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Equitable Coloring for Union of Graphs

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ABSTRACT

Let G be a finite, undirected and simple graph. A proper vertex coloring of a graph of G is equitable if the sizes of color classes differ by atmost 1. The equitable chromatic number of a graph G, denoted by $\chi_{=}(G)$, is the minimum k such that G is equitably k-colorable. We shall discuss about an equitable coloring for union of two graphs.

1. Introduction

All graphs considered in this paper are finite, undirected and without loops and multiple edges. Let $G = (V, E)$ be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. All the definitions which are not discussed in this paper one may refer [2, 3]. The origin of graph theory can be traced back to Euler's work on the Konigsberg bridge problems [1]. It has a wide range of applications such as finding communities in networks, solving shortest path problems, analyzing the chemical structures and so on. Beginning with the origin of the four color problem in 1852, the field of graph coloring has developed into one of the most interesting areas of graph theory [7]. In particular, graph coloring plays a central position in discrete mathematics and computer science. In past decades, many research articles have been devoted to exploring the applications of these coloring problems. A proper $k-$ coloring of a graph G is a function $f : V(G) \rightarrow \{1, 2, \ldots, k\}$ define in such a way that $f(x) \neq f(y)$ whenever $xy \in E(G)$. The vertices of the same color form a color class. The chromatic number $\chi(G)$ of a graph G, is the smallest integer k such that G has a proper $k-$ coloring. An edge coloring assigns a color to each so that no two adjacent edges share the same color. In the current paper, we focus on a typical version of graph coloring called equitable coloring [5]. The concept of equitable colorability was first introduced by Meyer [4]. His motivation came from the application given by Tucker where the vertices represented garbage collection routes and two such vertices were joined when the corresponding routes should not be run on the same day. If the set of vertices of a graph G can be partitioned into k classes $V_1, V_2, \ldots, V_k$ such that each V_i is an independent set and the condition $||V_i| - |V_j|| \leq 1$ holds for every pair (i, j), then G is said to be equitably $k -$ colorable. The smallest integer k for which G is equitably $k-$ colorable is known as the equitable chromatic number of [13–16] G and is denoted by $\chi_{=}(G)$. Since equitable coloring is a proper coloring with additional condition, $\chi(G) \leq \chi_{=}(G)$ for any graph G. It is interesting to note that if a graph G is equitably $k -$ colorable, it does not imply that it is equitably $k + 1-$ colorable. A counter example is the complete bipartite graph $K_{3,3}$ which can be equitably colored with two colors, but not with three. The equitable chromatic threshold of G is $\chi^*_{=}(G) = min\{t : G \text{ is equitably } k - \text{colorable for all } k \geq t\}$. In 1964, Erdos [8] conjectured that any graph G with maximum degree $\Delta(G) \leq k$ has an equitable $(k + 1) -$ coloring, or equivalently is $\chi^*_{=}(G) \leq \Delta(G) + 1$. This conjecture was proved in 1970 by Hajnal and Szemeredi [9]. Recently, Kierstead and Kostochka [10] gave a short proof of

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the theorem, and presented a polynomial algorithm for such a coloring. In 1973, Meyer [4] formulated the following conjecture: Equitable Coloring Conjecture [4]. For any connected graph G, other than a complete graph or an odd cycle, $\chi_{=}(G) \leq \Delta(G)$. This conjecture has been verified for all graphs on six or fewer vertices. Lih and Wu [16] proved that the Equitable Coloring Conjecture is true for all bipartite graphs. Wang and Zhang [19] considered a broader class of graphs, namely r −partite graphs. They proved that Meyer's conjecture is true for complete graphs from this class. Also, the conjecture was confirmed for outerplanar graphs [17] and planar graphs with maximum degree at least 13 [18]. We also have a stronger conjecture: Equitable $\Delta -$ Coloring conjecture [13], If G is a connected graph of degree Δ, other than a complete graph, an odd cycle or a complete bipartite graph $K_{2n+1,2n+1}$ for any n ≥ 1, then G is equitably $\Delta -$ Colorable. The Equitable Δ Coloring Conjecture holds for some classes of graphs, e.g., bipartite graphs [16], outerplanar graphs with $\Delta \geq 3$ [17] and planar graphs with $\Delta \geq 13$ [18]. The detailed survey of this type of coloring is found in Lih [6]. In the present paper, we study on equitable coloring for union of graphs.

