JOURNAL OF ALGEBRAIC STATISTICS
Volume 9, No. 1, 2018, p. 81-87
https://publishoa.com
ISSN: 1309-3452
DOI: https://doi.org/10.52783/jas.v9i1.1442

Mathematical Formulation of Arithmetic Surface (3, 5) Over Q

Deepak Khantwal
Department of Mathematics, Graphic Era Hill University, Dehradun, Uttarakhand, India 248002

Abstract

The primary arithmetic surface (3, 5) over Q is the focus of this work, which examines its construction and properties. Attaching the two symbols i,j where i2 = 3, j2 = 5, and ij = -ji yields a quaternion algebra over Q. This "example issue" involves the creation of a hyperbolic surface by linking the points (3, 5) over Q. If you want to demonstrate how to discover a Dirichlet domain, or how to generate a hyperbolic polygon, you may use this issue as an illustration. The list of generators that corresponds to the edge gluings is one possible motivation for seeking for a hyperbolic surface. The generators in this list have applications beyond only pure mathematics and science. Creating a co-compact Dirichlet domain in Hyperbolic 3-space is the focus of this article. Based on the concepts introduced in "Expository Note: An Arithmetic Surface," this article was written. We provide a brief summary of the material presented before constructing the Dirichlet domain in Hyperbolic 2-space associated with the quadratic form $Q(x) = x^2 + y^2 - 7z^2$. It is then shown that the new quadratic, $Q(x) = w^2 + x^2 + y^2 - 7z^2$, also has a Dirchlet Domain in Hyperbolic 3-space.

Keywords: Mathematical, Arithmetic Surface, Formulation, Hyperbolic, Dirichlet domain

Introduction

Multivariate polynomial equations over R or C are the focus of algebraic geometry. A hyperbola is defined by the equation x 2 + xy 5y 2 = 1. It makes use of both commutative algebra and intuitive geometry. In arithmetic geometry, however, the solutions in which the coordinates reside are of relevance, and these fields are often not algebraically closed. Q and Fp, as well as their finite extensions, are of particular importance. Z-coordinated solutions are also of interest.

Here we implement such an approach and provide numerical proof for Birch's and Swinnerton-Dyer's hypothesis on the Jacobian of the level-13 Cartan modular curve. Our key contribution is a novel method for calculating the local N'eron pairings that does not rely on archimedean geometry. Of possible independent relevance, we also provide a novel approach for finding the intersection pairing of two divisors without a shared component on a normal arithmetic surface. In brief, we utilize a saturation of the defining ideals to bring divisors up from the generic fibre to the arithmetic surface, and we use an inclusion-exclusion concept to handle divisors that intersect on several affine patches. Similar to the methods described, the archimedean local N'eron pairs D, E are calculated by inverting a translation of the Riemann theta function to C(C). New algorithms by Neurohr and Molin-Neurohr are used to compute period matrices and Abel-Jacobi maps on Riemann surfaces explicitly.

Literature Review

Ono, Ken & Trebat-Leder, Sarah (2016) Ramanujan's work on the mathematics behind the infamous "taxi-cab" figure 1729 is revisited. The $$ theme of Euler's writings is the diophantine equation a3+b3=c3+d3. Important aspects of geometry and number theory can be traced back to Ramanujan's work, as well as profound structures and occurrences. He found a $K3$ surface with Picard number 18, which allows for the construction of an endless number of cubic twists over $\mathbb{Q}$ of rank ≥ 2.

Bergeron, Nicolas. (2016) we reviewed the many methods available for building first-kind Fuchsian groups. From a number theoretic perspective, the arithmetic groups are the most important of these groups. Given the technical nature of the general description of these groups, we will instead focus on explaining one example family: the arithmetic groups arising from a quaternion algebra over Q. First, we prove some general facts about lattice space before discussing them.

Dória, Cayo. (2017) For each collection of arithmetic surfaces, there should be a consistent pattern in how their images of interesting functions are distributed. We look at how frequently the function systole occurs. Some of the findings we

JOURNAL OF ALGEBRAIC STATISTICS
Volume 9, No. 1, 2018, p. 81-87
https://publishoa.com
ISSN: 1309-3452
DOI: https://doi.org/10.52783/jas.v9i1.1442
provide suggest that the set of arithmetic surfaces cannot have its systoles concentrated. The proofs draw on several areas of mathematics, including combinatorics.

