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Bipolar L-fuzzy sub ℓ-HX group with their cosets

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ABSTRACT

In this paper, we define bipolar L-fuzzy cosets of bipolar L-fuzzy and bipolar anti L-fuzzy sub ℓ-HX groups. Level subset of bipolar L-fuzzy cosets and lower level subset of bipolar L-fuzzy cosets also discussed. We define the pseudo bipolar L-fuzzy cosets of bipolar L-fuzzy and bipolar anti L-fuzzy sub ℓ-HX group. We also discuss some properties of pseudo bipolar L-fuzzy cosets of bipolar L-fuzzy sub ℓ-HX group using the concepts of homomorphism and anti homomorphism.

Keywords: bipolar L-fuzzy ℓ-HX group, bipolar anti L-fuzzy ℓ-HX group, bipolar L-fuzzy cosets, Pseudo bipolar L-fuzzy cosets, ℓ-HX group homomorphism and ℓ-HX group anti homomorphism.

AMS Subject Classification (2010): 03F55, 06D72, 06F15, 06F35, 08A72.

I.Introduction:

The fuzzy set was initiated by L.A. Zadeh[13]. The membership degree of fuzzy set is defined in the interval of [0,1]. In continue, J.A. Goguen[1] introduced L-Fuzzy set. In L-Fuzzy set, the valuation set [0,1] replaced through a complete lattice. Complete lattice may be a poset within which all subsets have each a supremum(join) associated an infimum(meet). The membership degree[−1,1] of bipolar-valued fuzzy set contains two parts. That is, positive membership degree(0,1] and negative membership degree[−1,0). The membership degree (0,1] indicates that components somewhat satisfy the property and also the membership degree [−1,0) indicates that components somewhat satisfy the implicit counter-property. Li Hongxing[2] introduced the idea of HX group and also the authors Luo Chengzhong, Mi Honghai, Li Hongxing[3] introduced the idea of fuzzy HX group. G.S.V. Satya Saibaba[12] introduced the idea of fuzzy lattice ordered groups. Muthuraj.R, Sridharan.M[10,11] introduced the concepts of bipolar fuzzy cosets and pseudo bipolar fuzzy cosets. Muthuraj.R, Rakesh Kumar.T[6]defined pseudo L-fuzzy cosets. In this paper, we introduce the concepts of bipolar L-fuzzy cosets and pseudo bipolar L-fuzzy cosets.

II Preliminaries

In this paper G=(G,*,≤) could be a lattice ordered group or a ℓ-group, e is that the identity of G and mn we tend to mean m*n.

Definition 2.1[6]

Let α be a bipolar L-fuzzy subset defined on G. Let $\vartheta \subset 2^G - \{\phi\}$ be a ℓ-HX group on G. A bipolar L-fuzzy set ρ^α defined on ϑ is said to be a bipolar L-fuzzy sub ℓ-HX group on ϑ if for all P,Q$\in\vartheta$.

i) $(\rho^\alpha)^+(PQ) \geq (\rho^\alpha)^+(P) \wedge (\rho^\alpha)^+(Q)$
ii) $(\rho^\alpha)^-(PQ) \leq (\rho^\alpha)^-(P) \vee (\rho^\alpha)^-(Q)$

iii) $(\rho^{\alpha})^{+}(P) = (\rho^{\alpha})^{+}(P^{-1})$
iv) $(\rho^{\alpha})^{-}(P) = (\rho^{\alpha})^{-}(P^{-1})$
v) $(\rho^{\alpha})^{+}(P\vee Q) \geq (\rho^{\alpha})^{+}(P)\wedge(\rho^{\alpha})^{+}(Q)$
vi) $(\rho^{\alpha})^{-}(P\vee Q) \leq (\rho^{\alpha})^{-}(P)\vee(\rho^{\alpha})^{-}(Q)$
vii) $(\rho^{\alpha})^{+}(P\wedge Q) \geq (\rho^{\alpha})^{+}(P)\wedge(\rho^{\alpha})^{+}(Q)$
viii) $(\rho^{\alpha})^{-}(P\wedge Q) \leq (\rho^{\alpha})^{-}(P)\vee(\rho^{\alpha})^{-}(Q)$

Where $(\rho^{\alpha})^{+}(P) = \vee\{\alpha^{+}(m)$ / for all $m\in P \subseteq G\}$
and
$(\rho^{\alpha})^{-}(P) = \wedge\{\alpha^{-}(m)$ /for all $m\in P\subseteq G\}$

Definition 2.2[6]

Let α be a bipolar L-fuzzy subset defined on G. Let $\vartheta \subset 2^{G}-\{\phi\}$ be a ℓ-HX group on G. A bipolar L-fuzzy set ρ^{α} defined on ϑ is said to be a bipolar anti L-fuzzy sub ℓ-HX group on ϑ if for all $P,Q\in\vartheta$.

i) $(\rho^{\alpha})^{+}(PQ) \leq (\rho^{\alpha})^{+}(P)\wedge(\rho^{\alpha})^{+}(Q)$
ii) $(\rho^{\alpha})^{-}(PQ) \geq (\rho^{\alpha})^{-}(P)\vee(\rho^{\alpha})^{-}(Q)$
iii) $(\rho^{\alpha})^{+}(P) = (\rho^{\alpha})^{+}(P^{-1})$
iv) $(\rho^{\alpha})^{-}(P) = (\rho^{\alpha})^{-}(P^{-1})$
v) $(\rho^{\alpha})^{+}(P\vee Q) \leq (\rho^{\alpha})^{+}(P)\wedge(\rho^{\alpha})^{+}(Q)$
vi) $(\rho^{\alpha})^{-}(P\vee Q) \geq (\rho^{\alpha})^{-}(P)\vee(\rho^{\alpha})^{-}(Q)$
vii) $(\rho^{\alpha})^{+}(P\wedge Q) \leq (\rho^{\alpha})^{+}(P)\wedge(\rho^{\alpha})^{+}(Q)$
viii) $(\rho^{\alpha})^{-}(P\wedge Q) \geq (\rho^{\alpha})^{-}(P)\vee(\rho^{\alpha})^{-}(Q)$

Where $(\rho^{\alpha})^{+}(P) = \vee\{\alpha^{+}(m)$ /for all $m\in P\subseteq G\}$
and
$(\rho^{\alpha})^{-}(P) = \wedge\{\alpha^{-}(m)$ /for all $m\in P\subseteq G\}$

Definition 2.3[9]

Let ρ^{α} be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ. For any $<t_1,t_2>\in L^{+}\times L^{-}$ then $\rho^{\alpha}_{<t1,t2>} = \{X\in\vartheta /(\rho^{\alpha})^{+}(X) \geq t_1$ and $(\rho^{\alpha})^{-}(X) \leq t_2\}$ is called the $< t_1, t_2 >$ level subset of ρ^{α} or simply the level subset of ρ^{α}.

Definition 2.4[10]

Let ρ^{α} be an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ. For any $<t_1 ,t_2 >\in L^{+}\times L^{-}$ then $\rho^{\alpha}_{<t1,t2>} = \{X\in\vartheta$ / $(\rho^{\alpha})^{+}(X) \leq t_1$ and $(\rho^{\alpha})^{-}(X) \geq t_2\}$ is called the $< t_1 ,t_2>$ lower level subset of ρ^{α} or simply the lower level subset of ρ^{α}.

III. Bipolar L-fuzzy cosets of a bipolar L-fuzzy and bipolar anti L-fuzzy sub ℓ-HX groups.

Definition 3.1

Let $\alpha = (\alpha^{+},\alpha^{-})$ be an bipolar L-fuzzy subset of an ℓ-group G and let $\rho^{\alpha} = ((\rho^{\alpha})^{+},(\rho^{\alpha})^{-})$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ, where $(\rho^{\alpha})^{+}(P) = \vee\{\alpha^{+}(m)$ / for all $m\in P\subseteq G\}$ and $(\rho^{\alpha})^{-}(P) = \wedge\{\alpha^{-}(m)$ / for all $m\in P\subseteq G\}$. For any $X\in\vartheta$, the bipolar L-fuzzy subset $X\rho^{\alpha} = (X(\rho^{\alpha})^{+}, X(\rho^{\alpha})^{-})$ is defined by

i) $(X(\rho^{\alpha})^{+})(P) = (\rho^{\alpha})^{+}(X^{-1}P)$
ii) $(X(\rho^{\alpha})^{-})(P) = (\rho^{\alpha})^{-}(X^{-1}P)$, for every $P\in\vartheta$

is called an bipolar L-fuzzy coset of an bipolar L-fuzzy sub ℓ-HX group ϑ corresponding to X.

Definition 3.2

Let $\alpha = (\alpha^{+},\alpha^{-})$ be an bipolar L-fuzzy subset of an ℓ-group G and let $\rho^{\alpha} = ((\rho^{\alpha})^{+},(\rho^{\alpha})^{-})$ be an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ, where $\rho^{\alpha+}(P) = \wedge\{\alpha^{+}(m)$ / for all $m\in P\subseteq G\}$ and $\rho^{\alpha-}(P) = \vee\{\alpha^{-}(m)$ / for all $m\in P\subseteq G\}$. For any $X\in\vartheta$, the bipolar L-fuzzy subset $X\rho^{\alpha} = (X(\rho^{\alpha})^{+}, X(\rho^{\alpha})^{-})$ is defined by

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i) $(X(\rho^{\alpha})^{+})(P) = (\rho^{\alpha})^{+}(X^{-1}P)$

ii) $(X(\rho^{\alpha})^{-})(P) = (\rho^{\alpha})^{-}(X^{-1}P)$, for every $P \in \vartheta$

is called an bipolar L-fuzzy coset of an bipolar anti L-fuzzy sub ℓ-HX group ϑ corresponding to X.

Example 3.3

Let $(G,\cdot_{15},\leq)=(\{1,4,7,13\},\cdot_{15},\leq)$ be an ℓ-group where G is the non-negative integer relatively prime to 15. Let $\alpha=\{<m, \alpha^{+}(m), \alpha^{-}(m)>: m\in G\}$ be an bipolar L-fuzzy subset of sub ℓ-group of G and the mappings $\alpha^{+} : G \to L^{+}$ and $\alpha^{-} : G \to L^{-}$ are defined as,

$\alpha^{+}(1) = 0.7$	$\alpha^{-}(1) = -0.8$
$\alpha^{+}(4) = 0.6$	$\alpha^{-}(4) = -0.5$
$\alpha^{+}(7) = 0.4$	$\alpha^{-}(7) = -0.3$
$\alpha^{+}(13) = 0.4$	$\alpha^{-}(13) = -0.3$

Clearly, α is an bipolar L-fuzzy sub ℓ-group of G.

Consider $\vartheta=\{P,Q\}=\{\{1,4\},\{7,13\}\}$ be an ℓ-HX group of G with usual multiplication. Let ρ^{α} be an bipolar L-fuzzy subset of ϑ and the mappings $(\rho^{\alpha})^{+}: \vartheta\to L^{+}$ and $(\rho^{\alpha})^{-}: \vartheta\to L^{-}$ are defined as, $(\rho^{\alpha})^{+}(P) = \vee\{\alpha^{+}(m) / \text{for all } m\in P\subseteq G\}$ and

$$(\rho^{\alpha})^{-}(P) = \wedge\{\alpha^{-}(m) / \text{for all } m\in P \subseteq G\}$$

Now

$(\rho^{\alpha})^{+}(P) = 0.7$	$(\rho^{\alpha})^{-}(P) = -0.8$
$(\rho^{\alpha})^{+}(Q) = 0.4$	$(\rho^{\alpha})^{-}(Q) = -0.3$

Clearly, ρ^{α} is an bipolar L-fuzzy sub ℓ-HX group of ϑ.

