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Analysis of Balking & Probabilistically Modified Reneging in a M/M/1 Waiting Line under Multiple Differentiated Vacations

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Abstract

This paper discusses and investigates transient behavior of a M/M/1 waiting line undergoing multiple differentiated vacations in conjunction with impatient customers- manifested in the form of balking and probabilistically modified reneging. Special function, namely, confluent hypergeometric function, continued fractions, technique of probability generating function and Laplace transform are our mathematical tools to obtain relevant system size probabilities. Time varying mean, variance, throughput and delay time are the performance measures for which we have obtained explicit expressions. Steady-state expressions for system size probabilities and performance measures are also obtained. This paper is a generalized form of a model of Mishra et al. (2022).

Keywords and Phrases: M/M/1 waiting line, transient analysis, multiple differentiated vacations, probabilistically modified reneging, balking, continued fractions, confluent hypergeometric function, Laplace transform.

1. Introduction

Ever since the study of waiting lines aka queues begun, queue researchers have been trying to simulate real-life queues as close as possible. Result of these attempts is the introduction of new concepts in queueing theory e.g., vacation queues, feedback queues, queues with impatient customers etc. In queueing theory, a vacation queue has got a prominent place due to its practical applications. Vacation queues are characterized by the queueing systems wherein a server is absent from its primary task assigned to it—deliberatelyor non-deliberately. Levy & Yechially (1975) introduced the idea of vacation in queueing to exploit the sit-back phase (idle phase) of the server. They analyzed M/G/1 queue e.g., in recent communication and mobile technologies IEEE802.16e sleep condition is modelled as server vacation [(Baker et al. (2010), Sampath and Liu (2018)]. Doshi (1986, 1990) presented a methodological overview on queues with vacations and gave examples of real-life occasions which can be modelled as a queueing system with vacations. In this connection works of Takagi (1991) as well as of Tian & Zhang (2006) are also equally important. Vacation queues can be studied on two broad ways. One way is to study them on the basis of absence or presence of customers at the time of server undergoing a vacation. The other way is to study them on the basis of various mathematical techniques employed to analyze them. Details of these vacationing queues and an overview of them have been nicely discussed by Upadhyaya (2016). Shekhar et al. (2017) presented a survey paper for the period 2010 to 2017 in which they have described vacation (single, multiple and working) queueing model involving machining system. Panta et al. (2021) analyzed a vast and extensive look over vacation queues covering the period from 1975 to 2021. Very recently Wang et al. (2022) wrote a paper on vacationing queue where a customer is levied a fee, called "pay-to-activate-service" if he wanted to avail the services during a vacation period.

Queueing models in which severs undergone vacations by halting the services completely were upgraded to the queueing models in which severs work under "different" rate during their vacationing period by Servi & Finn (2002) and they gave the name "working" vacation for such vacation. They developed the model M/M/1/WV. Since then, researchers are engaged with varieties of queues having working vacations—single or multiple working vacations and with varieties of analytic methods [e.g., a survey on working vacation queue with matrix analytic method by Tian et al. (2009) and the references therein, a survey paper by Chandrasekaran et al. (2016) and references therein]. Jain et al. (2022) analyzed a Markovian retrial queue with WV (working vacation), imperfect service and balking. They have derived various performance measures, done sensitivity analysis, created a cost function and used quasi-Newton method (QNM) & genetic algorithm (GA) to minimize the cost function. Janani, &Vijayashree (2023) has analyzed a working vacations queue having applications (illustrated by a flow chart) in smart medical management system, the server being unreliable and capable of soft failures.

Ibe and Isijola (2014) introduced a new model of a M/M/1 vacation queue that differentiated between two sorts of vacations on the grounds of their span on time scale and gave steady-state stochastics. From this time, we call such vacations as differentiated vacations. Queue systems where server can take breaks of different time-length are suitable for modelling as differentiated vacations queues [see, e.g., Ibe & Isijola (2014), Phung-Duc (2015), Sampath and Liu (2018)]. In the same year 2014, Isijola-Adakeja & Ibe (2014) extended their model by embedding the concept of vacation interruptions in their previous model and obtained steady state stochastics. Vijayashree & Janani (2018) gave a time-dependent analysis of model of Ibe and Isijola (2014), found transient probabilities, important performance measures and showed that the limiting case probabilities tally with stationary state probabilities ofIbe and Isijola (2014). Sampath & Liu (2018) improved the model of Vijayashree & Janani (2018) by incorporating waiting server and impatience of customers to the model just mentioned. Sampath et al. (2020) improved extra one step further the model of Vijayashree & Janani (2018) by adding a fresh concept of multiple differentiated vacation to the model of Sampath & Liu (2018). Gupta and Kumar (2021) analyzed a retrial queue where server can take differentiated vacations and arrival rate depends on the states of the server, the analysis being steady state withprobability generating function technique. Gupta & Malik (2021) studied a feedback queue under the differentiated vacation scheme and doubtful waiting server. Differentiated vacations queue with server vacation interruptions is the investigation topic of Azhagappan & Deepa (2022). They have given transient analysis in this paper.

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The process of reneging and balking are the two prominent ways in which customers' impatience are evinced. As soon as the customer joins the queue and finds that the server is either busy in providing the services to the customers already present or is on the vacation, he becomes patient and turns on his individual impatient timer. After the expiration of this individual impatient time, he leaves the queue without availing the required services. This is called the reneging of the customer. In balking, customer leaves the system without joining the queue. Palm (1953), Udagava & Nakamura (1957), Haight (1957, 1959), Barrer (1957, 1957), Ancer & Gafarian (1963, 1963) are the earlier workers on the impatient behavior. Wang et al. (2010) produced a survey about the waiting lines with customer impatience that contains vast number of references to the relevant topic in various directions. During last decade scores of researchers studied and analyzed queues with discouragement [see, e.g., Padmavati et al. (2011), Goswami (2014, 2014), Vijayalaxmi & Jyothsna (2015, 2016), Ammar (2017), Sampath & Liu (2018), Manoharan and Jeeva (2019), Sampath et al (2020)]. Vijayashree & Ambika (2020) investigated a M/M/1 impatient vacation queue where entering customers wait for a fixed time-period and then abandon the queue permanently. Binay kumar (2022) has analyzedM/M/C/K (finite capacity) & M/M/C/L (finite population) waiting lines with reneging. For transient analysis, Runge-Kutta method (on MATLAB software) has been used for solving state differential equations. Recently, Mishra et al (2022) have introduced a new type of reneging, called probabilistically modified reneging, in a differentiated vacation M/M/1 queue. Zaid et al. (2023) dealt with a many-server Markovian queue with vacation interruption, waiting server, (k-variant) working vacation, reneging and balking, and presented a matrix geometric solution.

Here, in this paper, we intend to introduce a joint effect of probabilistically modified reneging and multiple differentiated vacation in a Markovian queue. We discuss and analyze a M/M/1 queue with probabilistically modified reneging, balking and multiple differentiated vacations under transient state. Our mathematical tools are—Laplace transform, continued fractions, probability generating functions, and a special function (confluent hypergeometric function). This queueing model is a generalized form of queueing model of Mishra et al. (2022)

This paper is organized in sections. Model is described in section 2. Section 3 contains transient solutions. Effectiveness measures are given in section 4. Steady state analysis is the subject matter of section 5. Finally, section 6 deals with conclusion & future scope of the model.