Surfaces

Classification of Surfaces

Surfaces are shapes that, at least in close proximity, resemble flat planes. The Earth, as an example, has a surface and is locally flat, which is why this misconception persisted for so long.

Definition 1. If the neighborhood of each point on the topological surface is homeomorphic to an open subset of R2, then the topological surface is a Hausdorff topological space with a countable basis.

Definition 2. We define a compact surface to be S. S is orientable if and only if there is a way to choose which direction is "outside" of the surface.

Definition 3. A group having discrete topology is said to be "discrete."

Proposition 5. The quotient is the area of the freshly adhered surface, and the result is a covering map made of |Γ|-sheets. The study of surface topology is our first step. Topology is the study of the intrinsic properties of surfaces. A surface may acquire cone points when it is a cover for another surface. Consider a literal pole, as in Example 1, intersecting with a surface; a fundamental domain is then rotated around the pole at some angle, and the resulting surface is the fundamental domain. Cone points are created at the locations where the pole and the surface meet.

Example 1. This is a genus 2 surface we can examine in detail.

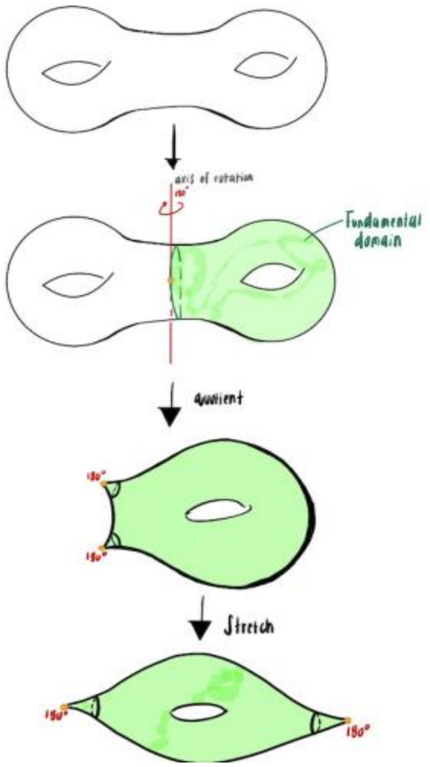

Figure 1: Surface-Covered Species 2 cone-shaped points on 1 flat surface.

When we put this pole in the exact center of this surface, we create two intersections. That's where you go in from the top and out the bottom. The surface is flipped 180 degrees around this axis. In this example, two cone points on a genus-2 surface wrap around a genus-1 surface, each of which has a radius of 180 degrees. The cover is double-sided.

JOURNAL OF ALGEBRAIC STATISTICS
Volume 9, No. 1, 2018, p. 81-87
https://publishoa.com
ISSN: 1309-3452
DOI: https://doi.org/10.52783/jas.v9i1.1442

Example 2. In this article, we examine the quotient orbifolds that emerge from rotating a genus 3 surface around two different axes.