Next we have to define the bipolar L-fuzzy cosets of bipolar L-fuzzy sub ℓ-HX group ϑ defined as for any $X\in\vartheta$, $(X(\rho^{\alpha})^{+})(P) = (\rho^{\alpha})^{+}(X^{-1}P)$ and

$$(X(\rho^{\alpha})^{-})(P) = (\rho^{\alpha})^{-}(X^{-1}P), \text{ for every } P\in\vartheta.$$

Now,

$$P(\rho^{\alpha})^{+}(P) = (\rho^{\alpha})^{+}(P^{-1}P) = (\rho^{\alpha})^{+}(PP) = (\rho^{\alpha})^{+}(P) = 0.7$$
$$P(\rho^{\alpha})^{+}(Q) = (\rho^{\alpha})^{+}(P^{-1}Q) = (\rho^{\alpha})^{+}(PQ) = (\rho^{\alpha})^{+}(Q) = 0.4$$
$$Q(\rho^{\alpha})^{+}(P) = (\rho^{\alpha})^{+}(Q^{-1}P) = (\rho^{\alpha})^{+}(QP) = (\rho^{\alpha})^{+}(Q) = 0.4$$
$$Q(\rho^{\alpha})^{+}(Q) = (\rho^{\alpha})^{+}(Q^{-1}Q) = (\rho^{\alpha})^{+}(QQ) = (\rho^{\alpha})^{+}(P) = 0.7$$
$$P(\rho^{\alpha})^{-}(P) = (\rho^{\alpha})^{-}(P^{-1}P) = (\rho^{\alpha})^{-}(PP) = (\rho^{\alpha})^{-}(P) = -0.8$$
$$P(\rho^{\alpha})^{-}(Q) = (\rho^{\alpha})^{-}(P^{-1}Q) = (\rho^{\alpha})^{-}(PQ) = (\rho^{\alpha})^{-}(Q) = -0.3$$
$$Q(\rho^{\alpha})^{-}(P) = (\rho^{\alpha})^{-}(Q^{-1}P) = (\rho^{\alpha})^{-}(QP) = (\rho^{\alpha})^{-}(Q) = -0.3$$
$$Q(\rho^{\alpha})^{-}(Q) = (\rho^{\alpha})^{-}(Q^{-1}Q) = (\rho^{\alpha})^{-}(QQ) = (\rho^{\alpha})^{-}(P) = -0.8$$

Hence, we can defined $X(\rho^{\alpha})$ is an bipolar L-fuzzy coset of an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ.
Similarly, we can prove that $X(\rho^{\alpha})$ is an bipolar L-fuzzy coset of an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ.

Definition 3.4

Let $\rho^{\alpha} = ((\rho^{\alpha})^{+},(\rho^{\alpha})^{-})$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ. Then for any $<t_1,t_2>\in L^{+}\times L^{-}$, the level subset of bipolar L-fuzzy coset $(X(\rho^{\alpha}))$ is defined as,

$U[X\rho^{\alpha}_{<t1,t2>}] = \{P\in\vartheta / (X(\rho^{\alpha})^{+})(P) = (\rho^{\alpha})^{+}(X^{-1}P) \geq t_1$ and

$(X(\rho^{\alpha})^{-})(P) = (\rho^{\alpha})^{-}(X^{-1}P) \leq t_2$, for some $X\in\vartheta\}$.

Definition 3.5

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Let $\rho^{\alpha} = ((\rho^{\alpha})^{+},(\rho^{\alpha})^{-})$ be an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ. Then for any $<t_1,t_2> \in L^{+} \times L^{-}$, the lower level subset of bipolar L-fuzzy coset $(X(\rho^{\alpha}))$ is defined as,

$L[X\rho^{\alpha}_{<t1,t2>}] = \{P\in\vartheta \ / \ (X(\rho^{\alpha})^{+})(P) = (\rho^{\alpha})^{+}(X^{-1}P) \le t_1$ and

$(X(\rho^{\alpha})^{-})(P) = (\rho^{\alpha})^{-}(X^{-1}P) \ge t_2$, for some $X\in\vartheta\}$.

Theorem 3.6

Let $\alpha = (\alpha^{+},\alpha^{-})$ be an bipolar L-fuzzy subset of an ℓ-group G and $(\rho^{\alpha}) = ((\rho^{\alpha})^{+},(\rho^{\alpha})^{-})$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ then the bipolar L-fuzzy coset $(X\rho^{\alpha}) = (X(\rho^{\alpha})^{+}, X(\rho^{\alpha})^{-})$ is an bipolar L-fuzzy sub ℓ-HX group of ϑ if

i. $(\rho^{\alpha})^{+}(X^{-1}Q) \wedge (\rho^{\alpha})^{+}(X) = (\rho^{\alpha})^{+}(X^{-1}Q)$

ii. $(\rho^{\alpha})^{-}(X^{-1}Q) \vee (\rho^{\alpha})^{-}(X) = (\rho^{\alpha})^{-}(X^{-1}Q)$, for all $Q\in\vartheta$.

Proof

Consider $(\rho^{\alpha}) = ((\rho^{\alpha})^{+},(\rho^{\alpha})^{-})$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ. For all $P,Q\in\vartheta$, we have,

i.
$$\begin{aligned}
(X(\rho^{\alpha})^{+})(PQ^{-1}) &= (\rho^{\alpha})^{+}(X^{-1}PQ^{-1})\\
&= (\rho^{\alpha})^{+}(X^{-1}PQ^{-1}XX^{-1})\\
&\ge (\rho^{\alpha})^{+}(X^{-1}P)\wedge(\rho^{\alpha})^{+}(Q^{-1}XX^{-1})\\
&\ge (\rho^{\alpha})^{+}(X^{-1}P)\wedge((\rho^{\alpha})^{+}(Q^{-1}X)\wedge(\rho^{\alpha})^{+}(X^{-1}))\\
&= (\rho^{\alpha})^{+}(X^{-1}P)\wedge((\rho^{\alpha})^{+}((Q^{-1}X)^{-1})\wedge(\rho^{\alpha})^{+}(X))\\
&= (\rho^{\alpha})^{+}(X^{-1}P)\wedge((\rho^{\alpha})^{+}(X^{-1}Q)\wedge(\rho^{\alpha})^{+}(X))\\
&= (\rho^{\alpha})^{+}(X^{-1}P)\wedge(\rho^{\alpha})^{+}(X^{-1}Q)\\
&= (X(\rho^{\alpha})^{+})(P)\wedge(X(\rho^{\alpha})^{+})(Q)\\
(X(\rho^{\alpha})^{+})(PQ^{-1}) &\ge (X(\rho^{\alpha})^{+})(P)\wedge(X(\rho^{\alpha})^{+})(Q)
\end{aligned}$$

ii)
$$\begin{aligned}
(X(\rho^{\alpha})^{-})(PQ^{-1}) &= (\rho^{\alpha})^{-}(X^{-1}PQ^{-1})\\
&= (\rho^{\alpha})^{-}(X^{-1}PQ^{-1}XX^{-1})\\
&\le (\rho^{\alpha})^{-}(X^{-1}P)\vee(\rho^{\alpha})^{-}(Q^{-1}XX^{-1})\\
&\le (\rho^{\alpha})^{-}(X^{-1}P)\vee((\rho^{\alpha})^{-}(Q^{-1}X)\vee(\rho^{\alpha})^{-}(X^{-1}))\\
&= (\rho^{\alpha})^{-}(X^{-1}P)\vee((\rho^{\alpha})^{-}((Q^{-1}X)^{-1})\vee(\rho^{\alpha})^{-}(X))\\
&= (\rho^{\alpha})^{-}(X^{-1}P)\vee((\rho^{\alpha})^{-}(X^{-1}Q)\vee(\rho^{\alpha})^{-}(X))\\
&= (\rho^{\alpha})^{-}(X^{-1}P)\vee(\rho^{\alpha})^{-}(X^{-1}Q)\\
&= (X(\rho^{\alpha})^{-})(P)\vee(X(\rho^{\alpha})^{-})(Q)\\
(X(\rho^{\alpha})^{-})(PQ^{-1}) &\le (X(\rho^{\alpha})^{-})(P)\vee(X(\rho^{\alpha})^{-})(Q)
\end{aligned}$$

iii)
$$\begin{aligned}
(X(\rho^{\alpha})^{+}(P\vee Q)) &= (\rho^{\alpha})^{+}(X^{-1}(P\vee Q))\\
&= (\rho^{\alpha})^{+}((X^{-1}P)\vee(X^{-1}Q))\\
&\ge (\rho^{\alpha})^{+}(X^{-1}P)\wedge(\rho^{\alpha})^{+}(X^{-1}Q)\\
&= (X(\rho^{\alpha})^{+})(P)\wedge(X(\rho^{\alpha})^{+})(Q)\\
(X(\rho^{\alpha})^{+}(P\vee Q)) &\ge (X(\rho^{\alpha})^{+}(P))\wedge(X(\rho^{\alpha})^{+}(Q))
\end{aligned}$$

iv)
$$\begin{aligned}
(X(\rho^{\alpha})^{-}(P\vee Q)) &= (\rho^{\alpha})^{-}(X^{-1}(P\vee Q))\\
&= (\rho^{\alpha})^{-}((X^{-1}P)\vee(X^{-1}Q))\\
&\le (\rho^{\alpha})^{-}((X^{-1}P)\vee(\rho^{\alpha})^{-}(X^{-1}Q))\\
&= (X(\rho^{\alpha})^{-}(P))\vee(X(\rho^{\alpha})^{-}(Q))\\
(X(\rho^{\alpha})^{-}(P\vee Q)) &\le (X(\rho^{\alpha})^{-}(P))\vee(X(\rho^{\alpha})^{-}(Q))
\end{aligned}$$

v)
$$\begin{aligned}
(X(\rho^{\alpha})^{+}(P\wedge Q)) &= (\rho^{\alpha})^{+}(X^{-1}(P\wedge Q))\\
&= (\rho^{\alpha})^{+}((X^{-1}P)\wedge(X^{-1}Q)\\
&\ge (\rho^{\alpha})^{+}(X^{-1}P)\wedge(\rho^{\alpha})^{+}(X^{-1}Q)\\
&= (X(\rho^{\alpha})^{+}(P))\wedge(X(\rho^{\alpha})^{+}(Q))\\
(X(\rho^{\alpha})^{+}(P\wedge Q)) &\ge (X(\rho^{\alpha})^{+}(P))\wedge(X(\rho^{\alpha})^{+}(Q))
\end{aligned}$$

vi)
$$\begin{aligned}
(X(\rho^{\alpha})^{-}(P\wedge Q)) &= (\rho^{\alpha})^{-}(X^{-1}(P\wedge Q))\\
&= (\rho^{\alpha})^{-}((X^{-1}P)\wedge(X^{-1}Q))\\
&\le (\rho^{\alpha})^{-}((X^{-1}P)\vee(\rho^{\alpha})^{-}(X^{-1}Q))
\end{aligned}$$

$= (X(\rho^{\alpha})^{-}(P))\vee(X(\rho^{\alpha})^{-}(Q))$

$(X(\rho^{\alpha})^{-}(P\wedge Q)) \leq (X(\rho^{\alpha})^{-}(P))\vee(X(\rho^{\alpha})^{-}(Q))$

Hence, $(X\rho^{\alpha}) = (X(\rho^{\alpha})^{+}, X(\rho^{\alpha})^{-})$ is an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ.

Theorem 3.7

Let $\alpha = (\alpha^{+},\alpha^{-})$ be an bipolar L-fuzzy subset of an ℓ-group G and $(\rho^{\alpha}) = ((\rho^{\alpha})^{+}, (\rho^{\alpha})^{-})$ be an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ then the bipolar L-fuzzy coset

$(X\rho^{\alpha}) = (X(\rho^{\alpha})^{+}, X(\rho^{\alpha})^{-})$ is an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ if

i. $(\rho^{\alpha})^{+}(X^{-1}Q)\vee(\rho^{\alpha})^{+}(X) = (\rho^{\alpha})^{+}(X^{-1}Q)$

ii. $(\rho^{\alpha})^{-}(X^{-1}Q)\wedge(\rho^{\alpha})^{-}(X) = (\rho^{\alpha})^{-}(X^{-1}Q)$, for all $Q\in\vartheta$.

