JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 1, 2022, p. 5480 - 5487
https://publishoa.com
ISSN: 1309-3452

Harmonious Coloring of Certain Graphs

[1]N. Ananthi[2]V.T. Chandrasekaran

[1]Research scholar
Affiliated to THIRUVALLUVAR UNIVERSITY-Vellore, India
[2]Associate Professor, Jawahar Science College Neyveli-607003
Affiliated to THIRUVALLUVAR UNIVERSITY-Vellore, India

Abstract
A Harmonious coloring of a graph G is a proper vertex coloring of G, in which each pair of colors performs on at most one pair of adjacent vertices and the harmonious chromatic number of graph G is the minimum number of colors needed for the harmonious coloring of G and it is denoted by $\chi_H(G)$. In this paper to find the harmonious chromatic number of Complement of some graphs, central graph of triangular cactus graph and middle graph of Firecracker graph.

Keywords: Coloring, Harmonious coloring, Cartesian product, Rooted product
Mathematics Subject Classification: 05C62, 68R10

Introduction:

Graph Theory is one of the most emerging branches of mathematics with wide-ranging applications to computer science. Graph Theory is applied in diverse areas such as communication engineering, social sciences, physical sciences, linguistics and others. Graph coloring is one of the eldest and an interesting problem that comes up in a lot of applications. Graph coloring is an assignment of colors to the vertices of a graph. A vertex coloring is called proper coloring if no two adjacent vertices of the graph get the same color and the graph is at that time called properly colored graph. Several problems can be communicated as a graph coloring problem including time tabling, scheduling, register allocation, channel assignment in radio stations such that no station has a conflict.

There are several problems in graph coloring, one such problem is harmonious coloring problem. A harmonious coloring of a simple graph is the proper vertex coloring such that each pair of colors appears together on at most one edge. The harmonious chromatic number of *G*, denoted by $\chi_H(G)$ is the least number of colors in a harmonious coloring of G [3,7]. Harmonious coloring was first introduced by Frank Harray and M. J. Plantholt in 1982. However the proper definition of this notion is due to Lee Hopcroft and Krishnamoorty [4]. The harmonious chromatic number of several different families of graph has been found by different author. The lower bound of harmonious chromatic number may be considered as an edge receiving a unique color pair. Moreover if $\left|E(G)\right| \leq^k C_2$, then a graph is harmoniously colored with k colors. Paths and cycles were the first graphs whose harmonious chromatic numbers have been established. Therefore, there be present to finding the exact bound for the graphs such as the collection of non-trivial disjoint of paths graph, cycle graph, complete graphs, trees, triangular snake graphs, double triangular snake graphs, and diamond snake graphs[1,4,6,8] Away from each other, Lee and Mitchem [5] provided an upper bound for harmonious chromatic number of a graph.

Theorem 2.1:

For any path graph P_n the harmonious coloring of complement of path graph is n.

i.e., If P_n is a path graph, then $\chi_H\left(\overline{P_n}\right) = n$, for $n \geq 4$

Proof:

Let us consider the vertices of path P_n are $v_1, v_2, v_3, \ldots, v_n$.

From the complement of path graph, we obtain two different set of degree sequences, one set has degree $n-2$, and another one has degree $n-3$.

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 1, 2022, p. 5480 - 5487
https://publishoa.com
ISSN: 1309-3452

That is $\{\deg(v_1) = \deg(v_n) = n-2\} \in I_1$ and $\{\deg(v_2) = ... = \deg(v_{n-1}) = n-3\} \in I_2$.