2. Model Description

Here, we are considering a solo service-dispenser queue system under discouragement (in the form of balking and reneging) and differentiated vacations. Suppositions regarding this queue system are as under:

1) Discipline of service is "First come, First served".
2) Customers approach the service in accordance with Poisson process so that probability distribution function for inter-arrival time is $\lambda e^{-\lambda t}$, and hence they arrive with mean rate λ. Number of upcoming customers is infinite.
3) Probability distribution function for inter-service time during normal busy spell is $\mu e^{-\mu t}$ so that mean service rate is μ.

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4) Impatient behavior happens only during vacations. Balking probabilities are ρ_1 and ρ_2 for vacations 1 and 2 respectively.
5) Probability distribution functions for inter-vacation times are $\nu_1 e^{-\nu_1 t}$ and $\nu_2 e^{-\nu_2 t}$ for vacations 1 and 2 respectively, and therefore state change rates from vacations 1 and 2 to normal busy period are ν_1 and ν_2 respectively.
6) Each service-seeker that lands at the service facility, while a vacation is going on, activates its own timer that are exponentially distributed having parameters ϵ_1 and ϵ_2 for vacations 1 and 2 respectively. After the expiration of time set by the timer, customer reneges—whether it has entered into the service or in waiting. In probabilistically modified reneging model, the customer faces a dilemma after the expiration of the time set by the timer—quit or not quit. This dilemma is accounted by Bernoulli type probabilities p and q during the vacations 1 and 2 respectively. Reneged and balked customers leave the system permanently.
7) When server finds an empty system after a busy period it goes to type 1 vacation (longer) and resumes its service when system becomes non-empty. If the server finds an empty system after vacation 1, it goes to vacation 2 (shorter). If the system is still empty then the server is permitted to take the vacation 1 (longer). This vacation process, called multiple differentiated vacation (Sampath et al. (2020)), goes on continuously until system becomes non-empty.
8) All the random variables involved are independent and identically distributed.

Let ₦ (t) represents total number of customers (waiting plus service receiving) in the system at time 't' and J(t)= 0, 1, 2 are representing busy period, vacation 1, vacation 2 respectively at any time 't'. Then, (₦ (t), J(t)) defines 2-dimensional CTMC (continuous time Markov chain) on the state space $S = \{(0,1) \cup (0,2) \cup (k,j): k = 1,2,3,\ldots; j = 0,1,2\}$.Figures 2.1 & 2.2 describes state transitions.

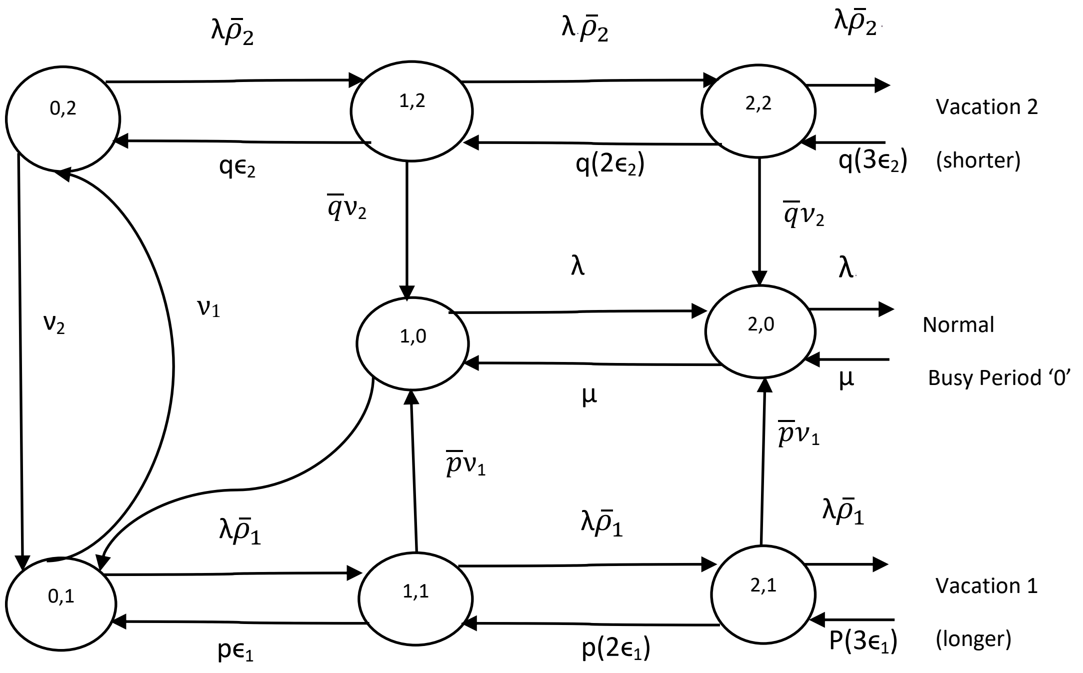

***Figure* 2.1**: Initial state-transition (↑)

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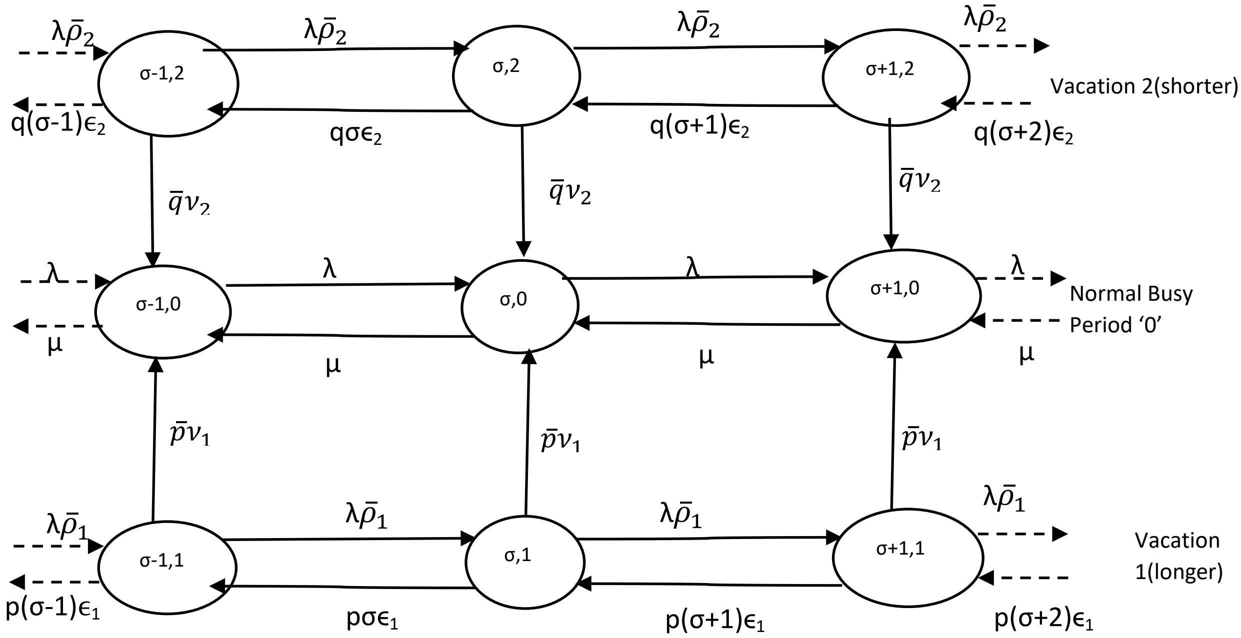

***Figure* 2.2:** State-transition at σ th stage (↑)

Mathematical Notations

Following notations are used in the paper—

λ= arrival rate; μ=service rate; ν_1=state change rate from vacation 1 to busy period;

ν_2=state change rate from vacation 2 to busy period; ϵ_1=reneging rate in vacation 1;

ϵ_2=reneging rate in vacation 2; p=reneging probability in vacation 1, $\bar{p}$=1-p;

q=reneging probability in vacation 2, $\bar{q}$=1-q; ρ_1=balking probability in vacation 1,

$\bar{\rho}_1$=1-ρ_1; ρ_2=balkingprobability in vacation 2, $\bar{\rho}_2$=1-ρ_2.