Proof

Consider $(\rho^{\alpha}) = ((\rho^{\alpha})^{+}, (\rho^{\alpha})^{-})$ be an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ. For all $P,Q\in\vartheta$, we have,

i) $(X(\rho^{\alpha})^{+})(PQ^{-1}) = (\rho^{\alpha})^{+}(X^{-1}PQ^{-1})$

$= (\rho^{\alpha})^{+}(X^{-1}PQ^{-1}XX^{-1})$

$\leq (\rho^{\alpha})^{+}(X^{-1}P)\vee(\rho^{\alpha})^{+}(Q^{-1}XX^{-1})$

$\leq (\rho^{\alpha})^{+}(X^{-1}P)\vee((\rho^{\alpha})^{+}(Q^{-1}X)\vee(\rho^{\alpha})^{+}(X^{-1}))$

$= (\rho^{\alpha})^{+}(X^{-1}P)\vee((\rho^{\alpha})^{+}((Q^{-1}X)^{-1})\vee(\rho^{\alpha})^{+}(X))$

$= (\rho^{\alpha})^{+}(X^{-1}P)\vee((\rho^{\alpha})^{+}(X^{-1}Q)\vee(\rho^{\alpha})^{+}(X))$

$= (\rho^{\alpha})^{+}(X^{-1}P)\vee(\rho^{\alpha})^{+}(X^{-1}Q)$

$= (X(\rho^{\alpha})^{+})(P)\vee(X(\rho^{\alpha})^{+})(Q)$

$(X(\rho^{\alpha})^{+})(PQ^{-1}) \leq (X(\rho^{\alpha})^{+})(P)\vee(X(\rho^{\alpha})^{+})(Q)$

ii) $(X(\rho^{\alpha})^{-})(PQ^{-1}) = (\rho^{\alpha})^{-}(X^{-1}PQ^{-1})$

$= (\rho^{\alpha})^{-}(X^{-1}PQ^{-1}XX^{-1})$

$\geq (\rho^{\alpha})^{-}(X^{-1}P)\wedge(\rho^{\alpha})^{-}(Q^{-1}XX^{-1})$

$\geq (\rho^{\alpha})^{-}(X^{-1}P)\wedge((\rho^{\alpha})^{-}(Q^{-1}X)\wedge(\rho^{\alpha})^{-}(X^{-1}))$

$= (\rho^{\alpha})^{-}(X^{-1}P)\wedge((\rho^{\alpha})^{-}((Q^{-1}X)^{-1})\wedge(\rho^{\alpha})^{-}(X))$

$= (\rho^{\alpha})^{-}(X^{-1}P)\wedge((\rho^{\alpha})^{-}(X^{-1}Q)\wedge(\rho^{\alpha})^{-}(X))$

$= (\rho^{\alpha})^{-}(X^{-1}P)\wedge(\rho^{\alpha})^{-}(X^{-1}Q)$

$= (X(\rho^{\alpha})^{-})(P)\wedge(X(\rho^{\alpha})^{-})(Q)$

$(X(\rho^{\alpha})^{-})(PQ^{-1}) \geq (X(\rho^{\alpha})^{-})(P)\wedge(X(\rho^{\alpha})^{-})(Q)$

iii) $(X(\rho^{\alpha})^{+}(P\vee Q)) = (\rho^{\alpha})^{+}(X^{-1}(P\vee Q))$

$= (\rho^{\alpha})^{+}((X^{-1}P)\vee(X^{-1}Q))$

$\leq (\rho^{\alpha})^{+}(X^{-1}P)\vee(\rho^{\alpha})^{+}(X^{-1}Q)$

$= (X(\rho^{\alpha})^{+})(P)\vee(X(\rho^{\alpha})^{+})(Q)$

$(X(\rho^{\alpha})^{+}(P\vee Q)) \leq (X(\rho^{\alpha})^{+}(P))\vee(X(\rho^{\alpha})^{+}(Q))$

iv) $(X(\rho^{\alpha})^{-}(P\vee Q)) = (\rho^{\alpha})^{-}(X^{-1}(P\vee Q))$

$= (\rho^{\alpha})^{-}((X^{-1}P)\vee(X^{-1}Q))$

$\geq (\rho^{\alpha})^{-}((X^{-1}P)\wedge(\rho^{\alpha})^{-}(X^{-1}Q))$

$= (X(\rho^{\alpha})^{-}(P))\wedge(X(\rho^{\alpha})^{-}(Q))$

$(X(\rho^{\alpha})^{-}(P\vee Q)) \geq (X(\rho^{\alpha})^{-}(P))\wedge(X(\rho^{\alpha})^{-}(Q))$

v) $(X(\rho^{\alpha})^{+}(P\wedge Q)) = (\rho^{\alpha})^{+}(X^{-1}(P\wedge Q))$

$= (\rho^{\alpha})^{+}((X^{-1}P)\wedge(X^{-1}Q)$

$\leq (\rho^{\alpha})^{+}(X^{-1}P)\vee(\rho^{\alpha})^{+}(X^{-1}Q)$

$= (X(\rho^{\alpha})^{+}(P))\vee(X(\rho^{\alpha})^{+}(Q))$

$(X(\rho^{\alpha})^{+}(P\wedge Q)) \leq (X(\rho^{\alpha})^{+}(P))\vee(X(\rho^{\alpha})^{+}(Q))$

vi) $(X(\rho^{\alpha})^{-}(P\wedge Q)) = (\rho^{\alpha})^{-}(X^{-1}(P\wedge Q))$

$= (\rho^{\alpha})^{-}((X^{-1}P)\wedge(X^{-1}Q))$

$\geq (\rho^{\alpha})^{-}((X^{-1}P)\wedge(\rho^{\alpha})^{-}(X^{-1}Q))$

$= (X(\rho^{\alpha})^{-}(P))\wedge(X(\rho^{\alpha})^{-}(Q))$

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$(X(\rho^{\alpha})^{-}(P \wedge Q)) \geq (X(\rho^{\alpha})^{-}(P)) \wedge (X(\rho^{\alpha})^{-}(Q))$

Hence, $(X\rho^{\alpha}) = (X(\rho^{\alpha})^{+}, X(\rho^{\alpha})^{-})$ is an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ.

Theorem 3.8

Let $\alpha = (\alpha^{+},\alpha^{-})$ be an bipolar L-fuzzy subset of an ℓ-group G and $(\rho^{\alpha}) = ((\rho^{\alpha})^{+}, (\rho^{\alpha})^{-})$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ then $P\rho^{\alpha} = Q\rho^{\alpha}$, for $P,Q \in \vartheta$ if and only if

i) $(\rho^{\alpha})^{+}(P^{-1}Q) = (\rho^{\alpha})^{+}(Q^{-1}P) = (\rho^{\alpha})^{+}(E)$

ii) $(\rho^{\alpha})^{-}(P^{-1}Q) = (\rho^{\alpha})^{-}(Q^{-1}P) = (\rho^{\alpha})^{-}(E)$

Proof

Let $(\rho^{\alpha}) = ((\rho^{\alpha})^{+}, (\rho^{\alpha})^{-})$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ.

Consider $P\rho^{\alpha} = Q\rho^{\alpha}$, for $P,Q \in \vartheta$, we get $P(\rho^{\alpha})^{+} = Q(\rho^{\alpha})^{+}$ and $P(\rho^{\alpha})^{-} = Q(\rho^{\alpha})^{-}$

So that $P(\rho^{\alpha})^{+}(P) = Q(\rho^{\alpha})^{+}(P)$, $P(\rho^{\alpha})^{+}(Q) = Q(\rho^{\alpha})^{+}(Q)$ and

$P(\rho^{\alpha})^{-}(P) = Q(\rho^{\alpha})^{-}(P)$, $P(\rho^{\alpha})^{-}(Q) = Q(\rho^{\alpha})^{-}(Q)$

Then by definition, $(\rho^{\alpha})^{+}(P^{-1}P) = (\rho^{\alpha})^{+}(Q^{-1}P)$, $(\rho^{\alpha})^{+}(P^{-1}Q) = (\rho^{\alpha})^{+}(Q^{-1}Q)$ and

$(\rho^{\alpha})^{-}(P^{-1}P) = (\rho^{\alpha})^{-}(Q^{-1}P)$, $(\rho^{\alpha})^{-}(P^{-1}Q) = (\rho^{\alpha})^{-}(Q^{-1}Q)$

Which implies that, $(\rho^{\alpha})^{+}(E) = (\rho^{\alpha})^{+}(Q^{-1}P)$, $(\rho^{\alpha})^{+}(P^{-1}Q) = (\rho^{\alpha})^{+}(E)$ and

$(\rho^{\alpha})^{-}(E) = (\rho^{\alpha})^{-}(Q^{-1}P)$, $(\rho^{\alpha})^{-}(P^{-1}Q) = (\rho^{\alpha})^{-}(E)$

Hence, $(\rho^{\alpha})^{+}(P^{-1}Q) = (\rho^{\alpha})^{+}(Q^{-1}P) = (\rho^{\alpha})^{+}(E)$ and $(\rho^{\alpha})^{-}(P^{-1}Q) = (\rho^{\alpha})^{-}(Q^{-1}P) = (\rho^{\alpha})^{-}(E)$.

Conversely,

Consider $(\rho^{\alpha})^{+}(P^{-1}Q) = (\rho^{\alpha})^{+}(Q^{-1}P) = (\rho^{\alpha})^{+}(E)$ and $(\rho^{\alpha})^{-}(P^{-1}Q) = (\rho^{\alpha})^{-}(Q^{-1}P) = (\rho^{\alpha})^{-}(E)$, for $P,Q \in \vartheta$. For every $X \in \vartheta$,

i)
$$\begin{aligned} P(\rho^{\alpha})^{+}(X) &= (\rho^{\alpha})^{+}(P^{-1}X) \\ &= (\rho^{\alpha})^{+}(P^{-1}QQ^{-1}X) \\ &\geq (\rho^{\alpha})^{+}(P^{-1}Q) \wedge (\rho^{\alpha})^{+}(Q^{-1}X) \\ &= (\rho^{\alpha})^{+}(E) \wedge (\rho^{\alpha})^{+}(Q^{-1}X) \\ &= (\rho^{\alpha})^{+}(Q^{-1}X) \\ &= Q(\rho^{\alpha})^{+}(X) \\ P(\rho^{\alpha})^{+}(X) &\geq Q(\rho^{\alpha})^{+}(X) \end{aligned}$$

Similarly, $Q(\rho^{\alpha})^{+}(X) \geq P(\rho^{\alpha})^{+}(X)$

So that, $P(\rho^{\alpha})^{+}(X) = Q(\rho^{\alpha})^{+}(X)$.

ii)
$$\begin{aligned} P(\rho^{\alpha})^{-}(X) &= (\rho^{\alpha})^{-}(P^{-1}X) \\ &= (\rho^{\alpha})^{-}(P^{-1}QQ^{-1}X) \\ &\leq (\rho^{\alpha})^{-}(P^{-1}Q) \vee (\rho^{\alpha})^{-}(Q^{-1}X) \\ &= (\rho^{\alpha})^{-}(E) \vee (\rho^{\alpha})^{-}(Q^{-1}X) \\ &= (\rho^{\alpha})^{-}(Q^{-1}X) \\ &= Q(\rho^{\alpha})^{-}(X) \\ P(\rho^{\alpha})^{-}(X) &\leq Q(\rho^{\alpha})^{-}(X) \end{aligned}$$

Similarly, $Q(\rho^{\alpha})^{-}(X) \leq P(\rho^{\alpha})^{-}(X)$

So that, $P(\rho^{\alpha})^{-}(X) = Q(\rho^{\alpha})^{-}(X)$.

Hence, $P\rho^{\alpha} = Q\rho^{\alpha}$, for $P,Q \in \vartheta$.

Theorem 3.9

Let $\alpha = (\alpha^{+},\alpha^{-})$ be an bipolar L-fuzzy subset of an ℓ-group G and $(\rho^{\alpha}) = ((\rho^{\alpha})^{+}, (\rho^{\alpha})^{-})$ be an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ then $P\rho^{\alpha} = Q\rho^{\alpha}$, for $P,Q \in \vartheta$ if and only if

i) $(\rho^{\alpha})^{+}(P^{-1}Q) = (\rho^{\alpha})^{+}(Q^{-1}P) = (\rho^{\alpha})^{+}(E)$

ii) $(\rho^{\alpha})^{-}(P^{-1}Q) = (\rho^{\alpha})^{-}(Q^{-1}P) = (\rho^{\alpha})^{-}(E)$

Proof

Let $(\rho^{\alpha}) = ((\rho^{\alpha})^{+}, (\rho^{\alpha})^{-})$ be an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ.

Consider $P\rho^{\alpha} = Q\rho^{\alpha}$, for $P,Q \in \vartheta$, we get $P(\rho^{\alpha})^{+} = Q(\rho^{\alpha})^{+}$ and $P(\rho^{\alpha})^{-} = Q(\rho^{\alpha})^{-}$

So that $P(\rho^{\alpha})^{+}(P) = Q(\rho^{\alpha})^{+}(P)$, $P(\rho^{\alpha})^{+}(Q) = Q(\rho^{\alpha})^{+}(Q)$ and

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$P(\rho^{\alpha})^{-}(P) = Q(\rho^{\alpha})^{-}(P)$, $P(\rho^{\alpha})^{-}(Q) = Q(\rho^{\alpha})^{-}(Q)$

Then by definition, $(\rho^{\alpha})^{+}(P^{-1}P) = (\rho^{\alpha})^{+}(Q^{-1}P)$, $(\rho^{\alpha})^{+}(P^{-1}Q) = (\rho^{\alpha})^{+}(Q^{-1}Q)$ and
$(\rho^{\alpha})^{-}(P^{-1}P) = (\rho^{\alpha})^{-}(Q^{-1}P)$, $(\rho^{\alpha})^{-}(P^{-1}Q) = (\rho^{\alpha})^{-}(Q^{-1}Q)$

Which implies that, $(\rho^{\alpha})^{+}(E) = (\rho^{\alpha})^{+}(Q^{-1}P)$, $(\rho^{\alpha})^{+}(P^{-1}Q) = (\rho^{\alpha})^{+}(E)$ and
$(\rho^{\alpha})^{-}(E) = (\rho^{\alpha})^{-}(Q^{-1}P)$, $(\rho^{\alpha})^{-}(P^{-1}Q) = (\rho^{\alpha})^{-}(E)$

Hence, $(\rho^{\alpha})^{+}(P^{-1}Q) = (\rho^{\alpha})^{+}(Q^{-1}P) = (\rho^{\alpha})^{+}(E)$ and $(\rho^{\alpha})^{-}(P^{-1}Q) = (\rho^{\alpha})^{-}(Q^{-1}P) = (\rho^{\alpha})^{-}(E)$.