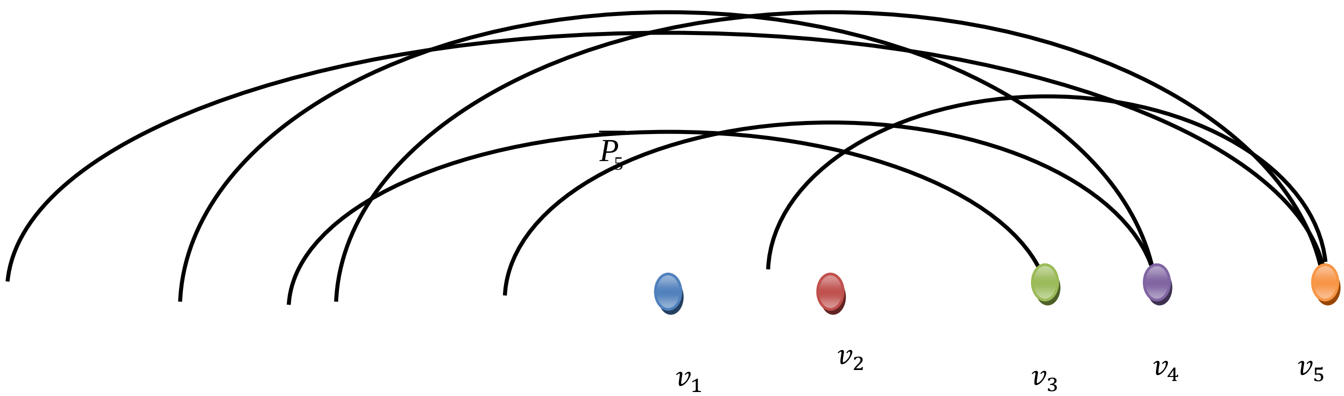
i.e., v_1 is adjacent to all the vertices other than v_2. Similarly v_n is adjacent to other than v_{n-1}.

Therefore, we need $n-1$ colors for v_1 and the neighbors of v_1 (or v_n and the neighbors of v_n) at this point let us choose n th color to the vertex v_2 since the neighbors of v_2 are all same as for v_1.

Suppose we assign v_2 and v_3 are same color which is contradiction to definition of harmonious coloring, since v_2 and v_3 are not adjacent as well as the neighbors of v_1 and the neighbors of v_3 are same.

Therefore, each vertex has assigned different color. That is, we can assign n colors to complement of path graph.

$\chi_H\left(\overline{P_n}\right) = n$



Theorem 2.2:

For any cycle graph C_n the harmonious coloring of complement of cycle is n, for $n > 5$.

i.e, If C_n is a cycle graph then $\chi_H\left(\overline{P_n}\right) = n$.

Proof:

Let C_n be a cycle graph the vertices of cycle graph is $1, 2, 3, ..., n$.

The complement C_n is a regular graph of degree $n-3$ for $n > 5$.

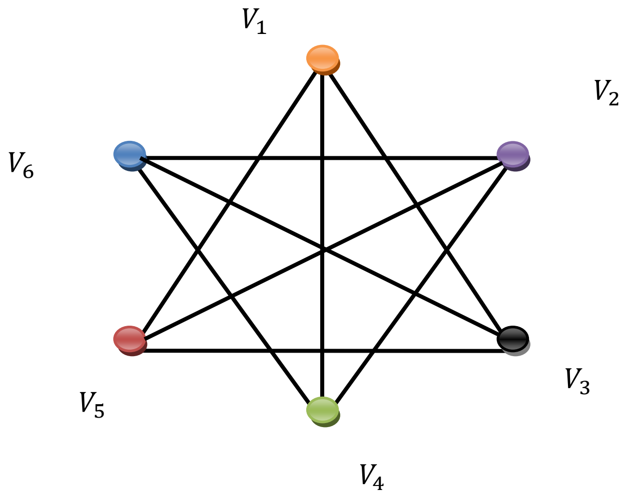
Therefore, we need $n-2$ colors for v_1 and some the neighbors of v_1, from $\overline{C_n}$ the vertex v_1 is not adjacent to v_2 and v_n as well as the neighbors of v_1 is adjacent to either v_2 or v_n. So the existing color is not possible to assign to v_2 and v_n.

Therefore we choose a color $n-1$ and n for v_2 and v_n respectively.

That is we can easily assign n color to $\overline{C_n}$.

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 1, 2022, p. 5480 - 5487
https://publishoa.com
ISSN: 1309-3452

$\overline{C_6}$:



Theorem 2.3:

For any wheel graph W_n, $n \geq 5$ the harmonious coloring on complement of wheel graph is $n-1$.

i.e., If W_n is a wheel graph then, $\chi_H\left(\overline{W_n}\right) = n-1$.

Proof:

Let W_n be a wheel graph the vertices of wheel graph is $1, 2, 3, \ldots, n-1$.

The complement of wheel graph $\overline{W_n}$ produces two components, one component covering a pendent vertex only, and the second components covering a graph with $n-1$ vertices which is a regular graph of degree $n-2$.