2. Preliminaries

Before we go through the main results, we want some preliminary results related for equitable coloring.

Definition 2.1. Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two simple graphs. $G_1 \cup G_2$ denotes the Union of two graphs G_1 and G_2 has the vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. When G_1 and G_2 are disjoint $G_1 \cup G_2$ is denoted by $G_1 + G_2$.

Lemma 2.2. [20] If two graphs G and H with disjoint vertex sets are both equitably $k -$ colorable, then $G + H$ is also equitably $k -$ colorable.

Lemma 2.3. [20] $mK_{n,n}$ is equitably $k -$ colorable for any $m \geq 2, n \geq 2 \; and \; k \geq 2$.

Lemma 2.4. [20] Let G be a graph and suppose that $|V(G)|$ is not divisible by a positive integer $n \geq 3$. If G is equitably $n -$ colorable, then $G + K_{n,n}$ is also equitably $n -$ colorable.

Lemma 2.5. [20] Let G be a graph and suppose that $|V(G)|$ is divisible by a positive integer $n \geq 3$. If there exists a proper $n -$ coloring of G such that the sizes of color classes in nondecreasing order are $\frac{|V(G)|}{n} - 1, \frac{|V(G)|}{n} - 2 \; . . . , \frac{|V(G)|}{n}, \frac{|V(G)|}{n} + 1$, then $G + K_{n,n}$ is equitably $n - \; colorable$.

Lemma 2.6. [20] Let $n \geq 2$ be a positive integer and let G be a graph with $\Delta(G) \leq n - 1$. Then $G + K_{n,n}$ is equitably $n -$ colorable if and only if n is even, or G is different from mK_n for all $m \geq 1$.

Lemma 2.7. [20] Let G be a graph with $\Delta(G) \geq \chi(G)$. If G is equtably Δ(G)− colorable, then at least one of the following statements holds.

1. $\Delta(G)$ is even.
2. No components or at least two components of G are isomorphic to $K_{\Delta(G),\Delta(G)}$.
3. Only one component of G is isomorphic to $K_{\Delta(G),\Delta(G)}$ and $\alpha(G - K_{\Delta(G),\Delta(G)}) > \frac{|V(G - K_{\Delta(G),\Delta(G)})|}{\Delta} > 0$.

2. Equitable coloring for union of two graphs

In this section, we obtained the gentralized formula for equitable chromatic number for union of any two graphs.

Theorem 3.1. Let G_1 and G_2 be two graphs. Let $k = max(\chi^*_{=}(G_1), \chi^*_{=}(G_2))$. Then $G_1 \cup G_2$ is equitably $k -$ colorable.

Proof. Let G_1 and G_2 be two graphs with $n_1$and n_2 vertices respectively. Let max $\left(\chi^*_{=}(G_1), \chi^*_{=}(G_2)\right) = k$. With out loss of generality, let $\chi^*_{=}(G_1) = k$. Case 1: Let $n_2 \geq k$. $\chi^*_{=}(G_2) \leq k \leq n_2$. Since G_2 is equitably k colorable. Let $\varphi_1 = \{V_1, V_2, \dots, V_k\}$ be an equitable color partition of G_1such that $|V_i| \leq |V_{i+1}|$ and $i = 1, 2, \dots, k - 1$. Let $\varphi_2 = \{W_1, W_2, \dots, W_k\}$ be an equitable color partition of G_2 such that $|W_i| \leq |W_{i+1}|$ and $i = 1, 2, \dots, k - 1$. Assume that the first t color classes in φ_1 has l elements and the remaining $k - t$ color classes has $l + 1$ elements. Similarly the