Conversely,

Consider $(\rho^{\alpha})^{+}(P^{-1}Q) = (\rho^{\alpha})^{+}(Q^{-1}P) = (\rho^{\alpha})^{+}(E)$ and $(\rho^{\alpha})^{-}(P^{-1}Q) = (\rho^{\alpha})^{-}(Q^{-1}P) = (\rho^{\alpha})^{-}(E)$, for $P,Q\in\vartheta$. For every $X\in\vartheta$,

i) $P(\rho^{\alpha})^{+}(X) = (\rho^{\alpha})^{+}(P^{-1}X)$
$= (\rho^{\alpha})^{+}(P^{-1}QQ^{-1}X)$
$\leq (\rho^{\alpha})^{+}(P^{-1}Q)\vee(\rho^{\alpha})^{+}(Q^{-1}X)$
$= (\rho^{\alpha})^{+}(E)\vee(\rho^{\alpha})^{+}(Q^{-1}X)$
$= (\rho^{\alpha})^{+}(Q^{-1}X)$
$= Q(\rho^{\alpha})^{+}(X)$

$P(\rho^{\alpha})^{+}(X) \leq Q(\rho^{\alpha})^{+}(X)$

Similarly, $Q(\rho^{\alpha})^{+}(X) \leq P(\rho^{\alpha})^{+}(X)$

So that, $P(\rho^{\alpha})^{+}(X) = Q(\rho^{\alpha})^{+}(X)$.

ii) $P(\rho^{\alpha})^{-}(X) = (\rho^{\alpha})^{-}(P^{-1}X)$
$= (\rho^{\alpha})^{-}(P^{-1}QQ^{-1}X)$
$\geq (\rho^{\alpha})^{-}(P^{-1}Q)\wedge(\rho^{\alpha})^{-}(Q^{-1}X)$
$= (\rho^{\alpha})^{-}(E)\wedge(\rho^{\alpha})^{-}(Q^{-1}X)$
$= (\rho^{\alpha})^{-}(Q^{-1}X)$
$= Q(\rho^{\alpha})^{-}(X)$

$P(\rho^{\alpha})^{-}(X) \geq Q(\rho^{\alpha})^{-}(X)$

Similarly, $Q(\rho^{\alpha})^{-}(X) \geq P(\rho^{\alpha})^{-}(X)$

So that, $P(\rho^{\alpha})^{-}(X) = Q(\rho^{\alpha})^{-}(X)$.

Hence, $P\rho^{\alpha} = Q\rho^{\alpha}$, for $P,Q\in\vartheta$.

Theorem 3.10

Let $\alpha = (\alpha^{+},\alpha^{-})$ be an bipolar L-fuzzy subset of an ℓ-group G and $(\rho^{\alpha}) = ((\rho^{\alpha})^{+}, (\rho^{\alpha})^{-})$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ and $P\rho^{\alpha} = Q\rho^{\alpha}$, for $P,Q\in\vartheta$ then $(\rho^{\alpha})(P) = (\rho^{\alpha})(Q)$.

Proof

Consider $(\rho^{\alpha}) = ((\rho^{\alpha})^{+}, (\rho^{\alpha})^{-})$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ and $P\rho^{\alpha} = Q\rho^{\alpha}$, for $P,Q\in\vartheta$.

i) $(\rho^{\alpha})^{+}(P) = (\rho^{\alpha})^{+}(QQ^{-1}P)$
$\geq (\rho^{\alpha})^{+}(Q)\wedge(\rho^{\alpha})^{+}(Q^{-1}P)$
$= (\rho^{\alpha})^{+}(Q)\wedge(\rho^{\alpha})^{+}(E)$

$(\rho^{\alpha})^{+}(P) \geq (\rho^{\alpha})^{+}(Q)$

Similarly, $(\rho^{\alpha})^{+}(Q) \geq (\rho^{\alpha})^{+}(P)$

So that, $(\rho^{\alpha})^{+}(P) = (\rho^{\alpha})^{+}(Q)$

ii) $(\rho^{\alpha})^{-}(P) = (\rho^{\alpha})^{-}(QQ^{-1}P)$
$\leq (\rho^{\alpha})^{-}(Q)\vee(\rho^{\alpha})^{-}(Q^{-1}P)$
$= (\rho^{\alpha})^{-}(Q)\vee(\rho^{\alpha})^{-}(E)$

$(\rho^{\alpha})^{-}(P) \leq (\rho^{\alpha})^{-}(Q)$

Similarly, $(\rho^{\alpha})^{-}(Q) \leq (\rho^{\alpha})^{-}(P)$

So that, $(\rho^{\alpha})^{-}(P) = (\rho^{\alpha})^{-}(Q)$.

Hence, $(\rho^{\alpha})(P) = (\rho^{\alpha})(Q)$.

Theorem 3.11

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Let $\alpha = (\alpha^+,\alpha^-)$ be an bipolar L-fuzzy subset of an ℓ-group G and $(\rho^\alpha) = ((\rho^\alpha)^+, (\rho^\alpha)^-)$ be an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ and $P\rho^\alpha = Q\rho^\alpha$, for $P,Q\in\vartheta$.
then $(\rho^\alpha)(P) = (\rho^\alpha)(Q)$.

Proof

Consider $(\rho^\alpha) = ((\rho^\alpha)^+, (\rho^\alpha)^-)$ be an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ and $P\rho^\alpha = Q\rho^\alpha$, for $P,Q\in\vartheta$.

i) $(\rho^\alpha)^+(P) = (\rho^\alpha)^+(QQ^{-1}P)$
$\leq (\rho^\alpha)^+(Q)\vee(\rho^\alpha)^+(Q^{-1}P)$
$= (\rho^\alpha)^+(Q)\vee(\rho^\alpha)^+(E)$
$(\rho^\alpha)^+(P) \leq (\rho^\alpha)^+(Q)$

Similarly, $(\rho^\alpha)^+(Q) \leq (\rho^\alpha)^+(P)$

So that, $(\rho^\alpha)^+(P) = (\rho^\alpha)^+(Q)$

ii) $(\rho^\alpha)^-(P) = (\rho^\alpha)^-(QQ^{-1}P)$
$\geq (\rho^\alpha)^-(Q)\wedge(\rho^\alpha)^-(Q^{-1}P)$
$= (\rho^\alpha)^-(Q)\wedge(\rho^\alpha)^-(E)$
$(\rho^\alpha)^-(P) \geq (\rho^\alpha)^-(Q)$

Similarly, $(\rho^\alpha)^-(Q) \geq (\rho^\alpha)^-(P)$

So that, $(\rho^\alpha)^-(P) = (\rho^\alpha)^-(Q)$

Hence, $(\rho^\alpha)(P) = (\rho^\alpha)(Q)$.

Theorem 3.12

Let $\alpha = (\alpha^+,\alpha^-)$ be an bipolar L-fuzzy subset of an ℓ-group G and $(\rho^\alpha) = ((\rho^\alpha)^+, (\rho^\alpha)^-)$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ then $U[X\rho^\alpha_{<t1,t2>}] = XU[\rho^\alpha_{<t1,t2>}]$ for every $X\in\vartheta$ and $<t_1,t_2>\in L^+\times L^-$.

Proof

Consider $(\rho^\alpha) = ((\rho^\alpha)^+, (\rho^\alpha)^-)$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ and $U[X\rho^\alpha_{<t1,t2>}] = U[(X(\rho^\alpha)^+ ; t_1), (X(\rho^\alpha)^- ; t_2)]$

Consider $P\in U[X\rho^\alpha_{<t1,t2>}] \Rightarrow P\in U[(X(\rho^\alpha)^+ ; t_1)]$ and $P\in U[(X(\rho^\alpha)^- ; t_2)]$

i) $P\in U[(X(\rho^\alpha)^+ ; t_1)] \Leftrightarrow (X(\rho^\alpha)^+(P)) \geq t_1$
$\Leftrightarrow (\rho^\alpha)^+(X^{-1}P) \geq t_1$
$\Leftrightarrow X^{-1}P\in U[(\rho^\alpha)^+ ; t_1]$
$\Leftrightarrow P\in XU[(\rho^\alpha)^+ ; t_1)]$
$U[(X(\rho^\alpha)^+ ; t_1)] = XU[(\rho^\alpha)^+ ; t_1)]$, for every $X\in\vartheta$.

ii) $P\in U[(X(\rho^\alpha)^- ; t_2)] \Leftrightarrow (X(\rho^\alpha)^-(P)) \leq t_2$
$\Leftrightarrow (\rho^\alpha)^-(X^{-1}P) \leq t_2$
$\Leftrightarrow X^{-1}P\in U[(\rho^\alpha)^- ; t_2]$
$\Leftrightarrow P\in XU[(\rho^\alpha)^- ; t_2)]$
$U[(X(\rho^\alpha)^- ; t_2)] = XU[(\rho^\alpha)^- ; t_2)]$, for every $X\in\vartheta$.

Hence, $U[X\rho^\alpha_{<t1,t2>}] = XU[\rho^\alpha_{<t1,t2>}]$.

Theorem 3.13

Let $\alpha = (\alpha^+,\alpha^-)$ be an bipolar L-fuzzy subset of an ℓ-group G and $(\rho^\alpha) = ((\rho^\alpha)^+, (\rho^\alpha)^-)$ be an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ then $L[X\rho^\alpha_{<t1,t2>}] = X\,L[\rho^\alpha_{<t1,t2>}]$ for every $X\in\vartheta$ and $<t_1,t_2>\in L^+\times L^-$.

Proof

Consider $(\rho^\alpha) = ((\rho^\alpha)^+, (\rho^\alpha)^-)$ be an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ and $L[X\rho^\alpha_{<t1,t2>}] = L[(X(\rho^\alpha)^+ ; t_1), (X(\rho^\alpha)^- ; t_2)]$

Consider $P\in U[X\rho^\alpha_{<t1,t2>}] \Rightarrow P\in U[(X(\rho^\alpha)^+ ; t_1)]$ and $P\in U[(X(\rho^\alpha)^- ; t_2)]$

i) $P\in L[(X(\rho^\alpha)^+ ; t_1)] \Leftrightarrow (X(\rho^\alpha)^+(P)) \leq t_1$
$\Leftrightarrow (\rho^\alpha)^+(X^{-1}P) \leq t_1$

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$\Leftrightarrow \quad X^{-1}P \in L[(\rho^{\alpha})^{+} ; t_1]$

$\Leftrightarrow \quad P \in XL[(\rho^{\alpha})^{+} ; t_1)]$

$L[(X(\rho^{\alpha})^{+} ; t_1)] = XL[(\rho^{\alpha})^{+} ; t_1)]$, for every $X \in \vartheta$.

ii) $P \in L[(X(\rho^{\alpha})^{-} ; t_2)] \Leftrightarrow (X(\rho^{\alpha})^{-}(P)) \geq t_2$

$\Leftrightarrow \quad (\rho^{\alpha})^{-}(X^{-1}P) \geq t_2$

$\Leftrightarrow \quad X^{-1}P \in L[(\rho^{\alpha})^{-} ; t_2]$

$\Leftrightarrow \quad P \in XL[(\rho^{\alpha})^{-} ; t_2)]$

$L[(X(\rho^{\alpha})^{-} ; t_2)] = XL[(\rho^{\alpha})^{-} ; t_2)]$, for every $X \in \vartheta$.

Hence, $L[X\rho^{\alpha}_{< t1,t2>}] = XL[\rho^{\alpha}_{< t1,t2>}]$.

IV Pseudo bipolar L-fuzzy cosets of an bipolar L-fuzzy sub ℓ-HX group and bipolar anti L-fuzzy sub ℓ-HX group.

Definition 4.1

Let $\alpha = (\alpha^{+},\alpha^{-})$ be an bipolar L-fuzzy subset of an ℓ-group G and Let $\vartheta \subset 2^{G} - \{\phi\}$ be an ℓ-HX group on G. Consider $\rho^{\alpha} = ((\rho^{\alpha})^{+},(\rho^{\alpha})^{-})$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ, where $(\rho^{\alpha})^{+}(P) = \wedge\{\alpha^{+}(m) \text{ / for all } m \in P \subseteq G\}$ and $(\rho^{\alpha})^{-}(P) = \vee\{\alpha^{-}(m) \text{ / for all } m \in P \subseteq G \}$. For $X \in \vartheta$, the bipolar L-fuzzy subset $(X\rho^{\alpha})^{T} =(((X\rho^{\alpha})^{T})^{+}, ((X\rho^{\alpha})^{T})^{-})$ is defined by

i) $(((X\rho^{\alpha})^{T})^{+})(P) = T(X)(\rho^{\alpha})^{+}(P)$

ii) $(((X\rho^{\alpha})^{T})^{-})(P) = T(X)(\rho^{\alpha})^{-}(P)$, for every $P \in \vartheta$ and for some $t \in T$,

Where $T = \{T(P)/T(P) \in L \text{ and } T(P) \neq 0 \text{ for all } P \in \vartheta\}$ is called a pseudo bipolar L-fuzzy coset of an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ.