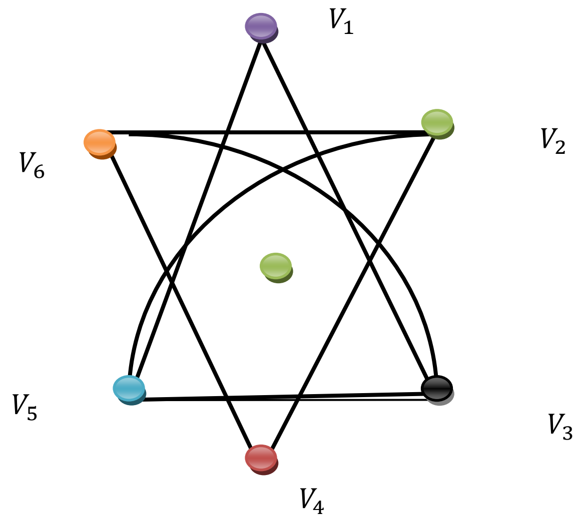
That is the second component is similar to the harmonious coloring of complement of cycle, so by the previous theorem, second components has $n-1$ colors for $n-1$ vertices.

Without loss generality once again we choose same color for the first component covering vertex v_1, since v_1 is not adjacent to $v_2, v_3, \ldots, v_n$.

$\therefore \chi_H\left(\overline{W_n}\right) = n-1$.

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 1, 2022, p. 5480 - 5487
https://publishoa.com
ISSN: 1309-3452

$\overline{W_7}$:



Theorem 2.4:

For any star graph S_n, $n \geq 3$ the harmonious coloring on complement of star graph is $n-1$.

i,e., If S_n is a star graph then $\chi_H\left(\overline{S_n}\right) = n-1$, for $n \geq 3$.

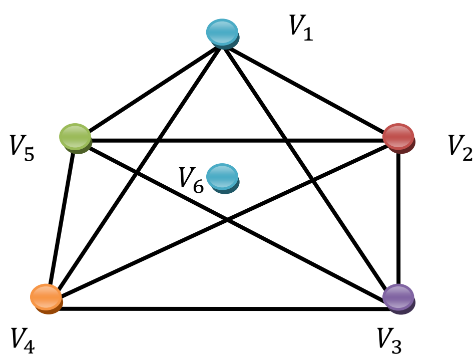
Proof:

Let S_n be a star graph the vertices of star graph $1, 2, \ldots, n-1$. By the definition of complement of star graph, we obtain one vertex has degree zero and remaining vertices has degree $n-2$.

For, $n \geq 4$, the complement of star graph produces two components, one is a pendent vertex, and another one has a complete graph of degree $n-1$. We can assign $n-1$ colors to the complement of star graph.

$\therefore \chi_H\left(\overline{S_n}\right) = n-1$.

$\overline{S_6}$:



Theorem 2.5:

For any ladder graph L_n, the harmonious coloring of complement of ladder graph is $2n-1$.

i.e., if L_n is a ladder graph, then $\chi_H\left(\overline{L_n}\right) = 2n-1$.

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 1, 2022, p. 5480 - 5487
https://publishoa.com
ISSN: 1309-3452

Proof:

Let L_n be a ladder graph the vertices of ladder graph are $v_{11}, v_{12}, ..., v_{1n}, v_{21}, v_{22}, ..., v_{2n}$. By the definition, the complement of ladder graph, we obtain each vertex has degree $n-1$, $\therefore$ we can get $2n$ vertices has degree $n-1$.

At this moment v_{1i} is adjacent to v_{2j} for $i \neq j$, we can assign a colors to the neighbours of v_1 is $n-1$ similarly for neighbour of v_2.

i.e, v_1 and v_2 has some color and neighbour are all distinct.

We can assign $2n-1$ colors to color the complement of ladder graph.

$\therefore \chi_H\left(\overline{L_n}\right) = 2n-1.$

Theorem:

The harmonious coloring of N copies of complete graph is $\chi_H\{G\} = N\left\{n - \frac{N-1}{2}\right\}$ for $n > 2$ and $N = n+1$.

.