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first s color classes in φ_2 has m elements and the remaining $k - s$ has color classes has $m + 1$ elements. Here $0 \leq s, t \leq k$. When either s or $t \in \{0, k\}$, $\{V_1 \cup W_1, V_2 \cup W_2, \ldots, V_k \cup W_k\}$ is an equitable coloring of $G_1 \cup G_2$. In this case $\chi_=(G_1 \cup G_2) \leq k$. Let $s, t \notin \{0, k\}$. Subcase 1: Let $s \leq k - t$. Consider the partition $\varphi_3 = \{V_1 \cup W_k, V_2 \cup W_{k-1}, \ldots, V_t \cup W_{k-(t-1)}, V_{t+1} \cup W_{k-t}, \ldots, V_k \cup W_1\}$. In this partition $s + t$ classes have $l + m + 1$ elements and the remaining classes contain $l + m + 2$ elements. Hence φ_3 is an equitable coloring of $G_1 \cup G_2$. Subcase 2: Let $t \geq k - s$. Clearly $s \geq k - t$. In this case φ_3 becomes an equitable coloring of $G_1 \cup G_2$ where each color class contains either $l + m$ elements or $l + m + 1$. Hence $\chi_=(G_1 \cup G_2) \leq k = max(\chi^*{}_=(G_1), \chi^*{}_=(G_2))$. Case 2: Let $n_2 \leq k$. Since G_2 is n_2 equitably colorable, we can always obtain an equitable color partition φ_2 for G_2 with n_2 color class when each color class contains single vertex. Let $\varphi_2 = \{\{W_1\}, \{W_2\}, \ldots, \{W_{n_2}\}\}$.Clearly $\{V_1 \cup \{W_1\}, V_2 \cup \{W_2\}, \ldots, V_{n_2} \cup \{W_{n_2}\}, V_{n_2+1}, \ldots, V_k\}$ is an equitable class partition for $G_1 \cup G_2$. Hence $\chi_=(G_1 \cup G_2) \leq k$.

Note1: The upper bound for $\chi_=(G_1 \cup G_2)$ given in the above theorem is attainable.
For example1: $\chi_=(K_{1,3} \cup K_{2,7}) \leq max(\chi^*{}_=(K_{1,3}), \chi^*{}_=(K_{2,7}))$. $\chi_=(K_{1,3} \cup K_{2,7}) = 3, \chi^*{}_=(K_{1,3}) = 3, \chi^*{}_=(K_{2,7}) = 4$
Therefore 3 < 4. The equitable chromatic number of $(K_{1,3} \cup K_{2,7})$ is given in the following figure.

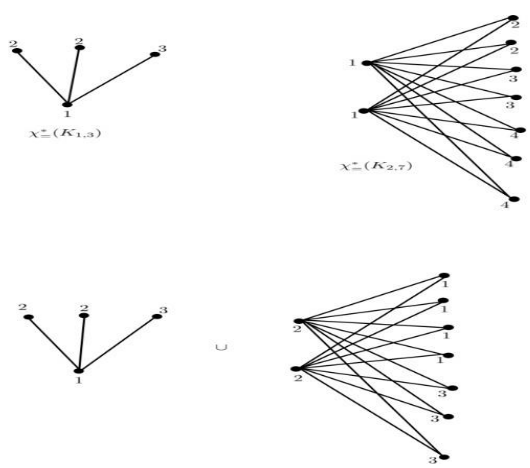

Figure 1: $\chi_=(K_{1,3} \cup K_{2,7}) = 3$.

Note2: The equality condition attains for the above theorem.

Corollary 1. $\chi_=(G_1 \cup G_2) \leq max(\chi^*{}_=(G_1), \chi^*{}_=(G_2))$.

Proof. In view of the above theorem, if $k = max(\chi^*{}_=(G_1), \chi^*{}_=(G_2))$, then $G_1 \cup G_2$ is equitably k colorable. Hence $\chi_=(G_1 \cup G_2) \leq k = max(\chi^*{}_=(G_1), \chi^*{}_=(G_2))$.

Theorem 3.2. If $G_1, G_2, \ldots, G_n$ are l disjoint graphs then $\chi_=(\cup G_i) \leq max\,\{\chi^*{}_=(G_1), \chi^*{}_=(G_2), \ldots, \chi^*{}_=(G_l)\}$.

Proof. We prove the theorem by the method of induction on n. The theorem is true for $n = 2$ in view of the above theorem. Assume that the theorem is true for $n < k$. We prove the theorem for $n = k$.

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$$\chi_{=}\left(\bigcup_{i=1}^{k} G_i\right) = \chi_{=}\left(\bigcup_{i=1}^{k-1} G_i \cup G_k\right) \leq \max\left(\chi^*_{=}\left(\bigcup_{i=1}^{k-1} G_i\right), \chi^*_{=}(G_k)\right)$$
$$\leq \max\left(\max\left(\chi^*_{=}(G_1), \chi^*_{=}(G_2), \dots, \chi^*_{=}(G_{k-1})\right), \chi^*_{=}(G_k)\right)$$
$$\leq \max\left(\chi^*_{=}(G_1), \chi^*_{=}(G_2), \dots \chi^*_{=}(G_{k-1}), \chi^*_{=}(G_k)\right).$$

Corollary 2. $\chi_{=}(\mathrm{P_m} \cup P_n) = 2$.

