (-1)^k=\left\langle {n \atop 0} \right\rangle-\left\langle {n \atop 1} \right\rangle+\left\langle {n \atop 2} \right\rangle-\left\langle {n \atop 3} \right\rangle+\cdots+\left\langle {n \atop n-4} \right\rangle-\left\langle {n \atop n-3} \right\rangle+\left\langle {n \atop n-2} \right\rangle-\left\langle {n \atop n-1} \right\rangle$$

By (3.1), we notice that $\left\langle {n \atop 0} \right\rangle=\left\langle {n \atop n-1} \right\rangle, \left\langle {n \atop 1} \right\rangle=\left\langle {n \atop n-2} \right\rangle, \left\langle {n \atop 2} \right\rangle=\left\langle {n \atop n-3} \right\rangle, \left\langle {n \atop 3} \right\rangle=\left\langle {n \atop n-4} \right\rangle, \cdots$

Thus all the terms in $E_n(-1)$ cancel each other if n is even. Hence, if n is even then $E_n(-1)=0$.

This completes the proof.

5. Eulerian Polynomials and Summation

Let $S_n(x)=1^n+2^n x+3^n x^2+4^n x^3+\cdots \quad (5.1)$ denote an expression whose coefficients are nth powers of natural numbers. I will now try to compute $S_n(x)$ for each whole number n and connect it with Eulerian Polynomial of corresponding index.

Using Binomial expansions and equations (4.2) to (4.8) we get

$$S_1(x)=1+2x+3x^2+4x^3+\cdots=\frac{1}{(1-x)^2}=\frac{E_1(x)}{(1-x)^2} \quad (5.2)$$

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Since, $(1+x)(1-x)^{-3}=(1+x)(1+3x+6x^2+10x^3+15x^4+\cdots)=1+4x+9x^2+16x^4+\cdots$ we have

$$S_2(x)=1^2+2^2x+3^2x^2+4^2x^3+\cdots=\frac{1+x}{(1-x)^3}=\frac{E_2(x)}{(1-x)^3} \quad (5.3)$$

Similarly, $(1+4x+x^2)(1-x)^{-4}=(1+4x+x^2)(1+4x+10x^2+20x^3+\cdots)=1+8x+27x^2+64x^3+\cdots$

$$S_3(x)=1^3+2^3x+3^3x^2+4^3x^3+\cdots=\frac{1+4x+x^2}{(1-x)^4}=\frac{E_3(x)}{(1-x)^4} \quad (5.4)$$

Proceeding in same fashion, we get

$$S_4(x)=1^4+2^4x+3^4x^2+4^4x^3+\cdots=\frac{1+11x+11x^2+x^3}{(1-x)^5}=\frac{E_4(x)}{(1-x)^5} \quad (5.5)$$

$$S_5(x)=1^5+2^5x+3^5x^2+4^5x^3+\cdots=\frac{1+26x+66x^2+26x^3+x^4}{(1-x)^6}=\frac{E_5(x)}{(1-x)^6} \quad (5.6)$$

$$S_6(x)=1^6+2^6x+3^6x^2+4^6x^3+\cdots=\frac{1+57x+302x^2+302x^3+57x^4+x^5}{(1-x)^7}=\frac{E_6(x)}{(1-x)^7} \quad (5.7)$$

$$S_7(x)=1^7+2^7x+3^7x^2+4^7x^3+\cdots=\frac{1+120x+1191x^2+2416x^3+1191x^4+120x^5+x^6}{(1-x)^8}$$
$$=\frac{E_7(x)}{(1-x)^8} \quad (5.8)$$

In general, for any natural number *n*, we find that $S_n(x)=\frac{E_n(x)}{(1-x)^{n+1}}, \ \forall\, x\in R-\{1\} \quad (5.9)$

6. Riemann Zeta Function

The Riemann Zeta Function is defined as $\varsigma(n)=\frac{1}{1^n}+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\cdots \ (6.1)$ where *n* is any complex number. From (6.1), we have $\varsigma(-n)=1^n+2^n+3^n+4^n+\cdots \ (6.2)$.