3.Time-dependent Analysis

System of Chapman—Kolmogorovdifferential -difference equations are asunder:

Vacation1.

$P'_{0,1}(\mathrm{t}) = -(\lambda\bar{\rho}_1 + \nu_1)P_{0,1}(t) + \mu P_{10}(t) + p\epsilon_1 P_{1,1}(t) + \nu_2 P_{0,2}(t)(1) P'_{\sigma,1}(t) = -(\lambda\bar{\rho}_1 + \bar{p}\nu_1 + \sigma p\epsilon_1)P_{\sigma,1}(t) + \lambda\bar{\rho}_1 P_{\sigma-1,1}(t) + p(\sigma+1)\epsilon_1 P_{\sigma+1,1}(t),$

$\sigma = 1,2,3, \ldots$.(2)

Vacation2.

$P'_{0,2}(\mathrm{t}) = -\lambda\bar{\rho}_2 P_{0,2}(t) + \nu_1 P_{01}(t) + q\epsilon_2 P_{1,2}(t),(3) P'_{\sigma,2} = -(\lambda\bar{\rho}_2 + \bar{q}\nu_2 + q\sigma\epsilon_2)P_{\sigma,2}(t) + \lambda\bar{\rho}_2 P_{\sigma-1,2}(t) + q(\sigma+1)\epsilon_2 P_{\sigma+1,2}(t), \sigma = 1,2,3, \ldots$

(4)

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Busy Period 0.

$$P'_{1,0}(t) = -(\lambda+\mu)P_{1,0}(t) + \mu P_{2,0}(t) + \bar{p}\nu_1 P_{1,1}(t) + \bar{q}\nu_2 P_{1,2}(t), \quad (5)$$

$$P'_{\sigma,0}(t) = -(\lambda+\mu)P_{\sigma,0}(t) + \lambda P_{\sigma-1,0} + \mu P_{\sigma+1,0}(t) + \bar{p}\nu_1 P_{\sigma,1}(t) + \bar{q}\nu_2 P_{\sigma,2}(t)\,,$$

$$\sigma = 1,2,3, \ldots(6)$$

Initially at $t = 0$, the system is empty and server is in vacation 1. Therefore, $P_{01}(0) = 1, P_{\sigma k}(0) = 0, \sigma = 1, 2, 3, \ldots \quad , \ k = 0, 1, 2.$

Laplace transform of equation (2) and some manipulation leads to

$$\frac{\bar{P}_{\sigma,1}(s)}{\bar{P}_{\sigma-1,1}(s)} = \frac{\left(\frac{\lambda\bar{\rho}_1}{p\epsilon_1}\right)}{\left(\frac{s+\lambda\bar{\rho}_1+\bar{p}\nu_1}{p\epsilon_1}+\sigma\right)-(\sigma+1)\frac{\bar{P}_{\sigma+1,1}(s)}{\bar{P}_{\sigma,1}(s)}} \ , \sigma = 1, 2, 3, \ldots \quad (A)$$

Iterating (A) and making use of confluent hypergeometric function and recursive reduction, we get

$\bar{P}_{\sigma 1}(s) = \bar{\Upsilon}_{\sigma}(s)\bar{P}_{01}(s)$(7)

$$where\ \ \bar{\Upsilon}_{\sigma}(s) = \left(\frac{\lambda\bar{\rho}_1}{p\epsilon_1}\right)^{\sigma} \frac{1}{\prod_{k=1}^{\sigma}\left(\frac{s+\bar{p}\nu_1}{p\epsilon_1}+k\right)} \frac{{}_1F_1\left(\sigma+1; \frac{s+\bar{p}\nu_1}{p\epsilon_1}+\sigma+1; -\left(\frac{\lambda\bar{\rho}_1}{p\epsilon_1}\right)\right)}{{}_1F_1\left(1; \frac{s+\bar{p}\nu_1}{p\epsilon_1}+1; -\left(\frac{\lambda\bar{\rho}_1}{p\epsilon_1}\right)\right)}$$

$\sigma = 1,2,3, \ldots$(8)

ϒis read as "Greek upsilon with hook symbol".

Similarly, from equation (4)

$\bar{P}_{\sigma 2}(s) = \bar{Ϙ}_{\sigma}(s)\bar{P}_{02}(s)$(9)

$$where\ \ \bar{Ϙ}_{\sigma}(s) = \left(\frac{\lambda\bar{\rho}_2}{q\epsilon_2}\right)^{\sigma} \frac{1}{\prod_{k=1}^{\sigma}\left(\frac{s+\bar{q}\nu_2}{q\epsilon_2}+k\right)} \frac{{}_1F_1\left(\sigma+1; \frac{s+\bar{q}\nu_2}{q\epsilon_2}+\sigma+1; -\left(\frac{\lambda\bar{\rho}_2}{q\epsilon_2}\right)\right)}{{}_1F_1\left(1; \frac{s+\bar{q}\nu_2}{q\epsilon_2}+1; -\left(\frac{\lambda\bar{\rho}_2}{q\epsilon_2}\right)\right)}$$

$\sigma = 1,2,3, \ldots$(10)

Ϙis read as "Greek letter archaic Koppa".

Evaluation of $P_{\sigma 0}(t), \sigma = 1, 2, 3, \ldots$

Define

$$P(\xi,t) = \sum_{\sigma=1}^{\infty} P_{\sigma,0}(t)\xi^{\sigma}, \qquad |\xi| < 1\ \ with\ \ P(\xi,0) = 0.$$

$$\therefore\ \ \frac{\partial P(\xi,t)}{\partial t} = P'_{1,0}(t)\xi + \sum_{\sigma=2}^{\infty} P'_{\sigma,0}(t)\xi^{\sigma}$$

Using equations (5), (6), and then simplifying, we have

$$\frac{\partial P(\xi,t)}{\partial t}+[\lambda(1-\xi)+\mu(1-\xi^{-1})]P(\xi,t)$$
$$=\bar{p}\nu_1\sum_{\sigma=1}^{\infty}P_{\sigma,1}(t)\xi^{\sigma}+\bar{q}\nu_2\sum_{\sigma=1}^{\infty}P_{\sigma,2}(t)\xi^{\sigma}-\mu P_{1,0}(t) \quad (11)$$

Equation (1.11) is a first-order linear differential equation whose integrating factor (I.F) is

$$e^{\int[\lambda(1-\xi)+\mu(1-\xi^{-1})]dt}=e^{[\lambda(1-\xi)+\mu(1-\xi^{-1})]t}=exp[\lambda(1-\xi)+\mu(1-\xi^{-1})]t$$

Multiplying the equation (11) by above I.F and integrating between the limits 0 to t, we obtain (remembering $\mathbb{P}(\xi,0)=0$)

$$\mathbb{P}(\xi,t)=\bar{p}\nu_1\int_0^t\left(\sum_{\sigma=1}^{\infty}P_{\sigma,1}(v)\xi^{\sigma}\right)e^{-[\lambda(1-\xi)+\mu(1-\xi^{-1})](t-v)}dv$$
$$+\bar{q}\nu_2\int_0^t\left(\sum_{\sigma=1}^{\infty}P_{\sigma,2}(v)\xi^{\sigma}\right)e^{-[\lambda(1-\xi)+\mu(1-\xi^{-1})](t-v)}dv$$
$$-\mu\int_0^t P_{10}(v)e^{-[\lambda(1-\xi)+\mu(1-\xi^{-1})](t-v)}dv \quad (12)$$

