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Anti-Vertex Covering and Its Characterizations on Intuitionistic Anti-Fuzzy Graph

Vijesh V. V[1]., R. Muthuraj[2].

[1]Department of Mathematics, Sree Narayana Gurukulam College of Engineering, Kolenchery, Ernakulam – 682 311, Kerala, India. Email: v.vijesh14@gmail.com

[1]PG & Research Department of Mathematics, H. H. The Rajah's College, Pudukkottai – 622 001, Tamil Nadu, Affiliated to Bharathidasan University, Tiruchirappalli, India.
Email: rmr1973@yahoo.co.in

ABSTRACT

Anti-fuzziness has a vital role in the field of intuitionistic fuzzy theory. In this paper we defined the notion of anti-vertex cover of an intuitionistic anti fuzzy graph. Further we discussed some properties of anti-vertex cover and anti-vertex covering number of intuitionistic anti-fuzzy graphs and derived some theorems too. Anti-Cartesian product of two intuitionistic anti-fuzzy graphs is defined and derived theorems based on that.

Keywords: Anti-vertex cover of intuitionistic anti-fuzzy graph, anti-vertex covering number, anti-Cartesian product, minimum anti-vertex covering, anti-independent set, neighbourhood degree of IAFG.

I. INTRODUCTION

The fundamental conception of fuzzy sets was introduced by Lotfi A. Zadeh [16] in 1965. His idea of fuzzy sets and fuzzy relations facilitate to carry out the vagueness, ambiguity and fuzziness of real life human problems. Kaufmann [2] introduced the idea of fuzzy graph in 1973 using the concepts of fuzzy sets and fuzzy relations introduced by L. A. Zadeh. Many mathematicians studied and discussed the notion of fuzzy graphs and developed different part of this theory. As graph theory has numerous applications in the field of computer science including communication networks, artificial intelligence, pattern clustering, image retrieval etc. the field of fuzzy graph theory accomplished a tremendous growth. Later Rosenfeld [13] imported several fuzzy based graph theoretic ideas such as bridges, paths, trees, connectedness etc. in 1975.

Krassimir Atanassov [1] popularized the concept of intuitionistic fuzzy relations and intuitionistic fuzzy graphs in 1994. Different operations on intuitionistic fuzzy graphs were discussed by Parvathy and Karunabigai [11]. A. Somasundaram and S. Somasundaram [14] gave an introductory discussion on domination in fuzzy graph using effective edges. But Nagoor Gani and Chandrasekharan [10] modified this definition of domination and independent domination in fuzzy graph using strong arcs. The concept of anti-fuzzy structure is first introduced by Muhammad Akram [4] in 2012. The anti-fuzzy structure in the theory of intuitionistic fuzzy graph was first introduced by R. Muthuraj, Vijesh V. V. et al. [7] and termed as intuitionistic anti-fuzzy graph. They defined different operations as anti-union, anti-join, anti-cartesian product etc. and discussed some properties on these operations. They also studied and explained some domination parameters on IAFG and derived some results. In this paper, our aim is to define anti-vertex covering of intuitionistic anti-fuzzy graph and develop some notable results with proof. The notion of neighbourhood degree of a vertex in an intuitionistic anti-fuzzy graph is defined and derived some theorems. We also discuss the relation between the intuitionistic fuzzy anti-vertex covering and anti-vertex independent set.

II. PRELIMINARIES

Definition 2.1: An intuitionistic anti-fuzzy graph is of the form $G = \langle V, E \rangle$ where
(i) $V = \{v_1, v_2, \ldots .. v_n\}$ such that $\mu_1 : V \to [0,1]$ and $\gamma_1 : V \to [0,1]$ denote the degree of membership and non-membership of the element $v_i \in V$ respectively and $0 \le \mu_1(v_i) + \gamma_1(v_i) \le 1$ ………… (1) for every $v_i \in V$, (i = 1, 2,….n),
(ii) $E \subseteq V \times V$ where $\mu_2 : V \times V \to [0,1]$ and $\gamma_2 : V \times V \to [0,1]$ are such that

$\mu_2(v_i, v_j) \ge \max\{\mu_1(v_i), \mu_1(v_j)\}$,……….(2)

$\gamma_2(v_i, v_j) \ge \min\{\gamma_1(v_i), \gamma_1(v_j)\}$ ………..(3)

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and $0 \leq \mu_2(v_i, v_j) + \gamma_2(v_i, v_j) \leq 1$………………(4) for every $(v_i, v_j) \in E$, (i, j = 1, 2, …n).