From (5.1) we have $S_n(-1)=1^n-2^n+3^n-4^n+5^n-6^n+\cdots \ (6.3)$

Now from (6.2) and (6.3) we have

$$(1-2^{n+1})\varsigma(-n)=\left(1^n+2^n+3^n+4^n+5^n+6^n+\cdots\right)-2\left(2^n+4^n+6^n+8^n+\cdots\right)$$
$$=1^n-2^n+3^n-4^n+5^n-6^n+\cdots=S_n(-1)$$

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ISSN: 1309-3452

Thus, $\varsigma(-n)=\dfrac{S_n(-1)}{1-2^{n+1}}$ (6.4)

7. Ramanujan Summation and Eulerian Polynomials

7.1 Theorem 3

If ς is the Riemann zeta function, then $\varsigma(-n)=\dfrac{E_n(-1)}{2^{n+1}(1-2^{n+1})}$ (7.1)

Proof: From (5.9), we have $S_n(-1)=\dfrac{E_n(-1)}{2^{n+1}}$ (7.2). Substituting (7.2) in (6.4), we have $\varsigma(-n)=\dfrac{S_n(-1)}{1-2^{n+1}}=\dfrac{E_n(-1)}{2^{n+1}(1-2^{n+1})}$. This completes the proof.

7.2 Theorem 4

If ς is the Riemann zeta function then

(a) $1^{2m}+2^{2m}+3^{2m}+4^{2m}+\cdots=0$ (7.2) (b) $1^{2m-1}+2^{2m-1}+3^{2m-1}+4^{2m-1}+\cdots=-\dfrac{B_{2m}}{2m}$ (7.3)

where B_m is the *m*th Bernoulli number.

Proof:

(a) If *n* is an even positive integer then by (4.9) of theorem 2, we know that $E_n(-1)=0$. Hence, if *n* is an even integer of the form say $n=2m$ then from (7.1), we get $\varsigma(-2m)=\dfrac{E_{2m}(-1)}{2^{2m+1}(1-2^{2m+1})}=0$. But from (6.2) we have $\varsigma(-2m)=1^{2m}+2^{2m}+3^{2m}+4^{2m}+\cdots$

Therefore $1^{2m}+2^{2m}+3^{2m}+4^{2m}+\cdots=0$ proving (7.2)

(b) To prove (7.3), first let us list first few Bernoulli numbers (see **[5]**)

$$B_0=1, B_1=-\frac{1}{2}, B_2=\frac{1}{6}, B_3=0, B_4=-\frac{1}{30}, B_5=0, B_6=\frac{1}{42},$$
$$B_7=0, B_8=-\frac{1}{30}, B_9=0, B_{10}=\frac{5}{66},\ldots \quad (7.4)$$

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ISSN: 1309-3452

Now from (4.2) and (7.1), we have $\varsigma(-1)=\dfrac{E_1(-1)}{2^2(1-2^2)}=-\dfrac{1}{12}$. Thus from (6.2) and (7.4), we have

$$\varsigma(-1)=1+2+3+\cdots=-\frac{1}{12}=-\frac{B_2}{2} \quad (7.5)$$

From (4.4) and (7.1) $\varsigma(-3)=\dfrac{E_3(-1)}{2^4(1-2^4)}=\dfrac{-2}{16\times-15}=\dfrac{1}{120}$. Hence from (6.2) and (7.4), we have

$$\varsigma(-3)=1^3+2^3+3^3+\cdots=\frac{1}{120}=-\frac{B_4}{4} \quad (7.6)$$

Similarly, we have the following equations

$$\varsigma(-5)=1^5+2^5+3^5+\cdots=\frac{E_5(-1)}{2^6(1-2^6)}=\frac{16}{64\times-63}=-\frac{1}{252}=-\frac{B_6}{6} \quad (7.7)$$