Using following known expansion involving modified Bessel's function of first kind $I_n(.)$

$$e^{\left(\frac{\mu}{\xi}+\lambda\xi\right)t}=\sum_{m=-\infty}^{\infty}(\beta\xi)^m I_{\sigma}(\gamma t),$$

where $\gamma=2\sqrt{\lambda\mu}$ and $\beta=\sqrt{\frac{\lambda}{\mu}}$, equation (12) becomes

$$\mathbb{P}(\xi,t)=\bar{p}\nu_1\int_0^t\left(\sum_{\sigma=1}^{\infty}P_{\sigma,1}(v)\xi^{\rho}\right)e^{-(\lambda+\mu)(t-v)}\sum_{m=-\infty}^{\infty}(\beta\xi)^m I_m(\gamma(t-v)dv$$
$$+\bar{q}\nu_2\int_0^t\left(\sum_{\sigma=1}^{\infty}P_{\rho,2}(v)\xi^{\rho}\right)e^{-(\lambda+\mu)(t-v)}\sum_{m=-\infty}^{\infty}(\beta\xi)^m I_m(\gamma(t-v))dv$$
$$-\mu\int_0^t P_{10}(v)e^{-(\lambda+\mu)(t-v)}\sum_{m=-\infty}^{\infty}(\beta\xi)^m I_m(\gamma(t-v)dv \quad (13)$$

Comparing the coefficients of ξ^{σ} and of$\xi^{-\sigma}$ on both the sides of equation (13) and then solving the two resulting equations (using $I_{-m}(\cdot)=I_m(\cdot)$), we obtain

$$P_{\sigma,0}(t)=\bar{p}\nu_1\int_0^t\left[\sum_{k=1}^{\infty}P_{k,1}(v)\beta^{\sigma-k}\{I_{\sigma-k}(\gamma(t-v))-I_{\sigma+k}(\gamma(t-v))\}\right]e^{-(\lambda+\mu)(t-v)}dv$$

$$+\bar{q}\nu_2\int_0^t\left[\sum_{k=1}^{\infty}P_{k,2}\,(v)\beta^{\sigma-k}\{I_{\sigma-k}(\gamma(t-v))-I_{\sigma+k}(\gamma(t-v))\}\right]e^{-(\lambda+\mu)(t-v)}dv,$$

$\sigma=1,2,3,\ldots$(14)

Laplace transformation of (14) gives us

$$\bar{P}_{\sigma,0}(s)=\bar{p}\nu_1\sum_{l=1}^{\infty}\bar{P}_{l,1}\,(s)\beta^{\sigma-l}\left\{\frac{\left(\theta-\sqrt{\theta^2-\gamma^2}\right)^{\sigma-l}}{\gamma^{\sigma-l}\sqrt{\theta^2-\gamma^2}}-\frac{\left(\theta-\sqrt{\theta^2-\gamma^2}\right)^{\sigma+l}}{\gamma^{\sigma+k}\sqrt{\theta^2-\gamma^2}}\right\}$$

$$+\bar{q}\nu_2\sum_{l=1}^{\infty}\bar{P}_{l,2}\,(s)\beta^{\sigma-l}\left\{\frac{\left(\theta-\sqrt{\theta^2-\gamma^2}\right)^{\sigma-l}}{\gamma^{\sigma-l}\sqrt{\theta^2-\gamma^2}}-\frac{\left(\theta-\sqrt{\theta^2-\gamma^2}\right)^{\sigma+l}}{\gamma^{\sigma+l}\sqrt{\theta^2-\gamma^2}}\right\},$$

$\sigma=1,2,3,\ldots\ ;\ \theta=s+\lambda+\mu\,.$ (15)

Evaluation of $P_{0,1}(t)$ and $P_{0,2}(t)$

Taking Laplace transform of (1), we get

$$s\bar{P}_{0,1}(s)-1=-(\nu_1+\lambda\bar{\rho}_1)\bar{P}_{0,1}(s)+\mu\bar{P}_{1,0}(s)+p\epsilon_1\bar{P}_{11}(s)+\nu_2\bar{P}_{02}(s),$$

as $P_{0,1}(0)=1$. Using (7) and simplifying, we get
$(s+\nu_1+\lambda\bar{\rho}_1-p\epsilon_1\bar{\Upsilon}_1(s))\bar{P}_{0,1}(s)=1+\mu\bar{P}_{1,0}(s)+\nu_2\bar{P}_{02}(s)$,(16)

Similarly, from equation (3) and (9) we get

$(s+\nu_2+\lambda\bar{\rho}_2-q\epsilon_2\bar{\Omega}_\sigma(s))\bar{P}_{0,2}(s)=\nu_1\bar{P}_{0,1}(s)$ (17)

Using (17) in (16), we have

$$\left[\left(s+\nu_1+\lambda\bar{\rho}_1-p\epsilon_1\bar{\Upsilon}_1(s)\right)-\frac{\nu_1\nu_2}{s+\nu_2+\lambda\bar{\rho}_2-q\epsilon_2\bar{\Omega}_1(s)}\right]\bar{P}_{0,1}(s)=1+\mu\bar{P}_{1,0}(s)$$

Above equation, after some algebra and manipulation leads to

$\bar{P}_{0,1}(s)=\left(1+\mu\bar{P}_{1,0}(s)\right)Q(s)$,(18)

where

$$Q(s)=\sum_{r=0}^{\infty}\sum_{n=0}^{r}\sum_{m=0}^{\infty}\binom{n+m}{m}\frac{(\nu_1\nu_2)^n\left(q\epsilon_2\bar{\Omega}_1(s)\right)^m\left(p\epsilon_1\bar{\Upsilon}_1(s)\right)^{r-n}}{(s+\nu_1+\lambda\bar{\rho}_1)^{r+1}(s+\nu_2+\lambda\bar{\rho}_2)^{n+m}}.$$

(19)

Inverting (18) & (19),

$P_{0,1}(t)=\left(1+\mu P_{1.0}(t)\right)*Q(t)\,,$ (20)

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where $Q(t)=\sum_{r=0}^{\infty}\sum_{n=0}^{r}\sum_{m=0}^{\infty}\binom{n+m}{m}(\nu_1\nu_2)^n(q\epsilon_2)^m(p\epsilon_1)^{r-n}\Upsilon_1^{*(r-n)}(t)*\left(\text{♀}_1(t)\right)^{*m}*\frac{t^r}{r!}e^{-(\nu_1+\lambda\bar{\rho}_1)t}*\frac{t^{n+m-1}}{(n+m-1)!}e^{-(\nu_2+\lambda\bar{\rho}_2)t},$ (21)

in which * denotes convolution and *(r-n) & *m are (r-n)-fold & m-fold convolutions respectively. Again from equations (17) and (18), we get

$$\bar{P}_{0,2}(s)=\nu_1(1+\mu\bar{P}_{1,0}(s))Q(s)\sum_{m=0}^{\infty}\frac{\left(q\epsilon_2\bar{\text{♀}}_1(s)\right)^m}{(s+\nu_2+\lambda\bar{\rho}_2)^{m+1}}, \tag{22}$$