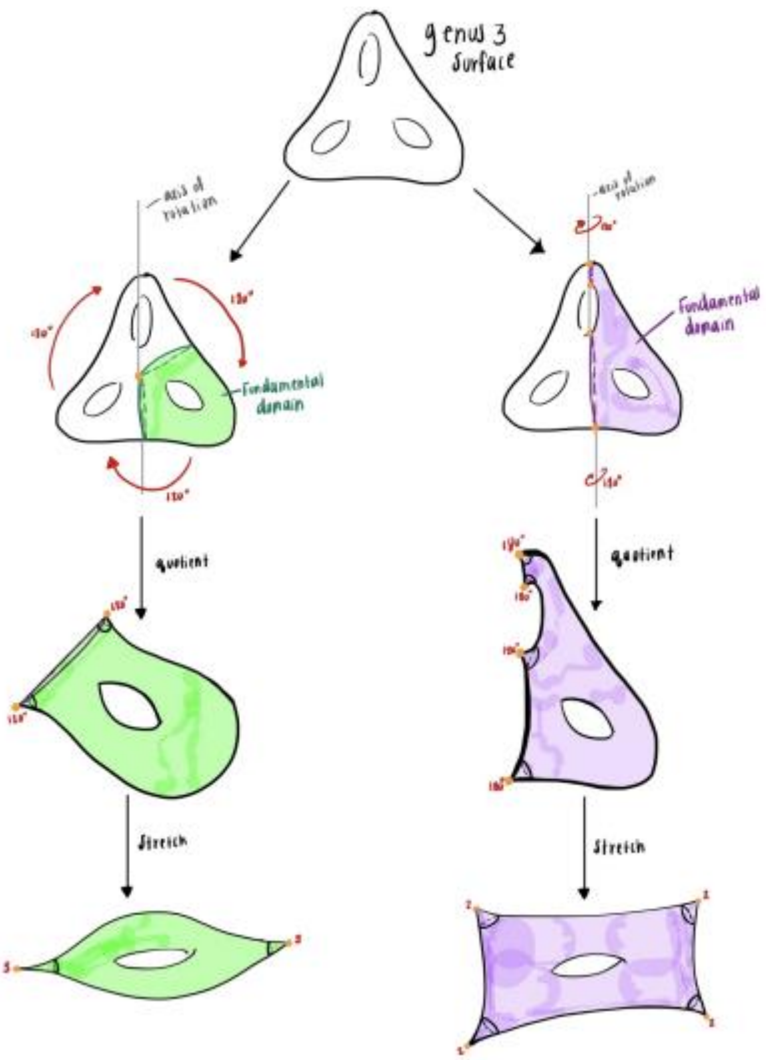

Figure 2: Surface of genus 3 with two possible orientations of the axis of rotation

The picture centers on the pole, while a 120-degree rotation of the fundamental domain places it at an angle. The degree of the two cone points is 3. The signature is the means through which different surfaces may be distinguished. Two surfaces of genus one with the signatures (1, 2, 2, 2, 2) are hidden under a genus three surface (1; 3, 3).

Examples of Compact Arithmetic Surfaces

Hyperbolic 2-Space

We'll provide a brief synopsis of Konorovich's explanatory letter, as promised. In order to spice up this part and provide the reader with two alternative examples to refer to, we will be working with a different quadratic form, $Q(x) = x^2 + y^2 - 7z^2$, than what was in the note. Anisotropic ($Q \neq 0$) and indefinite over Q describe this quadratic form.

We will begin by building G's spin representation. The goal is to build a pair of matrices mx that completely represent Q (x). Possible choices include,

JOURNAL OF ALGEBRAIC STATISTICS
Volume 9, No. 1, 2018, p. 81-87
https://publishoa.com
ISSN: 1309-3452
DOI: https://doi.org/10.52783/jas.v9i1.1442

$$m_{\mathbf{x}} = \begin{pmatrix} z\sqrt{7}-y & x \\ x & z\sqrt{7}+y \end{pmatrix}$$

Take into account that the determinant of this symmetric matrix is -Q (x). Then, we use

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{R})$$

the fact that

$$g \cdot m_{\mathbf{x}} \cdot g^t = \begin{pmatrix} -ya^2+\sqrt{7}za^2+2bxa+b^2y+\sqrt{7}b^2z & bcx+adx-acy+bdy+\sqrt{7}acz+\sqrt{7}bdz \\ bcx+adx-acy+bdy+\sqrt{7}acz+\sqrt{7}bdz & -yc^2+\sqrt{7}zc^2+2dxc+d^2y+\sqrt{7}d^2z \end{pmatrix}$$

whose determinant is -Q(x) and which is symmetric once more. New x' may be obtained directly from x and our operation g in the following way.

$$\mathbf{x}' = (x,y,z) \cdot \begin{pmatrix} bc+ad & cd-ab & \frac{1}{\sqrt{7}}(ad+cb) \\ bd-ac & \frac{1}{2}(a^2-b^2-c^2+d^2) & \frac{1}{2\sqrt{7}}(-a^2+b^2-c^2+d^2) \\ \sqrt{7}(ac+bd) & \frac{\sqrt{7}}{2}(-a^2-b^2+c^2+d^2) & \frac{1}{2}(a^2+b^2+c^2+d^2) \end{pmatrix}$$

The aforementioned 33G matrix acts as a road map.