$$\varsigma(-7)=1^7+2^7+3^7+\cdots=\frac{E_7(-1)}{2^8(1-2^8)}=\frac{-272}{256\times-255}=\frac{1}{240}=-\frac{B_8}{8} \quad (7.8)$$

$$\varsigma(-9)=1^9+2^9+3^9+\cdots=\frac{E_9(-1)}{2^{10}(1-2^{10})}=\frac{7936}{1024\times-1023}=-\frac{1}{132}=-\frac{B_{10}}{10} \quad (7.9)$$

Proceeding in same fashion and observing equations (7.5) to (7.9) we get (7.3).

This completes the proof.

7.3 Verifying Geometrically

Using the Eulerian Polynomials we can Geometrically verify (7.2) of part (a) in Theorem 4. For that we first try to plot the graphs of $E_2(x), E_4(x), E_6(x)$ obtained in (4.3), (4.5), (4.7) respectively. We can verify using (7.1) of Theorem 3.

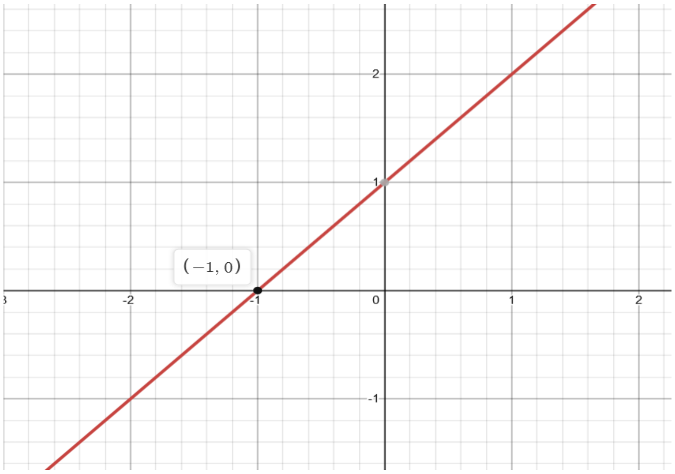

Figure 2: Graph of $E_2(x)=1+x$

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ISSN: 1309-3452

Since the graph of $E_2(x)=1+x$ pass through $(-1,0)$ we find that $\varsigma(-2)=1^2+2^2+3^2+\cdots=0$

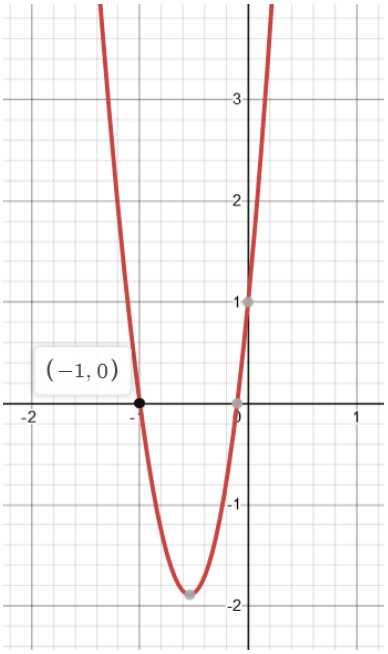

Figure 3: Graph of $E_4(x)=1+11x+11x^2+x^3$

Since the graph of $E_4(x)=1+11x+11x^2+x^3$ pass through $(-1,0)$ we find that $\varsigma(-4)=1^4+2^4+3^4+\cdots=0$.