Inverting equation (22),

$P_{0,2}(t)=\nu_1\left(1+\mu P_{1.0}(t)\right)*Q(t)*\sum_{m=0}^{\infty}(q\epsilon_2)^m\left(\text{♀}_1(t)\right)^{*m}*\frac{t^m}{m!}e^{-(\nu_2+\lambda\bar{\rho}_2)t}$ (23)Inverting equations (7) & (8) and then using equations (20) and (23) into these inverted equations, we get

$$P_{\sigma,1}(t)=\Upsilon_\sigma(t)*P_{0,1}(t)=\Upsilon_\sigma(t)*\left(1+\mu P_{1.0}(t)\right)*Q(t) \tag{24}$$

$$P_{\sigma,2}(t)=\text{♀}_\sigma(t)*P_{0,2}(t)$$

$$=\text{♀}_\sigma(t)*\nu_1\left(1+\mu P_{1.0}(t)\right)*Q(t)*\sum_{m=0}^{\infty}(q\epsilon_2)^m\left(\text{♀}_1(t)\right)^{*m}*\frac{t^m}{m!}e^{-(\nu_2+\lambda\bar{\rho}_2)t} \tag{25}$$

Equations (24) and (25) give explicit expressions for $P_{\sigma,1}(t)$ and $P_{\sigma,2}(t), \sigma=1,2,3,\ldots$ in terms of $P_{1.0}(t), Q(t), \Upsilon_\sigma(t) \& \text{♀}_\sigma(t)$, $\sigma=1,2,3,\ldots$, but equation (21) gives explicitly $Q(t)$ in terms of $\Upsilon_1(t) \& \text{♀}_1(t)$. Also, $P_{\sigma,0}(t), \sigma=1,2,3,\ldots$from equation (14) is explicitly given in terms of $P_{\sigma,1}(t)$ and $P_{\sigma,2}(t), \sigma=1,2,3,\ldots$. Hence, overall, it remains to evaluate $P_{1,0}(t), \Upsilon_\sigma(t) \& \text{♀}_\sigma(t)$, $\sigma=1,2,3,\ldots$.

Evaluation of $\Upsilon_\sigma(t) \& \text{♀}_\sigma(t)$, $\sigma=1,2,3,\ldots$

From the definition of confluent hypergeometric function

$$\frac{{}_1F_1\left(\sigma+1;\frac{s+\bar{p}\nu_1}{p\epsilon_1}+\sigma+1;-\left(\frac{\lambda\bar{\rho}_1}{p\epsilon_1}\right)\right)}{\prod_{j=1}^{\sigma}\left(\frac{s+\bar{p}\nu_1}{p\epsilon_1}+j\right)}$$

$$=(p\epsilon_1)^\sigma\sum_{k=0}^{\infty}\frac{\binom{\sigma+k}{k}(-\bar{\rho}_1\lambda)^k}{\prod_{j=1}^{\sigma+k}(s+\bar{p}\nu_1+p\epsilon_1 j)}$$

$$=(p\epsilon_1)^\sigma\sum_{k=0}^{\infty}\binom{\sigma+k}{k}\left(\frac{-\bar{\rho}_1\lambda}{p\epsilon_1}\right)^\sigma\sum_{r=1}^{\sigma+k}\frac{(-1)^{r-1}}{(r-1)!\,(\sigma+k-r)!\,(s+\bar{p}\nu_1+p\epsilon_1 r)}$$

(using partial fractions).

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Also, we have,

$$ {}_1F_1\left(1;\frac{s+\bar{p}\nu_1}{p\epsilon_1}+1;-\left(\frac{\bar{\rho}_1\lambda}{p\epsilon_1}\right)\right)=\sum_{m=0}^{\infty}\frac{(-\bar{\rho}_1\lambda)^m}{\prod_{l=1}^{m}(s+\bar{p}\nu_1+p\epsilon_1 l)}=\sum_{m=0}^{\infty}(-\bar{\rho}_1\lambda)^m\tilde{\alpha}_m(s)\,, $$

where $\tilde{\alpha}_0(s)=1$ and

$$ \tilde{\alpha}_m(s)=\frac{1}{\prod_{l=1}^{m}(s+\bar{p}\nu_1+p\epsilon_1 l)} $$

$$ =\frac{1}{(p\epsilon_1)^{m-1}}\sum_{l=1}^{m}\frac{(-1)^{l-1}}{(l-1)!\,(m-l)!\,(s+\bar{p}\nu_1+p\epsilon_1 l)}\quad ,m=1,2,3,\dots $$

and

$$ \left\{{}_1F_1\left(1;\frac{s+\bar{p}\nu_1}{p\epsilon_1}+1;-\left(\frac{\bar{\rho}_1\lambda}{p\epsilon_1}\right)\right)\right\}^{-1}=\sum_{m=0}^{\infty}(\bar{\rho}_1\lambda)^m\tilde{\delta}_m(s) $$

where $\tilde{\delta}_0(s)=1$ and

$$ \tilde{\delta}_m(s)=\begin{vmatrix} \tilde{\alpha}_1(s) & 1 & 0 & 0 & \cdots & \cdots & \cdots & 0\\ \tilde{\alpha}_2(s) & \tilde{\alpha}_1(s) & 1 & 0 & \cdots & \cdots & \cdots & 0\\ \tilde{\alpha}_3(s) & \tilde{\alpha}_2(s) & \tilde{\alpha}_1(s) & 1 & \cdots & \cdots & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & 0\\ \vdots & \vdots & \vdots & \vdots & \cdots & \cdots & \cdots & 0\\ \tilde{\alpha}_{m-1}(s) & \tilde{\alpha}_{m-2}(s) & \tilde{\alpha}_{m-3}(s) & \cdots & \cdots & \cdots & \tilde{\alpha}_1(s) & 1\\ \tilde{\alpha}_m(s) & \tilde{\alpha}_{m-1}(s) & \tilde{\alpha}_{m-2}(s) & \cdots & \cdots & \cdots & \tilde{\alpha}_2(s) & \tilde{\alpha}_1(s) \end{vmatrix} $$

$$ =\sum_{k=1}^{m}(-1)^{k+1}\tilde{\alpha}_k(s)\tilde{\delta}_{m-i}(s),\quad m=1,2,3,\dots. $$

Therefore, using all the above stuff into equation (8), we have

$$ \bar{\Upsilon}_\sigma(s)=\sum_{\boldsymbol{k=0}}^{\infty}(-1)^k\binom{\sigma+k}{k}(\lambda\bar{\rho}_1)^{\sigma+k}\tilde{\alpha}_{\sigma+k}(s)\sum_{l=0}^{\infty}(-\lambda\bar{\rho}_1)^l\tilde{\delta}_l(s),\sigma=1,2,3,\dots.(26), $$

Similarly,

$$ \bar{\text{Ϙ}}_{\boldsymbol{\sigma}}(s)=\sum_{\boldsymbol{l=0}}^{\infty}(-1)^l\binom{\sigma+l}{l}(\lambda\bar{\rho}_2)^{\sigma+l}\tilde{A}_{\sigma+l}(s)\sum_{k=0}^{\infty}(-\lambda\bar{\rho}_2)^k\tilde{\Delta}_k(s),\ \sigma=1,2,3,\dots.(27), $$

where $\tilde{A}_\sigma(s)\&\tilde{\Delta}_\sigma(s)$ have similar relations between them as that of between $\tilde{\alpha}_\sigma(s)\&\tilde{\delta}_\sigma(s)$, $\sigma=0,1,2,3,\dots$.