Example 4.2

Let $(G,\cdot_{12},\leq)=(\{ 1,5,7,11\},\cdot_{12},\leq)$ be an ℓ-group where G is the non-negative integer relatively prime to 12. Let $\alpha=\{<m, \alpha^{+}(m), \alpha^{-}(m)>: m \in G\}$ be the bipolar L-fuzzy subset of G. The mappings $\alpha^{+} : G \to L^{+}$, $\alpha^{-} : G \to L^{-}$ are defined as,

$\alpha^{+}(1) = 0.9$	$\alpha^{-}(1) = -0.7$
$\alpha^{+}(5) = 0.8$	$\alpha^{-}(5) = -0.4$
$\alpha^{+}(7) = 0.5$	$\alpha^{-}(7) = -0.3$
$\alpha^{+}(11) = 0.5$	$\alpha^{-}(11) = -0.3$

Assume $(\vartheta\cdot_{12}, \subseteq)=(\{P,Q\}\cdot_{12}, \subseteq)=(\{\{1,5\},\{7,11\}\}\cdot_{12}, \subseteq\}$ be an ℓ - HX group.

Let $\rho^{\alpha} =\{(m, (\rho^{\alpha})^{+}(m), (\rho^{\alpha})^{-}(m))\}/\text{for all } m \in \vartheta\}$ be an bipolar L-fuzzy subset of an ℓ-HX group ϑ and the mappings $(\rho^{\alpha})^{+}: \vartheta \to L^{+}$, $(\rho^{\alpha})^{-}: \vartheta \to L^{-}$ are defined as,

$(\rho^{\alpha})^{+}(P) = \vee\{\alpha^{+}(m) \text{ / for all } m \in P \subseteq G\}$ and $(\rho^{\alpha})^{-}(P) = \wedge\{\alpha^{-}(m) \text{ / for all } m \in P \subseteq G\}$

Now,

$(\rho^{\alpha})^{+}(P) = 0.9$	$(\rho^{\alpha})^{-}(P) = -0.7$
$(\rho^{\alpha})^{+}(Q) = 0.5$	$(\rho^{\alpha})^{-}(Q) = -0.3$

Clearly, ρ^{α} is an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ.

Next we have to calculate the pseudo bipolar L-fuzzy cosets of bipolar L-fuzzy sub ℓ-HX group ϑ defined as,

i) $((X\rho^{\alpha})^{T})^{+}(P) = T(X)(\rho^{\alpha})^{+}(P)$

ii) $((X\rho^{\alpha})^{T})^{-}(P) = T(X)(\rho^{\alpha})^{-}(P)$, for every $P \in \vartheta$ and for some $t \in T$,

Where $T = \{T(P)/T(P) \in L \text{ and } T(P) \neq 0 \text{ for all } P \in \vartheta\}$.

Consider T(P)=0.5 and T(Q)=0.8

Now,

$((P\rho^{\alpha})^{T})^{+}(P) = T(P)(\rho^{\alpha})^{+}(P) = (0.5)(0.9) = 0.45$

$((P\rho^{\alpha})^{T})^{+}(Q) = T(P)(\rho^{\alpha})^{+}(Q) = (0.5)(0.5) = 0.25$

$((Q\rho^{\alpha})^{T})^{+}(P) = T(Q)(\rho^{\alpha})^{+}(P) = (0.8)(0.9) = 0.72$

$((Q\rho^{\alpha})^{T})^{+}(Q) = T(Q)(\rho^{\alpha})^{+}(Q) = (0.8)(0.5) = 0.40$

$((P\rho^{\alpha})^{T})^{-}(P) = T(P)(\rho^{\alpha})^{-}(P) = (0.5)(-0.7) = -0.35$

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$((P\rho^{\alpha})^{T})^{-}(Q) = T(P)(\rho^{\alpha})^{-}(Q) = (0.5)(-0.3) = -0.15$
$((Q\rho^{\alpha})^{T})^{-}(P) = T(Q)(\rho^{\alpha})^{-}(P) = (0.8)(-0.7) = -0.56$
$(Q\rho^{\alpha})^{T})^{-}(Q) = T(Q)(\rho^{\alpha})^{-}(Q) = (0.8)(-0.3) = -0.24$

Hence, we can defined $(X\rho^{\alpha})^{T}$ is an pseudo bipolar L-fuzzy coset of an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ.

Theorem 4.3

Let $\rho^{\alpha} =((\rho^{\alpha})^{+}, (\rho^{\alpha})^{-})$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group of ϑ. Then the pseudo bipolar L-fuzzy coset $(X\rho^{\alpha})^{T} = (((X\rho^{\alpha})^{T})^{+}(P), ((X\rho^{\alpha})^{T})^{-}(P))$ is an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group of ϑ.

Proof

Let $\rho^{\alpha} =((\rho^{\alpha})^{+}, (\rho^{\alpha})^{-})$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ.

For every $P,Q\in\vartheta$

i) $(X\rho^{\alpha})^{T})^{+}(PQ^{-1}) = T(X)((\rho^{\alpha})^{+})(PQ^{-1})$
$\geq T(X)(((\rho^{\alpha})^{+})(P)\wedge((\rho^{\alpha})^{+})(Q))$
$= T(X)(\rho^{\alpha})^{+}(P)\wedge T(X)(\rho^{\alpha})^{+}(Q)$
$= ((X\rho^{\alpha})^{T})^{+}(P)\wedge((X\rho^{\alpha})^{T})^{+}(Q)$

$(X\rho^{\alpha})^{T})^{+}(PQ^{-1}) \geq ((X\rho^{\alpha})^{T})^{+}(P)\wedge((X\rho^{\alpha})^{T})^{+}(Q)$

ii) $(X\rho^{\alpha})^{T})^{-}(PQ^{-1}) = T(X)((\rho^{\alpha})^{-})(PQ^{-1})$
$\leq T(X)(((\rho^{\alpha})^{-})(P)\vee((\rho^{\alpha})^{-})(Q))$
$= T(X)((\rho^{\alpha})^{-})(P)\vee T(X)((\rho^{\alpha})^{-})(Q)$
$= ((X\rho^{\alpha})^{T})^{-}(P)\vee((X\rho^{\alpha})^{T})^{-}(Q)$

$(X\rho^{\alpha})^{T})^{-}(PQ^{-1}) \leq ((X\rho^{\alpha})^{T})^{-}(P)\vee((X\rho^{\alpha})^{T})^{-}(Q)$

iii) $(X\rho^{\alpha})^{T})^{+}(P\vee Q) = T(X)(\rho^{\alpha})^{+}(P\vee Q)$
$\geq T(X)((\rho^{\alpha})^{+}(P)\wedge(\rho^{\alpha})^{+}(Q))$
$= T(X)(\rho^{\alpha})^{+}(P)\wedge T(X)(\rho^{\alpha})^{+}(Q)$
$= ((X\rho^{\alpha})^{T})^{+}(P)\wedge((X\rho^{\alpha})^{T})^{+}(Q)$

$(X\rho^{\alpha})^{T})^{+}(P\vee Q) \geq ((X\rho^{\alpha})^{T})^{+}(P)\wedge((X\rho^{\alpha})^{T})^{+}(Q)$

iv) $(X\rho^{\alpha})^{T})^{-}(P\vee Q) = T(X)(\rho^{\alpha})^{-}(P\vee Q)$
$\leq T(X)((\rho^{\alpha})^{-}(P)\vee(\rho^{\alpha})^{-}(Q))$
$= T(X)(\rho^{\alpha})^{-}(P)\vee T(X)(\rho^{\alpha})^{-}(Q)$
$= ((X\rho^{\alpha})^{T})^{-}(P)\vee((X\rho^{\alpha})^{T})^{-}(Q)$

$(X\rho^{\alpha})^{T})^{-}(P\vee Q) \leq ((X\rho^{\alpha})^{T})^{-}(P)\vee((X\rho^{\alpha})^{T})^{-}(Q)$

v) $(X\rho^{\alpha})^{T})^{+}(P\wedge Q) = T(X)(\rho^{\alpha})^{+}(P\wedge Q)$
$\geq T(X)((\rho^{\alpha})^{+}(P)\wedge(\rho^{\alpha})^{+}(Q))$
$= T(X)(\rho^{\alpha})^{+}(P)\wedge T(X)(\rho^{\alpha})^{+}(Q)$
$= ((X\rho^{\alpha})^{T})^{+}(P)\wedge((X\rho^{\alpha})^{T})^{+}(Q)$

$(X\rho^{\alpha})^{T})^{+}(P\wedge Q) \geq ((X\rho^{\alpha})^{T})^{+}(P)\wedge((X\rho^{\alpha})^{T})^{+}(Q)$

vi) $(X\rho^{\alpha})^{T})^{-}(P\wedge Q) = T(X)(\rho^{\alpha})^{-}(P\wedge Q)$
$\leq T(X)((\rho^{\alpha})^{-}(P)\vee(\rho^{\alpha})^{-}(Q))$
$= T(X)(\rho^{\alpha})^{-}(P)\vee T(X)(\rho^{\alpha})^{-}(Q)$
$= ((X\rho^{\alpha})^{T})^{-}(P)\vee((X\rho^{\alpha})^{T})^{-}(Q)$

$(X\rho^{\alpha})^{T})^{-}(P\wedge Q) \leq ((X\rho^{\alpha})^{T})^{-}(P)\vee((X\rho^{\alpha})^{T})^{-}(Q)$

Hence, $(X\rho^{\alpha})^{T}$ is an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ.

Definition 4.4

Let $\alpha = (\alpha^{+},\alpha^{-})$ be an bipolar anti L-fuzzy subset of an ℓ-group G and Let $\vartheta\subset 2^{G}-\{\phi\}$ be an ℓ-HX group of an G. Consider $\rho^{\alpha} = ((\rho^{\alpha})^{+},(\rho^{\alpha})^{-})$ be an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ, where $(\rho^{\alpha})^{+}(P) = \vee\{\alpha^{+}(m)$ / for all $m\in P\subseteq G\}$ and $(\rho^{\alpha})^{-}(P) = \wedge\{\alpha^{-}(m)$ / for all $m\in P\subseteq G\}$. For $X\in\vartheta$, the bipolar L-fuzzy subset $(X\rho^{\alpha})^{T} =(((X\rho^{\alpha})^{T})^{+}, ((X\rho^{\alpha})^{T})^{-})$ is defined by

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i) $(((X\rho^{\alpha})^{T})^{+})(P) = T(X)(\rho^{\alpha})^{+}(P)$

ii) $(((X\rho^{\alpha})^{T})^{-})(P) = T(X)(\rho^{\alpha})^{-}(P)$, for every $P\in\vartheta$ and for some $t\in T$,

Where $T = \{T(P)/T(P) \in L$ and $T(P)\neq 0$ for all $P\in\vartheta\}$ is called an pseudo bipolar L-fuzzy coset of an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ.