Definition 2.2: Let $G_A = \langle V, E\rangle$ be an intuitionistic anti-fuzzy graph. Then the vertex cardinality of V is defined by

$$|V| = \sum_{v_i \in V} \left(\frac{1 + \mu_1(v_i) - \gamma_1(v_i)}{2}\right)$$

Definition 2.3: Let $G_A = \langle V, E\rangle$ be an intuitionistic anti-fuzzy graph. Then the edge cardinality of E is defined by

$$|E| = \sum_{(v_i, v_j) \in E} \left(\frac{1 + \mu_2(v_i, v_j) - \gamma_2(v_i, v_j)}{2}\right) = \sum_{e_i \in E} \left(\frac{1 + \mu_2(e_i) - \gamma_2(e_i)}{2}\right)$$

Definition 2.4: Let $G_A = \langle V, E\rangle$ be an intuitionistic anti-fuzzy graph. Then the cardinality of G_A is defined by

$$|G_A| = |\,|V| + |E|\,| = \left|\sum_{v_i \in V} \left(\frac{1 + \mu_1(v_i) - \gamma_1(v_i)}{2}\right) + \sum_{(v_i, v_j) \in E} \left(\frac{1 + \mu_2(v_i, v_j) - \gamma_2(v_i, v_j)}{2}\right)\right|$$

Definition 2.5: The vertices u and v are said to be the neighbours in an intuitionistic anti-fuzzy graph $\mathbf{G_A}$ if either of the following conditions hold.

(i) $\mu_2(u, v) > 0, \quad \gamma_2(u, v) > 0$
(ii) $\mu_2(u, v) = 0, \quad \gamma_2(u, v) > 0$
(iii) $\mu_2(u, v) > 0, \quad \gamma_2(u, v) = 0$

Definition 2.6: An edge $e = (u, v)$ of intuitionistic anti-fuzzy graph $G_A = \langle V, E\rangle$ is said to be an effective edge if $\mu_2(u, v) = \max\{\mu_1(u), \mu_1(v)\}$ and $\gamma_2(u, v) = \min\{\gamma_1(u), \gamma_1(v)\}$

Definition 2.7: An intuitionistic anti-fuzzy graph $G_A = \langle V, E\rangle$ is said to be complete if $\mu_{2ij} = \max\{\mu_{1i}, \mu_{1j}\}$ and $\gamma_{2ij} = \min\{\gamma_{1i}, \gamma_{1j}\}$, $\forall\, v_i, v_j \in V$.

Definition 2.8: An intuitionistic anti-fuzzy graph $\mathbf{G_A = \langle V, E\rangle}$ is said to be bipartite intuitionistic anti-fuzzy graph if the vertex set V can be partitioned into two non-empty sets $\mathbf{V_1}$ and $\mathbf{V_2}$ such that

(i) $\mathbf{\mu_{2ij} = 0}$ **and** $\mathbf{\gamma_{2ij} = 0}$, **if** $\mathbf{v_i, v_j \in V_1}$ **or** $\mathbf{v_i, v_j \in V_2}$
(ii) $\mathbf{\mu_{2ij} > 0}$ **and** $\mathbf{\gamma_{2ij} > 0}$, **if** $\mathbf{v_i \in V_1}$ **or** $\mathbf{v_j \in V_2}$ **for some i and j**
(or) $\mathbf{\mu_{2ij} = 0}$ **and** $\mathbf{\gamma_{2ij} > 0}$, ***if*** $\mathbf{v_i \in V_1}$ **or** $\mathbf{v_j \in V_2}$ **for some i and j**
(or) $\mathbf{\mu_{2ij} > 0}$ **and** $\mathbf{\gamma_{2ij} = 0}$, **if** $\mathbf{v_i \in V_1}$ **or** $\mathbf{v_j \in V_2}$ **for some i and j**

Definition 2.9: A bipartite IAFG graph $\mathbf{G_A = \langle V, E\rangle}$ is said to be complete if $\mu_2(u, v) = \max\{\mu_1(u), \mu_1(v)\}$ and $\gamma_2(u, v) = \min\{\gamma_1(u), \gamma_1(v)\}$ for all $u \in V_1$ and $v \in V_2$.