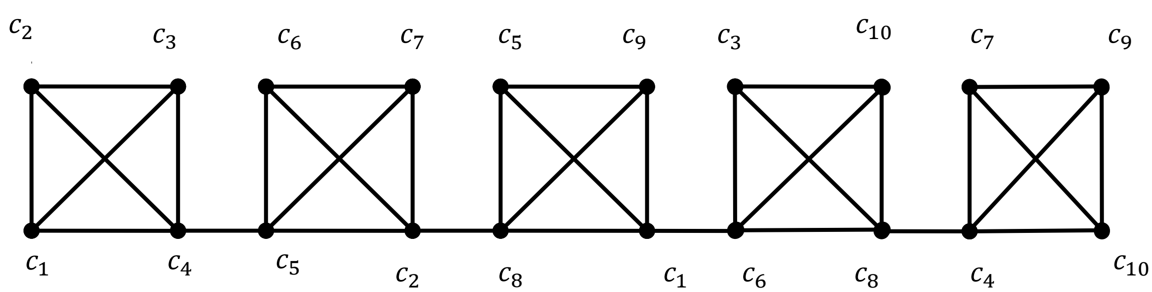
Proof :

Let $k_n^{(1)}, k_n^{(2)}, k_n^{(3)}, ..., k_n^{(N)}$ denote the N copies of complete graph k_n. For $N = 1, 2, 3, ..., n+1$ and $n > 2$ we allocate colors as follows. It is clear that in every complete graph, the harmonious coloring is implemented with separate n colors. Therefore $k_n^{(1)}$ is assigned with n colors, everywhere $k_n^{(2)}$ shares one bridge with $k_n^{(1)}$, it needs $(n-1)$ colors which are distinct from the set of colors assigned to $k_n^{(1)}$. Similarly $k_n^{(3)}$ shares one edge with $k_n^{(2)}$ it need $(n-2)$ colors.

Hence $k_n^{(N)}$ shares one edge with $k_n^{(N-1)}$ it need $(n-(N-1))$ colors.

Clearly we need $N_n - \frac{N(N-1)}{2}$ colors.

$\therefore \chi_H\{G\} = N\left\{n - \frac{N-1}{2}\right\}.$



Theorem 2.6

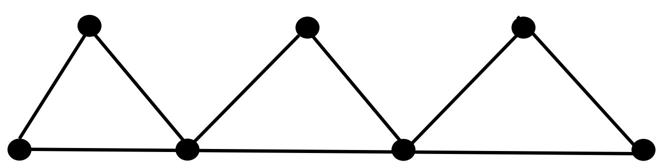
If G be a triangular cactus graph then the harmonious coloring on complement of Triangular cactus graph is $2n$.

Proof:

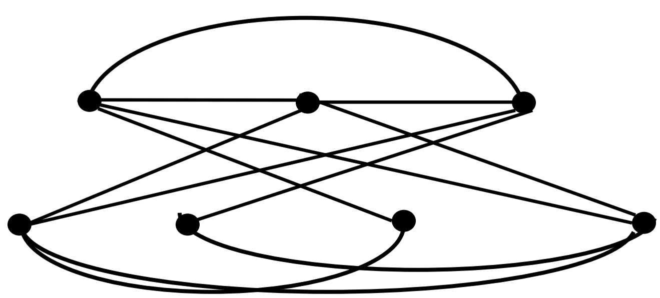
JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 1, 2022, p. 5480 - 5487
https://publishoa.com
ISSN: 1309-3452

Let us consider the triangular cactus graph with $2n+1$ vertices and $3n$ edges. Let us label the vertices of the triangular cactus graph are $\{v_1, v_2, v_3, \ldots, v_n, v_{n+1}, v_{n+2}, \ldots, v_{2n}\}$. Here $\{v_1, v_2, v_3, \ldots, v_n \,\&\, v_{2n}\}$ $n+1$ vertices has degree 2 and $\{v_{n+1}, v_{n+2}, \ldots, v_{2n-1}\}$ $n-1$ vertices has degree 4.

By the definition of complement of graph, the vertices $\{v_1, v_2, v_3, \ldots, v_n, v_{n+1}, v_{n+2}, \ldots, v_{2n}\}$ has degree $n-3$ and the vertices $\{v_{n+1}, v_{n+2}, \ldots, v_{2n-1}\}$ has degree $n-5$. Therefore the neighbors of $\{v_{n+1}, v_{n+2}, \ldots, v_{2n-1}\}$ are adjacent to neighbors of $\{v_{n+1}, v_{n+2}, \ldots, v_{2n-1}\}$, at this present we can allot each vertex as different colors. That is, the harmonious coloring on complement of Triangular cactus graph is $2n$.



Triangular cactus graph



Complement of Triangular cactus graph

Theorem 2.7
If G be a triangular cactus graph then the harmonious coloring on central graph of Triangular cactus graph is $2n+3$.