Example 4.5

Let $(G,._8,\leq)=(\{1,3,5,7\},._8,\leq)$ be an ℓ-group where G is the non-negative integer relatively prime to 8. Let $\alpha=\{\langle m, \alpha^{+}(m), \alpha^{-}(m)\rangle: m\in G\}$ be the bipolar L-fuzzy subset of G. The mappings $\alpha^{+} : G \to L^{+}$, $\alpha^{-} : G \to L^{-}$ are defined as,

$\alpha^{+}(1) = 0.4$	$\alpha^{-}(1) = -0.3$
$\alpha^{+}(3) = 0.6$	$\alpha^{-}(3) = -0.8$
$\alpha^{+}(5) = 0.7$	$\alpha^{-}(5) = -0.9$
$\alpha^{+}(7) = 0.7$	$\alpha^{-}(7) = -0.9$

Assume $(\vartheta._8, \subseteq)=(\{P,Q\}._8, \subseteq)=(\{\{1,3\},\{5,7\}\}._8, \subseteq\}$ be an ℓ - HX group.
Let $\rho^{\alpha} =\{(m, (\rho^{\alpha})^{+}(m), (\rho^{\alpha})^{-}(m))\}$/for all $m\in \vartheta\}$ is an bipolar L-fuzzy subset of ϑ and the mappings $(\rho^{\alpha})^{+}: \vartheta\to L^{+}$, $(\rho^{\alpha})^{-}: \vartheta\to L^{-}$ are defined as,
$(\rho^{\alpha})^{+}(P) = \vee\{\alpha^{+}(m)$ / for all $m\in P\subseteq G\}$ and $(\rho^{\alpha})^{-}(P) = \wedge\{\alpha^{-}(m)$ / for all $m\in P \subseteq G\}$
Now,

$(\rho^{\alpha})^{+}(P) = 0.6$	$(\rho^{\alpha})^{-}(P) = -0.8$
$(\rho^{\alpha})^{+}(Q) = 0.7$	$(\rho^{\alpha})^{-}(Q) = -0.9$

Clearly, ρ^{α} is an bipolar anti L-fuzzy sub ℓ-HX group of ϑ.
Next we have to define the pseudo bipolar L-fuzzy cosets of bipolar anti L-fuzzy sub ℓ-HX group ϑ defined as,

i) $((X\rho^{\alpha})^{T})^{+}(P) = T(X)(\rho^{\alpha})^{+}(P)$

ii) $((X\rho^{\alpha})^{T})^{-}(P) = T(X)(\rho^{\alpha})^{-}(P)$, for every $P\in\vartheta$ and for some $t\in T$,

Where $T = \{T(P)/T(P) \in L$ and $T(P)\neq 0$ for all $P\in\vartheta\}$.
Consider $T(P)=0.7$ and $T(Q)=0.4$
Now,

$$((P\rho^{\alpha})^{T})^{+}(P) = T(P)(\rho^{\alpha})^{+}(P) = (0.7)(0.6) = 0.42$$
$$((P\rho^{\alpha})^{T})^{+}(Q) = T(P)(\rho^{\alpha})^{+}(Q) = (0.7)(0.7) = 0.49$$
$$((Q\rho^{\alpha})^{T})^{+}(P) = T(Q)(\rho^{\alpha})^{+}(P) = (0.4)(0.6) = 0.24$$
$$((Q\rho^{\alpha})^{T})^{+}(Q) = T(Q)(\rho^{\alpha})^{+}(Q) = (0.4)(0.7) = 0.28$$
$$((P\rho^{\alpha})^{T})^{-}(P) = T(P)(\rho^{\alpha})^{-}(P) = (0.7)(-0.8) = -0.56$$
$$((P\rho^{\alpha})^{T})^{-}(Q) = T(P)(\rho^{\alpha})^{-}(Q) = (0.7)(-0.9) = -0.63$$
$$((Q\rho^{\alpha})^{T})^{-}(P) = T(Q)(\rho^{\alpha})^{-}(P) = (0.4)(-0.8) = -0.32$$
$$((Q\rho^{\alpha})^{T})^{-}(Q) = T(Q)(\rho^{\alpha})^{-}(Q) = (0.4)(-0.9) = -0.36$$

Hence, we can defined $(X\rho^{\alpha})^{T}$ is an pseudo bipolar L-fuzzy coset of an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group of ϑ.

Theorem 4.6

Let $\rho^{\alpha} =((\rho^{\alpha})^{+}, (\rho^{\alpha})^{-})$ be an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ. Then the pseudo bipolar L-fuzzy coset $(X\rho^{\alpha})^{T} = (((X\rho^{\alpha})^{T})^{+}(P), ((X\rho^{\alpha})^{T})^{-}(P))$ is an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ.

Proof

Let $\rho^{\alpha} =((\rho^{\alpha})^{+}, (\rho^{\alpha})^{-})$ be an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ.
For every $P,Q\in\vartheta$

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i) $((X\rho^{\alpha})^{T})^{+}(PQ^{-1}) = T(X)((\rho^{\alpha})^{+})(PQ^{-1})$
$\leq T(X)(((\rho^{\alpha})^{+})(P)\vee((\rho^{\alpha})^{+})(Q))$
$= T(X)((\rho^{\alpha})^{+})(P)\vee T(X)((\rho^{\alpha})^{+})(Q)$
$= ((X\rho^{\alpha})^{T})^{+}(P)\vee((X\rho^{\alpha})^{T})^{+}(Q)$
$((X\rho^{\alpha})^{T})^{+}(PQ^{-1}) \leq ((X\rho^{\alpha})^{T})^{+}(P)\vee((X\rho^{\alpha})^{T})^{+}(Q)$

ii) $((X\rho^{\alpha})^{T})^{-}(PQ^{-1}) = T(X)\,((\rho^{\alpha})^{-})(PQ^{-1})$
$\geq T(X)\,(((\rho^{\alpha})^{-})(P)\wedge((\rho^{\alpha})^{-})(Q))$
$= T(X)\,(\rho^{\alpha})^{-}(P)\wedge T(X)(\rho^{\alpha})^{-}(Q)$
$= ((X\rho^{\alpha})^{T})^{-}(P)\wedge((X\rho^{\alpha})^{T})^{-}(Q)$
$((X\rho^{\alpha})^{T})^{-}(PQ^{-1}) \geq ((X\rho^{\alpha})^{T})^{-}(P)\wedge((X\rho^{\alpha})^{T})^{-}(Q)$

iii) $((X\rho^{\alpha})^{T})^{+}(P\vee Q) = T(X)(\rho^{\alpha})^{+}(P\vee Q)$
$\leq T(X)((\rho^{\alpha})^{+}(P)\vee(\rho^{\alpha})^{+}(Q))$
$= T(X)(\rho^{\alpha})^{+}(P)\vee T(X)(\rho^{\alpha})^{+}(Q)$
$= ((X\rho^{\alpha})^{T})^{+}(P)\vee((X\rho^{\alpha})^{T})^{+}(Q)$
$((X\rho^{\alpha})^{T})^{+}(P\vee Q) \leq ((X\rho^{\alpha})^{T})^{+}(P)\vee((X\rho^{\alpha})^{T})^{+}(Q)$

iv) $((X\rho^{\alpha})^{T})^{-}(P\vee Q) = T(X)(\rho^{\alpha})^{-}(P\vee Q)$
$\geq T(X)((\rho^{\alpha})^{-}(P)\wedge(\rho^{\alpha})^{-}(Q))$
$= T(X)(\rho^{\alpha})^{-}(P)\wedge T(X)(\rho^{\alpha})^{-}(Q)$
$= ((X\rho^{\alpha})^{T})^{-}(P)\wedge((X\rho^{\alpha})^{T})^{-}(Q)$
$((X\rho^{\alpha})^{T})^{-}(P\vee Q) \geq ((X\rho^{\alpha})^{T})^{-}(P)\wedge((X\rho^{\alpha})^{T})^{-}(Q)$

v) $((X\rho^{\alpha})^{T})^{+}(P\wedge Q) = T(X)(\rho^{\alpha})^{+}(P\wedge Q)$
$\leq T(X)((\rho^{\alpha})^{+}(P)\vee(\rho^{\alpha})^{+}(Q))$
$= T(X)(\rho^{\alpha})^{+}(P)\vee T(X)(\rho^{\alpha})^{+}(Q)$
$= ((X\rho^{\alpha})^{T})^{+}(P)\vee((X\rho^{\alpha})^{T})^{+}(Q)$
$((X\rho^{\alpha})^{T})^{+}(P\wedge Q) \leq ((X\rho^{\alpha})^{T})^{+}(P)\vee((X\rho^{\alpha})^{T})^{+}(Q)$

vi) $((X\rho^{\alpha})^{T})^{-}(P\wedge Q) = T(X)(\rho^{\alpha})^{-}(P\wedge Q)$
$\geq T(X)((\rho^{\alpha})^{-}(P)\wedge(\rho^{\alpha})^{-}(Q))$
$= T(X)(\rho^{\alpha})^{-}(P)\wedge T(X)(\rho^{\alpha})^{-}(Q)$
$= ((X\rho^{\alpha})^{T})^{-}(P)\wedge((X\rho^{\alpha})^{T})^{-}(Q)$
$((X\rho^{\alpha})^{T})^{-}(P\wedge Q) \geq ((X\rho^{\alpha})^{T})^{-}(P)\wedge((X\rho^{\alpha})^{T})^{-}(Q)$

Hence, $(X\rho^{\alpha})^{T}$ is an bipolar anti L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ.

Theorem 4.7

Consider G be an ℓ-group. Let $\alpha = (\alpha^{+}, \alpha^{-})$ and $\beta = (\beta^{+}, \beta^{-})$ be bipolar L-fuzzy subsets of G. Let $\vartheta\subset 2^{G}-\{\phi\}$ be an ℓ-HX group of G. Let $\rho^{\alpha} = ((\rho^{\alpha})^{+}, (\rho^{\alpha})^{-})$ and $\omega^{\beta} = ((\omega^{\beta})^{+}, (\omega^{\beta})^{-})$ be bipolar L-fuzzy subsets of ϑ. Let $(X\rho^{\alpha})^{T} = (((X\rho^{\alpha})^{T})^{+}, ((X\rho^{\alpha})^{T})^{-})$ and $(Y\omega^{\beta})^{T} = (((Y\omega^{\beta})^{T})^{+}, (Y\omega^{\beta})^{T})^{-})$ be pseudo bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group determined by the element $X,Y\in\vartheta$. Then $((X\rho^{\alpha})^{T}\cap(Y\omega^{\beta})^{T}) = (((X\rho^{\alpha})^{T}\cap(Y\omega^{\beta})^{T})^{+}, ((X\rho^{\alpha})^{T}\cap(Y\omega^{\beta})^{T})^{-})$ is an pseudo bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ.

Proof

Let $(X\rho^{\alpha})^{T} = (((X\rho^{\alpha})^{T})^{+},((X\rho^{\alpha})^{T})^{-})$ and $(Y\omega^{\beta})^{T} = (((Y\omega^{\beta})^{T})^{+}, (Y\omega^{\beta})^{T})^{-})$ be pseudo bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group determined by the element $X,Y\in\vartheta$. Let $((X\rho^{\alpha})^{T}\cap(Y\omega^{\beta})^{T})=(((X\rho^{\alpha})^{T}\cap (Y\omega^{\beta})^{T})^{+}, ((X\rho^{\alpha})^{T}\cap(Y\omega^{\beta})^{T})^{-})$
For all $X,Y\in\vartheta$,

i) $((X\rho^{\alpha})^{T}\cap(Y\omega^{\beta})^{T})^{+}(PQ^{-1})= ((X\rho^{\alpha})^{T})^{+}(PQ^{-1})\wedge((Y\omega^{\beta})^{T})^{+}(PQ^{-1})$
$= T(X)((\rho^{\alpha})^{+}(PQ^{-1}))\wedge T(Y)((\omega^{\beta})^{+}(PQ^{-1}))$
$\geq (T(X)((\rho^{\alpha})^{+}(P)\wedge(\rho^{\alpha})^{+}(Q))\wedge(T(Y)((\omega^{\beta})^{+}(P)\wedge(\omega^{\beta})^{+}(Q)))$
$= (T(X)(\rho^{\alpha})^{+}(P)\wedge T(X)(\rho^{\alpha})^{+}(Q))\wedge(T(Y)(\omega^{\beta})^{+}(P)\wedge T(Y)(\omega^{\beta})^{+}(Q))$
$= T(X)\,(\rho^{\alpha})^{+}(P)\wedge T(Y)(\omega^{\beta})^{+}(P)\wedge T(X)(\rho^{\alpha})^{+}(Q)\wedge T(Y)(\omega^{\beta})^{+}(Q)$
$= (((X\rho^{\alpha})^{T})^{+}(P)\wedge((Y\omega^{\beta})^{T})^{+}(P))\wedge(((X\rho^{\alpha})^{T})^{+}(Q)\wedge(Y\omega^{\beta})^{T})^{+}(Q))$
$= (((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{+}(P))\wedge(((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{+}(Q))$
$((X\rho^{\alpha})^{T}\cap(Y\omega^{\beta})^{T})^{+}(PQ^{-1})\geq (((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{+}(P))\wedge(((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{+}(Q))$