Definition 2.10: An IAFG graph $\mathbf{G_A = \langle V, E\rangle}$ is said to be strong if $\mu_{2ij} = \max\{\mu_{1i}, \mu_{1j}\}$ and $\gamma_{2ij} = \min\{\gamma_{1i}, \gamma_{1j}\}$, $\forall\, (v_i, v_j) \in E$.

Definition 2.11: Let $\mathbf{G_A = \langle V, E\rangle}$ be an IAFG. The degree of a vertex u is denoted by $d_{G_A}(u)$ and defined as

$$d_{G_A}(u) = \sum_{(u,v) \in E}^{n} \left(\frac{1 + \mu_2(u, v) - \gamma_2(u, v)}{2}\right) = \sum_{v \neq u}^{n} \left(\frac{1 + \mu_2(u, v) - \gamma_2(u, v)}{2}\right)$$

Definition 2.12: The minimum degree of an intuitionistic anti-fuzzy graph $G_A = \langle V, E\rangle$ is $\delta(G_A) = \min\{d_{G_A}(u) / u \in V\}$.

Definition 2.13: The maximum degree of an intuitionistic anti-fuzzy graph $G_A = \langle V, E\rangle$ is $\Delta(G_A) = \max\{d_{G_A}(u) / u \in V\}$.

Definition 2.14: The total degree of a vertex u in an intuitionistic anti-fuzzy graph $G_A = \langle V, E\rangle$ is defined as

$$td(u) = d_{G_A}(u) + \left(\frac{1 + \mu_1(u) - \gamma_1(u)}{2}\right)$$

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Definition 2.15: Let $\mathbf{G_A} = \langle \mathbf{V}, \mathbf{E} \rangle$ be an intuitionistic anti-fuzzy graph. Let $\mathbf{u}, \mathbf{v} \in \mathbf{V}$, we say that u dominates v in $\mathbf{G_A}$ if $\boldsymbol{\mu_2(u,v) = \max\{\mu_1(u), \mu_1(v)\}}$ **and** $\boldsymbol{\gamma_2(u,v) = \min\{\gamma_1(u), \gamma_1(v)\}}$. A subset D of V is called a dominating set in an intuitionistic anti-fuzzy graph$\mathbf{G_A}$ if, for every vertex $\mathbf{v} \notin \mathbf{D}$, there exists $\mathbf{u} \in \mathbf{D}$ such that u dominates v. A dominating set D of an intuitionistic anti-fuzzy graph $\mathbf{G_A} = \langle \mathbf{V}, \mathbf{E} \rangle$ is said to be a minimal dominating set if no proper subset of D is a dominating set of $\mathbf{G_A}$. The maximum fuzzy cardinality taken over all minimal dominating set in an intuitionistic anti-fuzzy graph$\mathbf{G_A}$ is called the domination number of $\mathbf{G_A}$ and it is denoted by $\boldsymbol{\gamma(G_A)}$ or $\boldsymbol{\gamma_A}$.

Definition 2.16: Two vertices $\mathbf{u}, \mathbf{v} \in \mathbf{V}$ in an intuitionistic anti-fuzzy graph $\mathbf{G_A} = \langle \mathbf{V}, \mathbf{E} \rangle$ are said to be anti-independent if $\boldsymbol{\mu_2(u,v) \neq \max\{\mu_1(u), \mu_1(v)\}}$ **and** $\boldsymbol{\gamma_2(u,v) \neq \min\{\gamma_1(u), \gamma_1(v)\}}$