Inversion of equations (26) & (27) leads to

$$ \Upsilon_\sigma(t)=\sum_{\boldsymbol{l=0}}^{\infty}(-1)^l\binom{\sigma+l}{l}(\lambda\bar{\rho}_1)^{\sigma+l}\alpha_{\sigma+l}(t)*\sum_{k=0}^{\infty}(-\lambda\bar{\rho}_1)^k\delta_k(t),\ \sigma=1,2,3,\dots.(28) $$

$\text{Ϙ}_\sigma(t) = \sum_{\boldsymbol{l=0}}^{\infty}(-1)^l\binom{\sigma+l}{l}(\lambda\bar{\rho}_2)^{\sigma+l}A_{\sigma+l}(t) * \sum_{k=0}^{\infty}(-\lambda\bar{\rho}_2)^k\Delta_k(t),\ \sigma = 1,2,3,\ldots$ (29) respectively, where $\alpha_m(t) = \frac{1}{(p\epsilon_1)^{m-1}}\sum_{r=1}^{m}\frac{(-1)^{r-1}}{(r-1)!(m-r)!}e^{-(\bar{p}\nu_1+p\epsilon_1 r)t}, m = 1,2,3,\ldots,$

$$\delta_m(t) = \sum_{i=1}^{m}(-1)^{i+1}\alpha_i(t) * \delta_{m-i}(t), \qquad m = 2,3,4,\ldots$$

$$\delta_1(t) = \alpha_1(t), \qquad \tilde{\delta}_1(s) = \tilde{\alpha}_1(s)$$

and relations between $A_l(t)\&\Delta_l(t)$ are similar to the relations between $\alpha_l(t)\&\delta_l(t), l = 0,1,2,3,\ldots$.

Evaluation of $\boldsymbol{P_{1,0}(t)}$

From equation (17), for σ = 1, and using

$$I_{1-k}(\gamma(t-v)) - I_{1+k}(\gamma(t-v)) = \frac{2kI_k\big(\gamma(t-v)\big)}{\gamma(t-v)},$$

we obtain

$$P_{1,0}(t) = \bar{p}\nu_1\int_0^t\left[\sum_{k=1}^{\infty}P_{k,1}(v)\beta^{1-k}\frac{2kI_k(\gamma(t-v))}{\gamma(t-v)}\right]e^{-(\lambda+\mu)(t-v)}dv$$

$$+\bar{q}\nu_2\int_0^t\left[\sum_{k=1}^{\infty}P_{k,2}(v)\beta^{1-k}\frac{2kI_k(\gamma(t-v))}{\gamma(t-v)}\right]e^{-(\lambda+\mu)(t-v)}dv, \qquad (30)$$

Taking Laplace transform of (30),

$$\bar{P}_{1,0}(s) = \bar{p}\nu_1\sum_{l=1}^{\infty}\bar{P}_{l,1}(s)\beta^{1-l}\frac{1}{\gamma^{1-l}\left(\theta+\sqrt{\theta^2-\gamma^2}\right)^l} + \bar{q}\nu_2\sum_{l=1}^{\infty}\bar{P}_{l,2}(s)\beta^{1-l}\frac{1}{\gamma^{1-l}\left(\theta+\sqrt{\theta^2-\gamma^2}\right)^l},$$

$where\ \theta = s+\lambda+\mu.$ (31)

Taking advantage of equations (7) & (9) and after some algebra, equation (1.31) gives

$$\bar{P}_{1,0}(s) = \sum_{m=0}^{\infty}\mu^m[\bar{H}(s)]^{m+1}, \qquad (32)$$

where

$$\bar{H}(s) = 2\nu_1 Q(s)\sum_{k=1}^{\infty}\left[\bar{p}\bar{\Upsilon}_k(s) + \bar{\boldsymbol{q}}\bar{\text{Ϙ}}_{\boldsymbol{k}}(s)\sum_{m=0}^{\infty}\frac{\left(q\epsilon_2\bar{\text{Ϙ}}_1(s)\right)^m}{(s+\nu_2+\lambda\bar{\rho}_2)^{m+1}}\right]\frac{1}{\gamma^{1-k}\left(\theta+\sqrt{\theta^2-\gamma^2}\right)^k}$$

(33)

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(Q(s) is given by equation (19))

Taking Laplace Transform of the equation (32), we have

$$P_{1,0}(t) = \sum_{m=0}^{\infty} \mu^m [H(t)]^{*(m+1)} , \tag{34}$$

$* (m+1)\ being\ (m+1) - fold\ convolution$. And,

$$H(t) = 2\nu_1 Q(t) * \sum_{k=1}^{\infty} \left[\bar{p}\Upsilon_k(t) + \bar{q}\text{♀}_k(t) * \sum_{m=0}^{\infty} (q\epsilon_2)^m \big(\text{♀}_1(t)\big)^{*m} * \frac{t^m}{m!} e^{-(\nu_2+\lambda\bar{\rho}_2)t} \right] * \frac{2kI_k(\gamma t)}{\gamma t} e^{-(\lambda+\mu)t} , \tag{35}$$

and Q(t) is given by equation (21). Thus, we have calculated all the transient-state probabilities explicitly.

4.Effectiveness Measures

Time-dependent Mean $\Sigma(t)$

If $X(t)$=total aggregate of customers at any instant t in the system, then mean total aggregate of customers $\Sigma(t)$ is defined as

$$\Sigma(t) = E\big(X(t)\big) \overset{\text{def}}{=} \sum_{\sigma=1}^{\infty} \sigma\big[P_{\sigma,0}(t) + P_{\sigma,1}(t) + P_{\sigma,2}(t)\big] \tag{36}$$

Differentiating above equation with respect to t, we have

$$\Sigma'(t) = \sum_{\sigma=1}^{\infty} \sigma\big[P'_{\sigma,0}(t) + P'_{\sigma,1}(t) + P'_{\sigma,2}(t)\big] \tag{37}$$

Using equations (2), (4), (5) & (6) and doing some algebra, manipulations, groupings and adjustments in the equation (37), and then integrating, we obtain (using initial conditions)

$$\Sigma(t) = 1 - P_{0,1}(\mathrm{t}) - P_{0,2}(\mathrm{t}) + \sum_{\sigma=1}^{\infty} \int_0^t \big[\lambda\bar{\rho}_1 P_{\sigma,1}(t) + \lambda\bar{\rho}_2 P_{\sigma,2}(t)\big]\, dt - \sum_{\sigma=1}^{\infty} \sigma \int_0^t \big[p\epsilon_1 P_{\sigma,1}(t) + p\epsilon_2 P_{\sigma,2}(t)\big]\, dt \tag{38}$$

Time-dependent Variance $V(t)$

$$V(t) = \mathrm{Var}[X(t)] \overset{\text{def}}{=} E[X^2(t)] - \big[E[X(t)]\big]^2$$

Or,

$$V(t) = E[X^2(t)] - [\Sigma(t)]^2 = \mathrm{K(t)} - [\Sigma(t)]^2, \tag{39}$$

where $\mathrm{K(t)} = E[X^2(t)] = \sum_{\sigma=1}^{\infty} \sigma^2 [P_{\sigma,0}(t) + P_{\sigma,1}(t) + P_{\sigma,2}(t)], \mathrm{K}(0) = 0$ (40)

Differentiating equation (40) with respect to t, we have

$$\mathrm{K}'(\mathrm{t}) = \sum_{\sigma=1}^{\infty} \sigma^2 [P'_{\sigma,0}(t) + P'_{\sigma,1}(t) + P'_{\sigma,2}(t)]$$