$$\iota : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \rightarrow \begin{pmatrix} bc+ad & cd-ab & \frac{1}{\sqrt{7}}(ad+cb) \\ bd-ac & \frac{1}{2}(a^2-b^2-c^2+d^2) & \frac{1}{2\sqrt{7}}(-a^2+b^2-c^2+d^2) \\ \sqrt{7}(ac+bd) & \frac{\sqrt{7}}{2}(-a^2-b^2+c^2+d^2) & \frac{1}{2}(a^2+b^2+c^2+d^2) \end{pmatrix}$$

The properties $I^2 = 7$, $J^2 = 7$, $K^2 = -1$, and $\frac{1}{7}$ IJ = K are what we're after. Hence, the norm of u is found to be a + bi + cj + dk. We then see this algebra of division in the following forms:

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad I = \begin{pmatrix} \sqrt{7} & 0 \\ 0 & -\sqrt{7} \end{pmatrix} \qquad J = \begin{pmatrix} 0 & -\sqrt{7} \\ -\sqrt{7} & 0 \end{pmatrix} \qquad K = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Therefore

$$u = \begin{pmatrix} a+\sqrt{7}b & -d-\sqrt{7}c \\ d-\sqrt{7}c & a-\sqrt{7}b \end{pmatrix}$$

In this case, det(u) =N (u). That's why it's the same as having elements in SL (2, R) if u has norm 1. Applying the map symbol to them, we see that:

$$\iota : \begin{pmatrix} a+\sqrt{7}b & -d-\sqrt{3}c \\ d-\sqrt{7}c & a-\sqrt{3}b \end{pmatrix} \mapsto \begin{pmatrix} a^2-7b^2+7c^2-d^2 & 2ad+14bc & -2(ac+bd) \\ 14bc-2ad & a^2+7b^2-7c^2-d^2 & 2cd-2ab \\ 14bd-14ac & -14(ab+cd) & a^2+7b^2+7c^2+d^2 \end{pmatrix}$$

We have a discrete group if and only if a, b, c, and d are all elements of Z. That's why we're on the lookout for anything in the integer ring OK = Z[3] of the number field K = Q[3]. The next thing to do is a thorough search for "little". We then employ γ at a given point to create the Dirichlet domain. Keep in mind

JOURNAL OF ALGEBRAIC STATISTICS
Volume 9, No. 1, 2018, p. 81-87
https://publishoa.com
ISSN: 1309-3452
DOI: https://doi.org/10.52783/jas.v9i1.1442

$$tr(\begin{pmatrix} a+\sqrt{7}b & -d-\sqrt{7}c \\ d-\sqrt{7}c & a-\sqrt{7}b \end{pmatrix}) = 2a$$

Hence, we can only have a parabolic component if a = ±1. Because we went with an anisotropic Q, this situation never arises. As it lacks parabolic components, it is possible to build a co-compact Dirichlet domain without encountering any cusps.

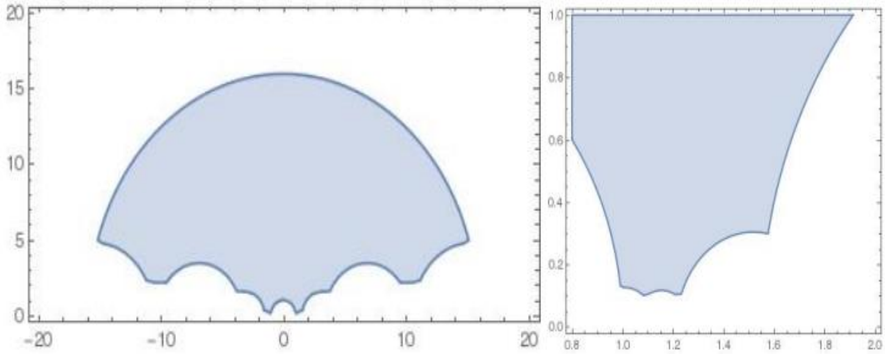

Figure 3: The Dirichlet domain associated with $Q(x) = x^2 + y^2 - 7z^2$

Hyperbolic 3-Space

The identical procedure will now be used to a different quadratic form. Let x = (w, x, y, z) and

$$Q(\mathbf{x}) = w^2 + x^2 + y^2 - 7z^2 = \mathbf{x}\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & & -7 \end{pmatrix}\mathbf{x}^t$$