Definition 2.17: Let $\mathbf{G_{A1}} = \langle \mathbf{V}_1, \mathbf{E_1} \rangle$ and $\mathbf{G_{A2}} = \langle \mathbf{V}_2, \mathbf{E_2} \rangle$ be two intuitionistic anti-fuzzy graphs. Then the anti-cartesian product of $\mathbf{G_{A1}}$ and $\mathbf{G_{A2}}$ is denoted by $\mathbf{G_{A1}} \times \mathbf{G_{A2}}$ is denoted by $\mathbf{G_{A1}} \times \mathbf{G_{A2}} = \langle \mathbf{V}, \mathbf{E'} \rangle$ where $\mathbf{V} = \mathbf{V}_1 \times \mathbf{V}_2$ and
$\mathbf{E'} = \{(\mathbf{u}, \mathbf{u_2})(\mathbf{u}, \mathbf{v_2}) : \mathbf{u} \in \mathbf{V}_1, (\mathbf{u_2}, \mathbf{v_2}) \in \mathbf{E_2}\} \cup \{(\mathbf{u_1}, \mathbf{w})(\mathbf{v_1}, \mathbf{w}) : \mathbf{w} \in \mathbf{V}_2, (\mathbf{u_1}, \mathbf{v_1}) \in \mathbf{E_1}\}$
and defined by

(i) $(\mu_1 \times \mu_1')(\mathbf{u_1}, \mathbf{u_2}) = \mathbf{max}\{\mu_1(\mathbf{u_1}), \mu_1'(\mathbf{u_2})\}, \forall\, \mathbf{u_1}, \mathbf{u_2} \in \mathbf{V}$ and
$(\gamma_1 \times \gamma_1')(\mathbf{u_1}, \mathbf{u_2}) = \mathbf{min}\{\gamma_1(\mathbf{u_1}), \gamma_1'(\mathbf{u_2})\}, \forall\, \mathbf{u_1}, \mathbf{u_2} \in \mathbf{V}$

(ii) $(\mu_2 \times \mu_2')(\mathbf{u}, \mathbf{u_2})(\mathbf{u}, \mathbf{v_2}) = \mathbf{max}\{\mu_1(\mathbf{u}), \mu_2'(\mathbf{u_2}, \mathbf{v_2})\}, \forall\, \mathbf{u} \in \mathbf{V}_1$ and $(\mathbf{u_2}, \mathbf{v_2}) \in \mathbf{E_2}$,
$(\gamma_2 \times \gamma_2')(\mathbf{u}, \mathbf{u_2})(\mathbf{u}, \mathbf{v_2}) = \mathbf{min}\{\gamma_1(\mathbf{u}), \gamma_2'(\mathbf{u_2}, \mathbf{v_2})\}, \forall\, \mathbf{u} \in \mathbf{V}_1$ and $(\mathbf{u_2}, \mathbf{v_2}) \in \mathbf{E_2}$ and $(\mu_2 \times \mu_2')(\mathbf{u_1}, \mathbf{w})(\mathbf{v_1}, \mathbf{w}) = \mathbf{max}\{\mu_1'(\mathbf{w}), \mu_2(\mathbf{u_1}, \mathbf{v_1})\}, \forall\, \mathbf{w} \in \mathbf{V}_2$ and $(\mathbf{u_1}, \mathbf{v_1}) \in \mathbf{E_1}$, $(\gamma_2 \times \gamma_2')(\mathbf{u_1}, \mathbf{w})(\mathbf{v_1}, \mathbf{w}) = \mathbf{min}\{\gamma_1'(\mathbf{w}), \gamma_2(\mathbf{u_1}, \mathbf{v_1})\}, \forall\, \mathbf{w} \in \mathbf{V}_2$ and $(\mathbf{u_1}, \mathbf{v_1}) \in \mathbf{E_1}$.

III. ANTI-VERTEX COVERING IN IAFG

Definition 3.1: Let $\mathbf{G_A} = \langle \mathbf{V}, \mathbf{E} \rangle$ be an IAFG. Then D ⊆ V in $\mathbf{G_A}$ is said to be an anti-vertex cover of $\mathbf{G_A}$ if every strong arc of $\mathbf{G_A}$ incident with at least one vertex of D.

The anti-vertex covering number of IAFG $\mathbf{G_A} = \langle \mathbf{V}, \mathbf{E} \rangle$ is the minimum cardinality among all anti-vertex covers of IAFG $\mathbf{G_A}$ and which is denoted by $\beta(\mathbf{G_A})$.