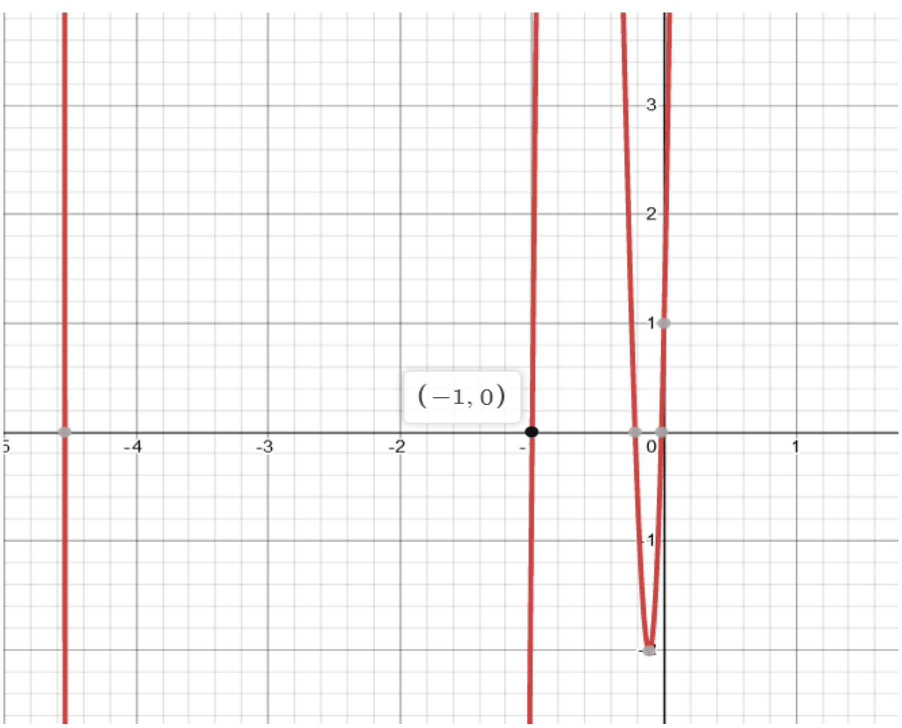

Figure 4: Graph of $E_6(x)=1+57x+302x^2+302x^3+57x^4+x^5$

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ISSN: 1309-3452

Since the graph of $E_6(x) = 1 + 57x + 302x^2 + 302x^3 + 57x^4 + x^5$ pass through $(-1, 0)$ we find that $\varsigma(-6) = 1^6 + 2^6 + 3^6 + \cdots = 0$.

In view of symmetry of Eulerian Numbers we find that, in general, the graph of $E_{2m}(x)$ always pass through $(-1, 0)$. Hence, from (4.9) and (7.1) we get $\varsigma(-2m) = 1^{2m} + 2^{2m} + 3^{2m} + 4^{2m} + \cdots = 0$ for all natural numbers *m*. These observations verify (7.2) geometrically.

8. Conclusion

In this paper, the Eulerian numbers are introduced through the concept of descents of permutations in permutation group. Using this idea, I had proved the symmetry property of Eulerian numbers through which we observe the palindromic pattern in each row of Figure 1 exhibiting Eulerian numbers. After introducing Eulerian Polynomials whose coefficients are Eulerian numbers, I connected these polynomials with the Riemann zeta function evaluated at negative integer values.

The well known fact that the Riemann zeta function has trivial zeros at negative integers was established in this paper using (4.9) of theorem 2 and (7.1) of theorem 3. These facts were also verified geometrically using Eulerian polynomials of even index through Figures 2,3,4 for better understanding.

Equation (7.1) of theorem was original and key result of this paper connecting Riemann zeta function at negative values with Eulerian polynomials evaluated at $x = -1$. Eventually using this equation, I could prove the two equations (7.2) and (7.3) of theorem 4 which are exactly Ramanujan Summation formulas provided by Srinivasa Ramanujan. Thus using Eulerian polynomials, I had proved Ramanujan Summation formulas through theorem 4.

By analyzing various other values of Eulerian polynomials we can try to explore few more properties in connection with Riemann zeta function and establish interesting combinatorial identities. Connecting completely unrelated concepts is one of the key aspects of mathematical research and this paper has exhibited this property thereby connecting the concepts provided by two great minds in mathematics history.

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