As a result, Q(x) is still anisotropic over Q despite being a quaternary quadratic form. The lack of solutions modulo 8 demonstrates its anisotropy. Once again, we may acquire the (w, x, y, z) values for Q(x) by storing them in a 22 matrix with the determinant equal to -Q(x). Keep in mind the following will do the trick:

$$m_{\mathbf{x}} = \begin{pmatrix} \sqrt{7}z - y & x + iw \\ x - iw & \sqrt{7}z + y \end{pmatrix}$$

It is possible to uniquely identify the point (w, x, y, z) from the matrix mx, where the determinant is -Q(x). Let be the conjugate transpose of g ∈ SL(2, C), and define $g \circ m_x = gm_xg^*$. After that, we get det(g ∘ mx) = $|ad - bc|^2 \det(m_x)$. Just as before, we can solve for x 0 = (w', x', y', z') by setting $g \circ m_x = m_{x'}$.

$$m_{\mathbf{x}'} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \sqrt{7}z - y & x + iw \\ x - iw & \sqrt{7}z + y \end{pmatrix}\begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \eta \end{pmatrix}$$

Where,

Volume 9, No. 1, 2018, p. 81-87
https://publishoa.com
ISSN: 1309-3452
DOI: https://doi.org/10.52783/jas.v9i1.1442

$$\alpha = -i\bar{a}bw + ia\bar{b}w + \bar{a}bx + a\bar{b}x - |a|^2y + \sqrt{7}|a|^2z + |b|^2y + \sqrt{7}|b|^2z$$
$$\beta = -a\bar{c}y + \sqrt{7}a\bar{c}z + ia\bar{d}w + a\bar{d}x - ib\bar{c}w + b\bar{c}x + b\bar{d}y + \sqrt{7}b\bar{d}z$$
$$\gamma = -\bar{a}cy + \sqrt{7}\bar{a}cz - i\bar{a}dw + \bar{a}dx + i\bar{b}cw + \bar{b}cx + \bar{b}dy + \sqrt{7}\bar{b}dz$$
$$\eta = -i\bar{c}dw + ic\bar{d}w + \bar{c}dx + c\bar{d}x - |c|^2y + \sqrt{7}|c|^2z + |d|^2y + \sqrt{7}|d|^2z$$

Define $\iota(g) = \{m_{i,j}\}_{i,j\leq 4}$ as the succeeding:

$$m_{1,1} = \frac{1}{2}\left(d\bar{a} + a\bar{d} - c\bar{b} - b\bar{c}\right) \qquad m_{1,2} = -\frac{1}{2}i\left(d\bar{a} - a\bar{d} - c\bar{b} + b\bar{c}\right)$$
$$m_{1,3} = \frac{1}{2}i\left(b\bar{a} - a\bar{b} - d\bar{c} + c\bar{d}\right) \qquad m_{1,4} = -\frac{i}{2\sqrt{7}}\left(b\bar{a} - a\bar{b} + d\bar{c} - c\bar{d}\right)$$
$$m_{2,1} = \frac{1}{2}i\left(d\bar{a} - a\bar{d} + c\bar{b} - b\bar{c}\right) \qquad m_{2,2} = \frac{1}{2}\left(d\bar{a} + a\bar{d} + c\bar{b} + b\bar{c}\right)$$
$$m_{2,3} = \frac{1}{2}\left(-b\bar{a} - a\bar{b} + d\bar{c} + c\bar{d}\right) \qquad m_{2,4} = \frac{1}{2\sqrt{7}}\left(b\bar{a} + a\bar{b} + d\bar{c} + c\bar{d}\right)$$
$$m_{3,1} = -\frac{1}{2}i\left(c\bar{a} - a\bar{c} - d\bar{b} + b\bar{d}\right) \qquad m_{3,2} = \frac{1}{2}\left(-c\bar{a} - a\bar{c} + d\bar{b} + b\bar{d}\right)$$
$$m_{3,3} = \frac{1}{2}\left(|a|^2 - |b|^2 - |c|^2 + |d|^2\right) \qquad m_{3,4} = \frac{1}{2\sqrt{7}}\left(-|a|^2 + |b|^2 - |c|^2 + |d|^2\right)$$
$$m_{4,1} = \frac{1}{2}i\sqrt{7}\left(c\bar{a} - a\bar{c} + d\bar{b} - b\bar{d}\right) \qquad m_{4,2} = \frac{1}{2}\sqrt{7}\left(c\bar{a} + a\bar{c} + d\bar{b} + b\bar{d}\right)$$
$$m_{4,3} = \frac{1}{2}\sqrt{7}\left(-|a|^2 - |b|^2 + |c|^2 + |d|^2\right) \qquad m_{4,4} = \frac{1}{2}\left(|a|^2 + |b|^2 + |c|^2 + |d|^2\right)$$

As in the preceding section, we may apply the map ι to these matrices; however, we will not do so here. After that, we resort to a simple brute-force search for $\gamma \in \Gamma$. Once again, there are no cusps, therefore we have built a co-compact Dirichlet domain.