Using equations through (1) to (6) at appropriate places, doing some algebra and simplifying some series summations, and then integrating and using initial conditions, we obtain explicit expression for K(t) as given below

$$\mathrm{K(t)} = 1 - P_{0,1}(t) - P_{0,2}(t) + \mu \int_0^t P_{1,0}(t)\, dt + (\lambda + \mu) \sum_{\sigma=1}^{\infty} \int_0^t P_{\sigma,0}(t)\, dt$$

$$+2(\lambda - \mu) \sum_{\sigma=1}^{\infty} \sigma \int_0^t P_{\sigma,0}(t)\, dt + \sum_{\sigma=1}^{\infty} (2\sigma + 1) \int_0^t [\lambda \bar{\rho}_1 P_{\sigma,1}(t) + \lambda \bar{\rho}_2 P_{\sigma,2}(t)]\, dt$$

$$-\sum_{\sigma=2}^{\infty} \sigma(2\sigma - 1) \int_0^t [p\epsilon_1 P_{\sigma,1}(t) + p\epsilon_2 P_{\sigma,2}(t)]\, dt \quad (41)$$

Using equations (38) & (41) in equation (39), we get explicit expression for the variance Ѵ(t). (Ѵ is read as “Cyrillic capital letter izhitsa”)

Throughput of the System *Ϫ(t)*

$$\text{Ϫ}(t) \stackrel{\text{def}}{=} \sum_{\sigma=1}^{\infty} \mu [P_{\sigma,0}(t) + P_{\sigma,1}(t) + P_{\sigma,2}(t)]$$

Differentiating above equation and proceeding in the same manner as in the case of mean and variance, we getthroughput of the system as

$$\text{Ϫ}(t) = \mu - \mu [P_{0,1}(t) + P_{0,2}(t)] \quad (42)$$

(Ϫ is read as “Coptic capital letter gangia”)

Delay Time ᶑ(t)

$$\text{ᶑ}(t) \stackrel{\text{def}}{=} \frac{mean}{throughput} = \frac{\Sigma(t)}{\text{Ϫ}(t)} \quad (43)$$

Mean and throughput at any time t is given by equations (38) and (42) respectively.

(ᶑ is read as “Latin small letter D with hook and tail”)

Server-state at Any Time

Let

k=0, 1, 2 denote busy state, vacation 1, vacation 2 respectively,

$\wp_{\blacktriangle,\mathrm{k}}(\mathrm{t})$ = Probability that the server is in the state k at any time t, then

$$\wp_{\blacktriangle,\mathrm{k}}(t) = \begin{cases} \sum_{\sigma=1}^{\infty} P_{\sigma,0}(t), & k = 0 \\ \sum_{\sigma=0}^{\infty} P_{\sigma,1}(t), & k = 1 \\ \sum_{\sigma=0}^{\infty} P_{\sigma,2}(t), & k = 2 \end{cases}$$

Now, $\bar{\wp}_{\blacktriangle,1}(s)$ = Laplacetransform of $\wp_{\blacktriangle,\mathrm{k}}(\mathrm{t}) = \sum_{\sigma=0}^{\infty} \bar{P}_{\sigma,1}(s)$

$$= \bar{P}_{0,1}(s) + \sum_{\sigma=1}^{\infty} \bar{P}_{\sigma,1}(s) = \left(1 + \mu\bar{P}_{1,0}(s)\right) Q(s) \left[1 + \sum_{\sigma=1}^{\infty} \bar{\Upsilon}_{\sigma}(s)\right] \quad (44)$$

(due to equations (7) & (18) and Q(s) is given by (19))

$$\therefore\ \wp_{\blacktriangle,1}(\mathrm{t}) = \left(1 + \mu P_{1,0}(t)\right) * Q(t) * \left[1 + \sum_{\sigma=1}^{\infty} \Upsilon_{\sigma}(t)\right] \quad (45)$$

Similarly, from equation (22)

$$\bar{\wp}_{\blacktriangle,2}(s) = \nu_1(1 + \mu\bar{P}_{1,0}(s))Q(s) \left(\sum_{m=0}^{\infty} \frac{\left(q\epsilon_2 \bar{\text{♀}}_1(s)\right)^m}{(s + \nu_2 + \lambda\bar{\rho}_2)^{m+1}}\right) \left[1 + \sum_{\sigma=1}^{\infty} \bar{\text{♀}}_{\sigma}(s)\right] \quad (46)$$

$$\therefore\ \wp_{\blacktriangle,2}(\mathrm{t}) = \nu_1 \left(1 + \mu P_{1,0}(t)\right) * Q(t) * \sum_{m=0}^{\infty} (q\epsilon_2)^m \left(\text{♀}_1(t)\right)^{*m} * \frac{t^m}{m!} e^{-(\nu_2 + \lambda\bar{\rho}_2)t} * \left[1 + \sum_{\sigma=1}^{\infty} \text{♀}_{\sigma}(t)\right] \quad (47)$$

in which * denotes convolution and *m is m-fold convolution and Q(t) is given by (21).

Again, using equation (15), we have (by using equations (9), (11), (24) & (28) and some rearrangements)

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$$\bar{P}_{\sigma,0}(s) = \left(1 + \mu\bar{P}_{1,0}(s)\right) Q(s) \sum_{l=1}^{\infty} \left[\bar{p}\nu_1 \overline{\Upsilon}_l(s) + \bar{q}\nu_1\nu_2 \overline{\text{♀}}_\sigma(s) \left(\sum_{m=0}^{\infty} \frac{\left(q\epsilon_2 \overline{\text{♀}}_1(s)\right)^m}{(s+\nu_2+\lambda\bar{\rho}_2)^{m+1}}\right)\right] \beta^{\sigma-l} \left\{\frac{\left(\theta-\sqrt{\theta^2-\gamma^2}\right)^{\sigma-l}}{\gamma^{\sigma-l}\sqrt{\theta^2-\gamma^2}} - \frac{\left(\theta-\sqrt{\theta^2-\gamma^2}\right)^{\sigma+l}}{\gamma^{\sigma+l}\sqrt{\theta^2-\gamma^2}}\right\} \quad (48)$$

Therefore,

$$\wp_{\blacktriangle,0}(\mathrm{t}) = \left(1 + \mu P_{1,0}(t)\right) * Q(t) * \sum_{\sigma=1}^{\infty} \left[\sum_{l=0}^{\infty} \left[\beta^{\sigma-l} \left\{\bar{p}\nu_1\Upsilon_l(t) + \bar{q}\nu_1\nu_2 \text{♀}_l(t) * \left(\sum_{m=0}^{\infty} (q\epsilon_2)^m \left(\text{♀}_1(t)\right)^{*m} * \frac{t^m}{m!} e^{-(\nu_2+\lambda\bar{\rho}_2)t}\right)\right\} * \{I_{\sigma-l}(\gamma t) - I_{\sigma+l}(\gamma t)\} e^{-(\lambda+\mu)t}\right]\right] \quad (49)$$

5. Steady-state Analysis

Suppose $\mathbb{III}_{\sigma,j} \stackrel{\text{def}}{=} \lim_{t\to\infty} P_{\sigma,j}(t) = \lim_{s\to 0} s\bar{P}_{\sigma,j}(s)$, then

$$\mathbb{III}_{0,1} = \lim_{s\to 0} s\bar{P}_{0,1}(s) = \mu \mathbb{III}_{1,0} Q \quad (50)$$

where $Q(0) = Q = \lim_{s\to 0} Q(s)$

$$= \sum_{r=0}^{\infty}\sum_{n=0}^{r}\sum_{m=0}^{\infty} \binom{n+m}{m} \frac{(\nu_1\nu_2)^n (q\epsilon_2 \text{♀}_1)^m (p\epsilon_1\Upsilon_1)^{r-n}}{(\nu_1+\lambda\bar{\rho}_1)^{r+1}(\nu_2+\lambda\bar{\rho}_2)^{n+m}} \quad (51)$$