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ii)
$$\begin{aligned}
((X\rho^{\alpha})^{T}\cap(Y\omega^{\beta})^{T})^{-}(PQ^{-1}) &= ((X\rho^{\alpha})^{T})^{-}(PQ^{-1})\vee((Y\omega^{\beta})^{T})^{-}(PQ^{-1})\\
&= T(X)((\rho^{\alpha})^{-}(PQ^{-1}))\vee(Y)((\omega^{\beta})^{-}(PQ^{-1}))\\
&\leq (T(X)((\rho^{\alpha})^{-}(P)\vee(\rho^{\alpha})^{-}(Q))\vee(T(Y)((\omega^{\beta})^{-}(P)\vee(\omega^{\beta})^{-}(Q)))\\
&= (T(X)(\rho^{\alpha})^{-}(P)\vee T(X)(\rho^{\alpha})^{-}(Q))\vee(T(Y)(\omega^{\beta})^{-}(P)\vee T(Y)(\omega^{\beta})^{-}(Q))\\
&= T(X)\,(\rho^{\alpha})^{-}(P)\vee T(Y)(\omega^{\beta})^{-}(P)\vee T(X)(\rho^{\alpha})^{-}(Q)\vee T(Y)(\omega^{\beta})^{-}(Q)\\
&= (((X\rho^{\alpha})^{T})^{-}(P)\vee((Y\omega^{\beta})^{T})^{-}(P))\vee(((X\rho^{\alpha})^{T})^{-}(Q)\vee(Y\omega^{\beta})^{T})^{-}(Q))\\
&= (((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{-}(P))\vee(((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{-}(Q))\\
((X\rho^{\alpha})^{T}\cap(Y\omega^{\beta})^{T})^{-}(PQ^{-1}) &\leq (((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{-}(P))\vee(((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{-}(Q))
\end{aligned}$$

iii)
$$\begin{aligned}
((X\rho^{\alpha})^{T}\cap(Y\omega^{\beta})^{T})^{+}(P\vee Q) &= ((X\rho^{\alpha})^{T})^{+}(P\vee Q)\wedge((Y\omega^{\beta})^{T})^{+}(P\vee Q)\\
&= T(X)((\rho^{\alpha})^{+}(P\vee Q))\wedge T(Y)((\omega^{\beta})^{+}(P\vee Q))\\
&\geq (T(X)((\rho^{\alpha})^{+}(P)\wedge(\rho^{\alpha})^{+}(Q))\wedge(T(Y)((\omega^{\beta})^{+}(P)\wedge(\omega^{\beta})^{+}(Q)))\\
&= (T(X)(\rho^{\alpha})^{+}(P)\wedge T(X)(\rho^{\alpha})^{+}(Q))\wedge(T(Y)(\omega^{\beta})^{+}(P)\wedge T(Y)(\omega^{\beta})^{+}(Q))\\
&= T(X)\,(\rho^{\alpha})^{+}(P)\wedge T(Y)(\omega^{\beta})^{+}(P)\wedge T(X)(\rho^{\alpha})^{+}(Q)\wedge T(Y)(\omega^{\beta})^{+}(Q)\\
&= (((X\rho^{\alpha})^{T})^{+}(P)\wedge((Y\omega^{\beta})^{T})^{+}(P))\wedge(((X\rho^{\alpha})^{T})^{+}(Q)\wedge(Y\omega^{\beta})^{T})^{+}(Q))\\
&= (((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{+}(P))\wedge(((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{+}(Q))\\
((X\rho^{\alpha})^{T}\cap(Y\omega^{\beta})^{T})^{+}(P\vee Q) &\geq (((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{+}(P))\wedge(((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{+}(Q))
\end{aligned}$$

iv)
$$\begin{aligned}
((X\rho^{\alpha})^{T}\cap(Y\omega^{\beta})^{T})^{-}(P\vee Q) &= ((X\rho^{\alpha})^{T})^{-}(P\vee Q)\vee((Y\omega^{\beta})^{T})^{-}(P\vee Q)\\
&= T(X)((\rho^{\alpha})^{-}(P\vee Q))\vee T(Y)((\omega^{\beta})^{-}(P\vee Q))\\
&\leq (T(X)((\rho^{\alpha})^{-}(P)\vee(\rho^{\alpha})^{-}(Q))\vee(T(Y)((\omega^{\beta})^{-}(P)\vee(\omega^{\beta})^{-}(Q))\\
&= (T(X)(\rho^{\alpha})^{-}(P)\vee T(X)(\rho^{\alpha})^{-}(Q))\vee(T(Y)(\omega^{\beta})^{-}(P)\vee T(Y)(\omega^{\beta})^{-}(Q))\\
&= T(X)\,(\rho^{\alpha})^{-}(P)\vee T(Y)(\omega^{\beta})^{-}(P)\vee T(X)(\rho^{\alpha})^{-}(Q)\vee T(Y)(\omega^{\beta})^{-}(Q)\\
&= (((X\rho^{\alpha})^{T})^{-}(P)\vee((Y\omega^{\beta})^{T})^{-}(P))\vee(((X\rho^{\alpha})^{T})^{-}(Q)\vee(Y\omega^{\beta})^{T})^{-}(Q))\\
&= (((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{-}(P))\vee(((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{-}(Q))\\
((X\rho^{\alpha})^{T}\cap(Y\omega^{\beta})^{T})^{-}(P\vee Q) &\leq (((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{-}(P))\vee(((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{-}(Q))
\end{aligned}$$

v)
$$\begin{aligned}
((X\rho^{\alpha})^{T}\cap(Y\omega^{\beta})^{T})^{+}(P\wedge Q) &= ((X\rho^{\alpha})^{T})^{+}(P\wedge Q)\wedge((Y\omega^{\beta})^{T})^{+}(P\wedge Q)\\
&= T(X)((\rho^{\alpha})^{+}(P\wedge Q))\wedge T(Y)((\omega^{\beta})^{+}(P\wedge Q))\\
&\geq (T(X)((\rho^{\alpha})^{+}(P)\wedge(\rho^{\alpha})^{+}(Q))\wedge(T(Y)((\omega^{\beta})^{+}(P)\wedge(\omega^{\beta})^{+}(Q))\\
&= (T(X)(\rho^{\alpha})^{+}(P)\wedge T(X)(\rho^{\alpha})^{+}(Q))\wedge(T(Y)(\omega^{\beta})^{+}(P)\wedge T(Y)(\omega^{\beta})^{+}(Q))\\
&= T(X)\,(\rho^{\alpha})^{+}(P)\wedge T(Y)(\omega^{\beta})^{+}(P)\wedge T(X)(\rho^{\alpha})^{+}(Q)\wedge T(Y)(\omega^{\beta})^{+}(Q)\\
&= (((X\rho^{\alpha})^{T})^{+}(P)\wedge((Y\omega^{\beta})^{T})^{+}(P))\wedge(((X\rho^{\alpha})^{T})^{+}(Q)\wedge(Y\omega^{\beta})^{T})^{+}(Q))\\
&= (((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{+}(P))\wedge(((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{+}(Q))\\
((X\rho^{\alpha})^{T}\cap(Y\omega^{\beta})^{T})^{+}(P\wedge Q) &\geq (((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{+}(P))\wedge(((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{+}(Q))
\end{aligned}$$

vi)
$$\begin{aligned}
((X\rho^{\alpha})^{T}\cap(Y\omega^{\beta})^{T})^{-}(P\wedge Q) &= ((X\rho^{\alpha})^{T})^{-}(P\wedge Q)\vee((Y\omega^{\beta})^{T})^{-}(P\wedge Q)\\
&= T(X)((\rho^{\alpha})^{-}(P\wedge Q))\vee T(Y)((\omega^{\beta})^{-}(P\wedge Q))\\
&\leq (T(X)((\rho^{\alpha})^{-}(P)\vee(\rho^{\alpha})^{-}(Q))\vee(T(Y)((\omega^{\beta})^{-}(P)\vee(\omega^{\beta})^{-}(Q))\\
&= (T(X)(\rho^{\alpha})^{-}(P)\vee T(X)(\rho^{\alpha})^{-}(Q))\vee(T(Y)(\omega^{\beta})^{-}(P)\vee T(Y)(\omega^{\beta})^{-}(Q))\\
&= T(X)\,(\rho^{\alpha})^{-}(P)\vee T(Y)(\omega^{\beta})^{-}(P)\vee T(X)(\rho^{\alpha})^{-}(Q)\vee T(Y)(\omega^{\beta})^{-}(Q)\\
&= (((X\rho^{\alpha})^{T})^{-}(P)\vee((Y\omega^{\beta})^{T})^{-}(P))\vee(((X\rho^{\alpha})^{T})^{-}(Q)\vee(Y\omega^{\beta})^{T})^{-}(Q))\\
&= (((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{-}(P))\vee(((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{-}(Q))\\
((X\rho^{\alpha})^{T}\cap(Y\omega^{\beta})^{T})^{-}(P\vee Q) &\leq (((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{-}(P))\vee(((X\rho^{\alpha})^{T})\cap(Y\omega^{\beta})^{T}))^{-}(Q))
\end{aligned}$$

Hence, $((X\rho^{\alpha})^{T}\cap(Y\omega^{\beta})^{T})=(((X\rho^{\alpha})^{T}\cap\ (Y\omega^{\beta})^{T})^{+},\ ((X\rho^{\alpha})^{T}\cap(Y\omega^{\beta})^{T})^{-})$ is an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ.

V. Properties of pseudo bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group homomorphism and ℓ-HX group anti homomorphism.

Theorem 5.1

Let ϑ_1 and ϑ_2 be any two ℓ-HX groups on G_1 and G_2. Let $\rho^{\alpha}=((\rho^{\alpha})^{+},(\rho^{\alpha})^{-})$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ_1. Let $\varphi: \vartheta_1 \rightarrow \vartheta_2$ be an onto ℓ-HX group homomorphism. Let $(X\rho^{\alpha})^{T} = (((X\rho^{\alpha})^{T})^{+},((X\rho^{\alpha})^{T})^{-})$ be an pseudo bipolar L-fuzzy coset of an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group determined by the element $X\in\vartheta_1$.

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Then $\varphi(X\rho^{\alpha})=((\varphi(X\rho^{\alpha}))^{+},(\varphi(X\rho^{\alpha}))^{-})$ is a pseudo bipolar L-fuzzy coset of an bipolar L-fuzzy sub ℓ-HX group $\varphi(\rho^{\alpha}) = ((\varphi(\rho^{\alpha}))^{+},(\varphi(\rho^{\alpha}))^{-})$ of an ℓ-HX group determined by the element $\varphi(X)\in\vartheta_2$ and $\varphi((X\rho^{\alpha})^{T})=(\varphi(X)\ \varphi(\rho^{\alpha}))^{T}$ if $\rho^{\alpha}=((\rho^{\alpha})^{+},(\rho^{\alpha})^{-})$ has supremum property and $\rho^{\alpha} = ((\rho^{\alpha})^{+},(\rho^{\alpha})^{-})$ is φ-invariant.

Proof

Let $\rho^{\alpha}=((\rho^{\alpha})^{+},(\rho^{\alpha})^{-})$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ and let $(X\rho^{\alpha})^{T} = (((X\rho^{\alpha})^{T})^{+},((X\rho^{\alpha})^{T})^{-})$ be an pseudo bipolar L-fuzzy coset of an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group determined by the element $X\in\vartheta_1$. Clearly, $\varphi(\rho^{\alpha})=(\varphi(\rho^{\alpha})^{+}, \varphi(\rho^{\alpha})^{-})$ is an bipolar L-fuzzy sub ℓ-HX group of ϑ_2 and $(\varphi((X\rho^{\alpha})^{T}))= ((\varphi((X\rho^{\alpha})^{T}))^{+},(\varphi((X\rho^{\alpha})^{T}))^{-})$ is an pseudo bipolar L-fuzzy coset of an bipolar L-fuzzy sub ℓ-HX group $\varphi(\rho^{\alpha})=(\varphi(\rho^{\alpha})^{+},\varphi(\rho^{\alpha})^{-})$ of an ℓ-HX group ϑ_2 determined by the element $\varphi(X)\in\vartheta_2$.

For any $P\in\vartheta_1$, $\varphi(P)\in\vartheta_2$

i) $$((\varphi(X)\ \varphi(\rho^{\alpha}))^{T})^{+}(\varphi(P)) = T(\varphi(X))(\varphi(\rho^{\alpha}))^{+}(\varphi(P))$$
$$= T(X)\ (\rho^{\alpha})^{+}(P)$$
$$= ((X\rho^{\alpha})^{T})^{+}(P)$$
$$= (\varphi(X\rho^{\alpha})^{T})^{+}(\varphi(P))$$
$$((\varphi(X)\ \varphi(\rho^{\alpha}))^{T})^{+}(\varphi(P)) = (\varphi(X\rho^{\alpha})^{T})^{+}(\varphi(P))$$

ii) $$((\varphi(X)\ \varphi(\rho^{\alpha}))^{T})^{-}(\varphi(P)) = T(\varphi(X))(\varphi(\rho^{\alpha}))^{-}(\varphi(P))$$
$$= T(X)(\rho^{\alpha})^{-}(P)$$
$$= ((X\rho^{\alpha})^{T})^{-}(P)$$
$$= (\varphi(X\rho^{\alpha})^{T})^{-}(\varphi(P))$$
$$((\varphi(X)\ \varphi(\rho^{\alpha}))^{T})^{-}(\varphi(P)) = (\varphi(X\rho^{\alpha})^{T})^{-}(\varphi(P))$$

Hence, $\varphi((X\rho^{\alpha})^{T}) = (\varphi(X)\ \varphi(\rho^{\alpha}))^{T}$.