Proof:

Let us consider the triangular cactus graph with $2n+1$ vertices and $3n$ edges. Let the vertices of the triangular cactus graph are $\{v_1, v_2, v_3, \ldots, v_n, v_{n+1}, u_1, \ldots, u_n\}$.

.

By the definition of central graph, each edge of graph is subdivided by a new vertex. Therefore assume that each edge (v_i, v_{i+1}) and the line joining v_i and v_{i+1} to a vertex u_i, $i = 1, 2, 3, \ldots, n$ are subdivided by the vertices w_i, e_{jj} and $e_{j,j+1}$, $j = 1, 2, 3, \ldots, n$ respectively. Assign colors to the vertices as follows:

$$v_i = i\,(1 \le i \le n+1)$$

$$u_i \,\&\, w_i = (n+1) + i\,(1 \le i \le n)$$

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 1, 2022, p. 5480 - 5487
https://publishoa.com
ISSN: 1309-3452

$$e_{jj} = (2n+1)+1(1 \le j \le n)$$
$$e_{j,j+1} = (2n+2)+1(1 \le j \le n)$$

Clearly, T_n has $2n+3$ coloring.

Clearly it is a proper coloring. Since every v_i, u_i and its neighbors obtain distinct colors. Further color of $v_i \neq \text{color of } u_i$. This is the minimum number of colors in a harmonious coloring.

Suppose Case (i): The color of v_i and u_i covers $2n+1$ colors. If we assign $2n$ colors, then the colors will not be distinct colors which are contradiction to the definition of harmonious colors.

Case (ii): The neighbors of u_i and v_i are assumed same color, then the color pair duplications which is a contradiction to the definition of harmonious coloring.

Therefore the minimum number of colors in a harmonious coloring for central graph of triangular cactus graph is $2n+3$.

Harmonious coloring on Middle graph of firecracker graph $(F_n, 3)$:

Theorem:3.1

Let G be a middle graph of $(F_n, 3)$, $n \ge 5$ graph, then the harmonious coloring on middle graph of $(F_n, 3)$ is $2n$.

Proof:

By the definition of Middle graph of $(F_n, 3)$, G has $2n$ vertices which is attained by attaching n pendent edges to the cycle of length n. Let us consider the vertices of C_n are $v_1, v_2, ..., v_n$.

As well as the pendent edges are adjacent with v_i and $v_i u_i$ where u_i are the pendent vertices of $(F_n, 3)$.

Now allot the coloring to the vertices as follows,let us consider the colors are $\{1, 2, 3, \ldots, 2n\}$.

(i) Assign the color i to u_i for $(1 \le i \le n)$.

(ii) Allot color i to v_i for $(2 \le i \le n-1)$.

(iii) Assign color $n+i$ to v_i, v_{i+1} for $(1 \le i \le n-1)$.

Therefore, the number of colors used in middle graph of $(F_n, 3)$ is $2n$. Thus above assumed coloring is clearly harmonious coloring with minimum number of colors, suppose if we replace any color, which is already used then the resulting coloring will be improper or not harmonious.

References:

1. AkhlakMansuri and R. S. Chandel A note on Harmonious Coloring of n-sunlet Graph *International Journal of Computational Science and Mathematics*. Volume 4, Number 3 (2012), pp. 267-270.
2. Aflaki, A. Akbari, S.; Edwards, K.J.; Eskandani, D.S.; Jamaali, M.; Ravanbod, H. (2012) On harmonious colouring of trees. *Electron. J. Comb*. 19, 3–11.
3. Georges, J.P. On the harmonious coloring of collections of graphs. (1995) *J. Graph Theory*, 20, 241–254.
4. Hopcroft, J. Krishnamoorthy, M.S.(1983) On the harmonious coloring of graphs. SIAM *J. Algebr. Discrete Methods*, 20, 306–311.
5. Lee, S. Mitchem, J.(1987) An upper bound for the harmonious chromatic number of a graph. *J. Graph Theory*, 11, 565–567.
6. Miller, Z. Pritikin, D. (1991) The harmonious coloring number of a graph. *Discrete Math*, 93, 211–228.

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 1, 2022, p. 5480 - 5487
https://publishoa.com
ISSN: 1309-3452

7. Selvi, M.S.F.T.(2015) Harmonious coloring of central graphs of certain snake graphs. *Appl. Math. Sci*, 9, 569–578.
8. Zhang, P.(2015) *Color-Induced Graph Colorings*; Springer: London, UK.