Proof. We know that $\chi_{=}(P_m) = \chi_{=}(P_n) = 2, \chi^*_{=}(P_m) = \chi^*_{=}(P_n) = 2$ for all m, n. So $2 = \chi_{=}(\mathrm{P_m}) \leq \chi_{=}(\mathrm{P_m} \cup P_n) \leq \max\{2,2\}$. Hence $\chi_{=}(\mathrm{P_m} \cup P_n) = 2$.

Corollary 3. $\chi_{=}(\mathrm{P_n} \cup C_m) = \begin{cases} 2, & if\ m = even \\ 3, & if\ m = odd \end{cases}$.

Proof. It is easy to verify that $\chi_{=}(\mathrm{P_n}) = \chi^*_{=}(P_n) = 2$, $\chi_{=}(C_m) = \chi^*_{=}(C_m) = \begin{cases} 2, & if\ m = even \\ 3, & if\ m = odd \end{cases}$. By the above theorem , $\chi_{=}(\mathrm{P_n} \cup C_m) = \begin{cases} 2, & if\ m = even \\ 3, & if\ m = odd \end{cases}$.

Corollary 4. $\chi_{=}(\mathrm{C_n} \cup C_m) = \begin{cases} 2, & if\ m = even \\ 3, & if\ m = odd \end{cases}$.

Proof. Similarly from the above corollary, $\chi_{=}(\mathrm{C_n} \cup C_m) = \begin{cases} 2, & if\ m = even \\ 3, & if\ m = odd \end{cases}$.

The equitable chromatic number of $(\mathrm{C_5} \cup C_6)$ is given in the following figure.

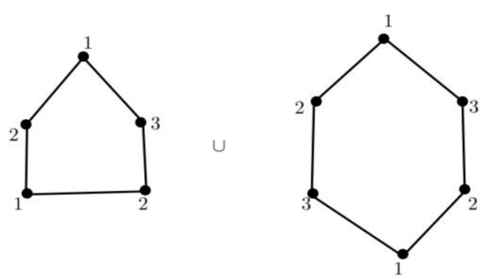

Figure 2: $\chi_{=}(\mathrm{C_5} \cup C_6) = 3$

Corollary 5. $\chi_{=}(\mathrm{K_n} \cup P_n) = n$.

Proof. It is easy to verify that $\chi_{=}(P_n) = \chi^*_{=}(P_n) = 2$ and $\chi_{=}(K_n) = \chi^*_{=}(K_n) = n$. So $n = \chi_{=}(\mathrm{K_n}) \leq \chi_{=}(\mathrm{K_n} \cup P_n) \leq \max\{n, 2\}$. Hence $\chi_{=}(\mathrm{K_n} \cup P_n) = n$.

Corollary 6. $\chi_{=}(\mathrm{K_n} \cup C_m) = n$.

Proof. It is easy to verify that $\chi_{=}(\mathrm{K_n}) = \chi^*_{=}(K_n) = n$, $\chi_{=}(C_m) = \chi^*_{=}(C_m) = \begin{cases} 2, & if\ m = even \\ 3, & if\ m = odd \end{cases}$. By the above theorem, $\chi_{=}(\mathrm{K_n} \cup C_m) = n$.

Corollary 7. $\chi_{=}(\mathrm{K_n} \cup K_n) = n$.

Proof. It is easy to verify that $\chi_{=}(\mathrm{K_n}) = \chi^*_{=}(K_n) = n$. So $n = \chi_{=}(\mathrm{K_n}) \leq \chi_{=}(\mathrm{K_n} \cup K_n) \leq \max\{n, n\}$. Hence $\chi_{=}(\mathrm{K_n} \cup$

$K_n) = n$.

4. Conclusion

In this paper, we have tried to obtain generalized formula for equitable chromatic number of union of any two graphs. Also we obtained the equitable chromatic number of path union path, path union cycle, cycle union cycle, complete graph union path, complete graph union cycle and complete graph union complete graph.

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