Theorem 5.2

Let ϑ_1 and ϑ_2 be any two ℓ-HX groups on G_1 and G_2. Let $\omega^{\beta}=((\omega^{\beta})^{+},(\omega^{\beta})^{-})$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ_2. Let $\varphi:\vartheta_1 \rightarrow \vartheta_2$ be an onto ℓ - HX group homomorphism. Let $(Y\omega^{\beta})^{T} = (((Y\omega^{\beta})^{T})^{+},((Y\omega^{\beta})^{T})^{-})$ be an pseudo bipolar L-fuzzy coset of an bipolar L-fuzzy sub ℓ-HX group ω^{β} of an ℓ-HX group determined by the element $Y\in\vartheta_2$. Then $\varphi^{-1}(Y\omega^{\beta})=((\varphi^{-1}(Y\omega^{\beta}))^{+},(\varphi^{-1}(Y\omega^{\beta}))^{-})$ is a pseudo bipolar L-fuzzy coset of an bipolar L-fuzzy sub ℓ-HX group $\varphi^{-1}(\omega^{\beta}) = ((\varphi^{-1}(\omega^{\beta}))^{+},(\varphi^{-1}(\omega^{\beta}))^{-})$ of an ℓ-HX group determined by the element $\varphi^{-1}(Y)\in\vartheta_1$ and $\varphi^{-1}((Y\omega^{\beta})^{T})=(\varphi^{-1}(Y)\ \varphi^{-1}(\omega^{\beta}))^{T}$.

Proof

Let $\omega^{\beta}=((\omega^{\beta})^{+},(\omega^{\beta})^{-})$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ_2 and let $(Y\omega^{\beta})^{T} = (((Y\omega^{\beta})^{T})^{+},((Y\omega^{\beta})^{T})^{-})$ be an pseudo bipolar L-fuzzy coset of an bipolar L-fuzzy sub ℓ-HX group ω^{β} of an ℓ-HX group determined by the element $Y\in\vartheta_2$. Clearly, $\varphi^{-1}(\omega^{\beta}) = (\varphi^{-1}(\omega^{\beta})^{+}, \varphi^{-1}(\omega^{\beta})^{-})$ is an bipolar L-fuzzy sub ℓ-HX group of ϑ_1 and $(\varphi^{-1}((Y\omega^{\beta})^{T})) = ((\varphi^{-1}((Y\omega^{\beta})^{T}))^{+},(\varphi^{-1}((Y\omega^{\beta})^{T}))^{-})$ is an pseudo bipolar L-fuzzy coset of an bipolar L-fuzzy sub ℓ-HX group $\varphi^{-1}(\omega^{\beta})=(\varphi^{-1}(\omega^{\beta})^{+},\varphi^{-1}(\omega^{\beta})^{-})$ of an ℓ-HX group determined by the element $\varphi^{-1}(Y)\in\vartheta_1$.

For any $P\in\vartheta_1$, $\varphi(P)\in\vartheta_2$

i) $$((\varphi^{-1}(Y)\ \varphi^{-1}(\omega^{\beta}))^{T})^{+}(P) = T(\varphi^{-1}(Y))(\varphi^{-1}(\omega^{\beta}))^{+}(P)$$
$$= T(Y)\ (\omega^{\beta})^{+}(\varphi(P))$$
$$= ((Y\omega^{\beta})^{T})^{+}(\varphi(P))$$
$$= (\varphi^{-1}(Y\omega^{\beta})^{T})^{+}(P)$$
$$((\varphi^{-1}(Y)\ \varphi^{-1}(\omega^{\beta}))^{T})^{+}(P) = (\varphi^{-1}(Y\omega^{\beta})^{T})^{+}(P)$$

ii) $$((\varphi^{-1}(Y)\ \varphi^{-1}(\omega^{\beta}))^{T})^{-}(P) = T(\varphi^{-1}(Y))(\varphi^{-1}(\omega^{\beta}))^{-}(P)$$
$$= T(Y)(\omega^{\beta})^{-}(\varphi(P))$$
$$= (Y\omega^{\beta})^{T})^{-}(\varphi(P))$$
$$= (\varphi^{-1}(Y\omega^{\beta})^{T})^{-}(P)$$
$$((\varphi^{-1}(Y)\ \varphi^{-1}(\omega^{\beta}))^{T})^{-}(P) = (\varphi^{-1}(Y\omega^{\beta})^{T})^{-}(P)$$

Hence, $\varphi^{-1}((Y\omega^{\beta})^{T}) = (\varphi^{-1}(Y)\ \varphi^{-1}(\omega^{\beta}))^{T}$.

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Theorem 5.3

Let ϑ_1 and ϑ_2 be any two ℓ-HX groups on G_1 and G_2. Let $\rho^\alpha=((\rho^\alpha)^+, (\rho^\alpha)^-)$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ_1. Let $\varphi: \vartheta_1 \rightarrow \vartheta_2$ be an onto ℓ - HX group anti homomorphism. Let $(X\rho^\alpha)^T = (((X\rho^\alpha)^T)^+, ((X\rho^\alpha)^T)^-)$ be an pseudo bipolar L-fuzzy coset of an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group determined by the element $X\in\vartheta_1$. Then $\varphi(X\rho^\alpha)=((\varphi(X\rho^\alpha))^+, (\varphi(X\rho^\alpha))^-)$ is a pseudo bipolar L-fuzzy coset of an bipolar L-fuzzy sub ℓ-HX group $\varphi(\rho^\alpha)=((\varphi(\rho^\alpha))^+, (\varphi(\rho^\alpha))^-)$ of an ℓ-HX group determined by the element $\varphi(X)\in\vartheta_2$ and $\varphi((X\rho^\alpha)^T)=(\varphi(X)\ \varphi(\rho^\alpha))^T$ if $\rho^\alpha=((\rho^\alpha)^+, (\rho^\alpha)^-)$ has supremum property and $\rho^\alpha = ((\rho^\alpha)^+, (\rho^\alpha)^-)$ is φ-invariant.

Proof

Let $\rho^\alpha=((\rho^\alpha)^+,(\rho^\alpha)^-)$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ and let $(X\rho^\alpha)^T = (((X\rho^\alpha)^T)^+,((X\rho^\alpha)^T)^-)$ be an pseudo bipolar L-fuzzy coset of an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group determined by the element $X\in\vartheta_1$. Clearly, $\varphi(\rho^\alpha)=(\varphi(\rho^\alpha)^+, \varphi(\rho^\alpha)^-)$ is an bipolar L-fuzzy sub ℓ-HX group of ϑ_2 and $(\varphi((X\rho^\alpha)^T))= ((\varphi((X\rho^\alpha)^T))^+ , (\varphi((X\rho^\alpha)^T))^-)$ is an pseudo bipolar L-fuzzy coset of an bipolar L-fuzzy sub ℓ-HX group $\varphi(\rho^\alpha)=(\varphi(\rho^\alpha)^+,\varphi(\rho^\alpha)^-)$ of an ℓ-HX group ϑ_2 determined by the element $\varphi(X)\in\vartheta_2$.

For any $P\in\vartheta_1$, $\varphi(P)\in\vartheta_2$

i) $((\varphi(X)\ \varphi(\rho^\alpha))^T)^+(\varphi(P)) = T(\varphi(X))(\varphi(\rho^\alpha))^+(\varphi(P))$
$= T(X)\ (\rho^\alpha)^+(P)$
$= ((X\rho^\alpha)^T)^+(P)$
$= (\varphi(X\rho^\alpha)^T)^+(\varphi(P))$

$((\varphi(X)\ \varphi(\rho^\alpha))^T)^+(\varphi(P)) = (\varphi(X\rho^\alpha)^T)^+(\varphi(P))$

ii) $((\varphi(X)\ \varphi(\rho^\alpha))^T)^-(\varphi(P)) = T(\varphi(X))(\varphi(\rho^\alpha))^-(\varphi(P))$
$= T(X)(\rho^\alpha)^-(P)$
$= ((X\rho^\alpha)^T)^-(P)$
$= (\varphi(X\rho^\alpha)^T)^-(\varphi(P))$

$((\varphi(X)\ \varphi(\rho^\alpha))^T)^-(\varphi(P)) = (\varphi(X\rho^\alpha)^T)^-(\varphi(P))$

Hence, $(\varphi(X)\ \varphi(\rho^\alpha))^T = \varphi((X\rho^\alpha)^T)$.

Theorem 5.4

Let ϑ_1 and ϑ_2 be any two ℓ-HX groups on G_1 and G_2. Let $\omega^\beta=((\omega^\beta)^+, (\omega^\beta)^-)$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ_2. Let $\varphi^{-1}: \vartheta_1 \rightarrow \vartheta_2$ be an onto ℓ - HX group anti homomorphism. Let $(Y\omega^\beta)^T = (((Y\omega^\beta)^T)^+, ((Y\omega^\beta)^T)^-)$ be an pseudo bipolar L-fuzzy coset of an bipolar L-fuzzy sub ℓ-HX group ω^β of an ℓ-HX group determined by the element $Y\in\vartheta_2$. Then $\varphi^{-1}(Y\omega^\beta)=((\varphi^{-1}(Y\omega^\beta))^+, (\varphi^{-1}(Y\omega^\beta))^-)$ is a pseudo bipolar L-fuzzy coset of an bipolar L-fuzzy sub ℓ-HX group $\varphi^{-1}(\omega^\beta) = ((\varphi^{-1}(\omega^\beta))^+, (\varphi^{-1}(\omega^\beta))^-)$ of an ℓ-HX group determined by the element $\varphi^{-1}(Y)\in\vartheta_1$ and $\varphi^{-1}((Y\omega^\beta)^T)=(\varphi^{-1}(Y)\ \varphi^{-1}(\omega^\beta))^T$.

Proof

Let $\omega^\beta=((\omega^\beta)^+, (\omega^\beta)^-)$ be an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group ϑ_2 and let $(Y\omega^\beta)^T = (((Y\omega^\beta)^T)^+, ((Y\omega^\beta)^T)^-)$ be an pseudo bipolar L-fuzzy coset of an bipolar L-fuzzy sub ℓ-HX group of an ℓ-HX group determined by the element $Y\in\vartheta_2$. Clearly, $\varphi^{-1}(\omega^\beta) = (\varphi^{-1}(\omega^\beta)^+,\varphi^{-1}(\omega^\beta)^-)$ is an bipolar L-fuzzy sub ℓ-HX group of ϑ_1 and $(\varphi^{-1}((Y\omega^\beta)^T))=((\varphi^{-1}((Y\omega^\beta)^T))^+, (\varphi^{-1}((Y\omega^\beta)^T))^-)$ is an pseudo bipolar L-fuzzy coset of an bipolar L-fuzzy sub ℓ-HX group $\varphi^{-1}(\omega^\beta)=(\varphi^{-1}(\omega^\beta)^+, \varphi^{-1}(\omega^\beta)^-)$ of an ℓ-HX group ϑ_2 determined by the element $\varphi^{-1}(Y)\in\vartheta_1$.

For any $P\in\vartheta_1$, $\varphi(P)\in\vartheta_2$

i) $((\varphi^{-1}(Y)\ \varphi^{-1}(\omega^\beta))^T)^+(P) = T(\varphi^{-1}(Y))(\varphi^{-1}(\omega^\beta))^+(P)$
$= T(Y)\ (\omega^\beta)^+(\varphi(P))$
$= ((Y\omega^\beta)^T)^+(\varphi(P))$
$= (\varphi^{-1}(Y\omega^\beta)^T)^+(P)$

$((\varphi^{-1}(Y)\ \varphi^{-1}(\omega^\beta))^T)^+(P) = (\varphi^{-1}(Y\omega^\beta)^T)^+(P)$

ii) $((\varphi^{-1}(Y)\ \varphi^{-1}(\omega^\beta))^T)^-(P) = T(\varphi^{-1}(Y))(\varphi^{-1}(\omega^\beta))^-(P)$
$= T(Y)(\omega^\beta)^-(\varphi(P))$
$= ((Y\omega^\beta)^T)^-(\varphi(P))$
$= (\varphi^{-1}(Y\omega^\beta)^T)^-(P)$

$((\varphi^{-1}(Y)\ \varphi^{-1}(\omega^{\beta}))^{T})^{-}(P) \quad = \quad (\varphi^{-1}(Y\omega^{\beta})^{T})^{-}(P)$

Hence, $\varphi^{-1}((Y\omega^{\beta})^{T}) = (\varphi^{-1}(Y)\ \varphi^{-1}(\omega^{\beta}))^{T}$.

VI. Conclusion

Bipolar L-fuzzy coset and pseudo bipolar L-fuzzy coset, are discussed in bipolar L-fuzzy sub ℓ-HX group and bipolar anti L-fuzzy sub ℓ-HX group and also have been proved their related properties as well as suitable examples.

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