♀_1 and Υ_1 are calculated by the following general formulae which are obtained from equations (8) and (10) respectively

$$\Upsilon_\sigma = \lim_{s\to 0} \overline{\Upsilon}_\sigma(s) = \left(\frac{\lambda\bar{\rho}_1}{p\epsilon_1}\right)^\sigma \frac{1}{\prod_{k=1}^{\sigma}\left(\frac{\bar{p}\nu_1}{p\epsilon_1}+k\right)} \frac{{}_1F_1\left(\sigma+1; \frac{\bar{p}\nu_1}{p\epsilon_1}+\sigma+1; -\left(\frac{\lambda\bar{\rho}_1}{p\epsilon_1}\right)\right)}{{}_1F_1\left(1; \frac{\bar{p}\nu_1}{p\epsilon_1}+1; -\left(\frac{\lambda\bar{\rho}_1}{p\epsilon_1}\right)\right)},$$

$\sigma = 1, 2, 3, \ldots$(52)

$$\text{♀}_\sigma = \lim_{s\to 0} \overline{\text{♀}}_\sigma(s) = \left(\frac{\lambda\bar{\rho}_2}{q\epsilon_2}\right)^\sigma \frac{1}{\prod_{k=1}^{\sigma}\left(\frac{\bar{q}\nu_2}{q\epsilon_2}+k\right)} \frac{{}_1F_1\left(\sigma+1; \frac{\bar{q}\nu_2}{q\epsilon_2}+\sigma+1; -\left(\frac{\lambda\bar{\rho}_2}{q\epsilon_2}\right)\right)}{{}_1F_1\left(1; \frac{\bar{q}\nu_2}{q\epsilon_2}+1; -\left(\frac{\lambda\bar{\rho}_2}{q\epsilon_2}\right)\right)},$$

$\sigma = 1,2,3, \ldots$(53)

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Similarly,

$$\mathbb{II}_{0,2} = \mu\nu_1\mathbb{II}_{1,0}Q\sum_{m=0}^{\infty}\frac{(q\epsilon_2\text{♀}_1)^m}{(\nu_2+\lambda\bar{\rho}_2)^{m+1}} \quad (54)$$

$$\mathbb{II}_{\sigma,1} = \mu\Upsilon_\sigma\mathbb{II}_{1,0}Q\,,\ \ \sigma = 1,2,3,\dots \quad (55)$$

$$\mathbb{II}_{\sigma,2} = \mu\nu_1\text{♀}_\sigma\mathbb{II}_{1,0}Q\sum_{m=0}^{\infty}\frac{(q\epsilon_2\text{♀}_1)^m}{(\nu_2+\lambda\bar{\rho}_2)^{m+1}}\ ,\sigma = 1,2,3,\dots \quad (56)$$

$$\mathbb{II}_{\sigma,0} = \mu\mathbb{II}_{1,0}Q\sum_{l=1}^{\infty}\beta^{\sigma-l}\text{Ɲ}_{\sigma,l}\left[\bar{p}\nu_1\Upsilon_l+\bar{q}\nu_1\nu_2\text{♀}_\sigma\left(\sum_{m=0}^{\infty}\frac{(q\epsilon_2\text{♀}_1)^m}{(\nu_2+\lambda\bar{\rho}_2)^{m+1}}\right)\right],$$

$\sigma = 1,2,3,\dots$(57)

where $\text{Ɲ}_{\sigma,l} = \frac{1}{\mu-\lambda}\left\{\left(\sqrt{\frac{\lambda}{\mu}}\right)^{\sigma-l}-\left(\sqrt{\frac{\lambda}{\mu}}\right)^{\sigma+l}\right\}$ (58)

Now, we have all the steady-state probabilities evaluated in terms of known quantities and one unknown quantity viz., $\mathbb{II}_{1,0}$. So, we proceed to evaluate it. We have, for t > 0, a normalizing condition viz.,

$$\sum_{\sigma=1}^{\infty}\mathbb{II}_{\sigma,0}+\sum_{\sigma=0}^{\infty}\mathbb{II}_{\sigma,1}+\sum_{\infty}^{\infty}\mathbb{II}_{\sigma,2} = 1$$

Making use of equations (50), (57) in above normalizing condition and then rearranging and simplifying, we obtain

$$\mathbb{II}_{1,0} = \frac{1-\beta^2}{Q\Omega}\,, \quad (59)$$

where

$$R = \sum_{m=0}^{\infty}\frac{(q\epsilon_2\text{♀}_1)^m}{(\nu_2+\lambda\bar{\rho}_2)^{m+1}} \quad (60)$$

$$\Omega = \left[\sum_{\sigma=1}^{\infty}\left\{\beta^{2\sigma}(\bar{p}\nu_1\Upsilon_\sigma+\bar{q}\nu_1\nu_2\text{♀}_\sigma R)\left(\sum_{k=1}^{\infty}\frac{1-\beta^{2k}}{\beta^{2k}}\right)+\Upsilon_\sigma+\nu_1\text{♀}_\sigma R\right\}+(1+\nu_1R)(1-\beta^2)\right]$$

(61)

Q is given by equation (51).Again, let

$$\mathbb{II}_{\blacktriangle,\text{k}} = \lim_{t\to\infty}\wp_{\blacktriangle,\text{k}}(\text{t}) = \lim_{s\to 0}s\bar{\wp}_{\blacktriangle,\text{k}}(\text{s})\,, \qquad k = 0,1,,2,3.$$

Then from equations (44), (46) and (48), we obtain

$$\mathbb{III}_{\blacktriangle,1} = \mu Q\left[1+\sum_{\sigma=1}^{\infty}\Upsilon_{\sigma}\right]\left[\frac{1-\beta^2}{Q\Omega}\right] = \frac{\mu(1-\beta^2)\{1+\sum_{\sigma=1}^{\infty}\Upsilon_{\sigma}\}}{\Omega} \quad (62)$$

$$\mathbb{III}_{\blacktriangle,2} = \frac{\mu\nu_1(1-\beta^2)\{1+\sum_{\sigma=1}^{\infty}\Upsilon_{\sigma}\}}{\Omega}\left(\sum_{m=0}^{\infty}\frac{(q\epsilon_2\text{♀}_1)^m}{(\nu_2+\lambda\bar{\rho}_2)^{m+1}}\right) \quad (63)$$

$$\mathbb{III}_{\blacktriangle,0} = \frac{\mu(1-\beta^2)}{\Omega}\sum_{l=1}^{\infty}\beta^{\sigma-l}\mathrm{N}_{\sigma,l}\left[\bar{p}\nu_1\Upsilon_l+\bar{q}\nu_1\nu_2\text{♀}_{\sigma}\left(\sum_{m=0}^{\infty}\frac{(q\epsilon_2\text{♀}_1)^m}{(\nu_2+\lambda\bar{\rho}_2)^{m+1}}\right)\right] \quad (64)$$

Thus, all the steady state probabilities have been evaluated.

6. Conclusion and Future Research

An M/M/1 waiting line model having Poisson arrival, exponential service times, multiple differentiated vacations, balking and reneging (modified probabilistically) is analyzed. Both transient and steady-state system size probabilities are obtained. Transient analysis is done with the help of Laplace transform, continued fractions, (confluent) hypergeometric functions and PGF (probability generating function). Steady-state probabilities are obtained from transient state probabilities by the use of final value theorem of Laplace transform. Various effectiveness measures under transient state, viz., mean, variance, throughput and delay time are also presented. Additionally, the probabilities (transient and steady-state) of the server to be in a particular state—busy, vacation 1 or vacation 2—are also obtained. Futureprospect of the model is—fuzzification of the model with single server/multi-servers, batch arrivals, cost analysis, waiting server with/without server setup time, numerical and graphical analysis etc.

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