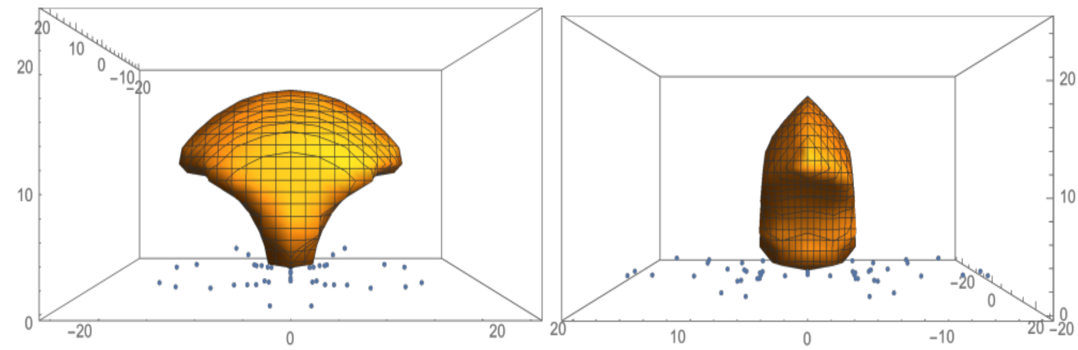

Figure 4: The Dirichlet domain associated with $Q(x) = w^2 + x^2 + y^2 - 7z^2$

Conclusion

Fibered surfaces over algebraic curves have a mathematical counterpart in arithmetic surfaces over Dedekind domains. There is a generic fiber of an arithmetic surface over the ring of integers OK that is isomorphic to a curve over a number field S. It's possible to think about arithmetic systems in greater dimensions. The study of numbers is where arithmetic surfaces first appear. My Dirichlet domain centers on the coordinates (0, 2), therefore I use this quaternion algebra to generate a long series of matrices and apply them using M¨obius transformations. My orbit sampling is this set of locations after transformation. After compiling a complete set of perpendicular bisectors, we next plot them. The list of generators that corresponds to the edge gluings is one possible motivation for seeking for a hyperbolic surface.

JOURNAL OF ALGEBRAIC STATISTICS
Volume 9, No. 1, 2018, p. 81-87
https://publishoa.com
ISSN: 1309-3452
DOI: https://doi.org/10.52783/jas.v9i1.1442

References

1. Ono, Ken & Trebat-Leder, Sarah. (2016). The 1729 K3 surface. Research in Number Theory. 2. 10.1007/s40993-016-0058-2.
2. Bergeron, Nicolas. (2016). Arithmetic Hyperbolic Surfaces. 10.1007/978-3-319-27666-3_2.
3. Dória, Cayo. (2017). Asymptotic Properties of the Set of Systoles of Arithmetic Riemann Surfaces. International Mathematics Research Notices. 2020. 10.1093/imrn/rny246.
4. Oyono, Roger & Thériault, Nicolas. (2013). Group Arithmetic on C_{3,5} curves.
5. Lieblich, Max. (2011). Period and index in the Brauer group of an arithmetic surface. Journal für die Reine und Angewandte Mathematik. 659. 10.1515/CRELLE.2011.059.
6. Eric Lehman. Hyperbolic Geometry,emeritus, Universit´e de Caen, France, 2014.
7. Michelle Chu and Han Li. Small Generators of Cocompact Arithmetic Fuchsian Groups, American Mathematical Society, 2016.
8. Benson Farb and Dan Margalit. A primer on mapping class groups, Princeton University Press, 2012.
9. Eric Lehman. Hyperbolic Geometry,emeritus, Universit´e de Caen, France, 2014.
10. Michelle Chu and Han Li. Small Generators of Cocompact Arithmetic Fuchsian Groups, American Mathematical Society, 2016.