Theorem 3.1: In an IAFG $\mathbf{G_A} = \langle \mathbf{V}, \mathbf{E} \rangle$ without isolated vertices, the anti-vertex covering number is at most half of the vertex cardinality. That is, $\beta(\mathbf{G_A}) \leq \frac{|V|}{2}$.

Proof: Let $\mathbf{G_A} = \langle \mathbf{V}, \mathbf{E} \rangle$ be an IAFG without isolated vertices. Let D be the anti-vertex cover of $\mathbf{G_A}$.
So D covers all the strong arcs of $\mathbf{G_A}$ and hence V \ D is also an anti-vertex cover of $\mathbf{G_A}$, because $\mathbf{G_A} = \langle \mathbf{V}, \mathbf{E} \rangle$ is having no isolated vertex.
Therefore, $\beta(\mathbf{G_A}) = \min\{|D|, |V \setminus D|\} \leq \frac{|V|}{2}$
Hence, $\beta(\mathbf{G_A}) \leq \frac{|V|}{2}$. Thus anti-vertex covering number $\beta(\mathbf{G_A})$ is at most half of the vertex cardinality of an IAFG $\mathbf{G_A} = \langle \mathbf{V}, \mathbf{E} \rangle$ without any isolated vertex.

Example 3.1:

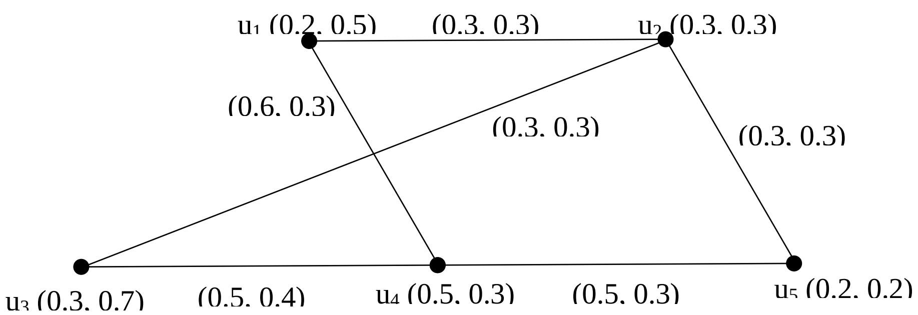

Figure 1: Intuitionistic anti-fuzzy graph $G_A\langle V, E \rangle$

In figure 1, (u_1, u_2), (u_2, u_3) and (u_4, u_5) are strong arcs.

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Thus the anti-vertex cover of $G_A\langle V, E\rangle$ is $D = \{u_2, u_5\}$ and hence the anti-vertex covering number $\beta(\mathbf{G_A}) = 1$.

Theorem 3.2: Let $\mathbf{G_{A1}} = \langle \mathbf{V_1}, \mathbf{E_1}\rangle$ and $\mathbf{G_{A2}} = \langle \mathbf{V_2}, \mathbf{E_2}\rangle$ be two intuitionistic anti-fuzzy graphs with anti- vertex covering sets D_1 and D_2 , respectively. Then the minimum anti-vertex covering of the union $\mathbf{G_{A1}} \cup \mathbf{G_{A2}}$ is $D_1 \cup D_2$.

Proof: Let $\mathbf{G_{A1}} = \langle \mathbf{V_1}, \mathbf{E_1}\rangle$ and $\mathbf{G_{A2}} = \langle \mathbf{V_2}, \mathbf{E_2}\rangle$ be two intuitionistic anti-fuzzy graphs with anti- vertex covering sets D_1 and D_2 , respectively.
The union of $\mathbf{G_{A1}}$ and $\mathbf{G_{A2}}$ is $\mathbf{G_{A1}} \cup \mathbf{G_{A2}} = \langle \mathbf{V_1} \cup \mathbf{V_2}, \mathbf{E_1} \cup \mathbf{E_2}\rangle$ such that

$$(\mu_1 \cup \mu_1')(v) = \begin{cases} \mu_1(v), \text{if } v \in V_1 \setminus V_2 \\ \mu_1'(v), \text{if } v \in V_2 \setminus V_1 \\ \min\{\mu_1(v), \mu_1'(v)\}, \text{if } v \in V_1 \cap V_2 \end{cases}$$

$$(\gamma_1 \cup \gamma_1')(v) = \begin{cases} \gamma_1(v), \text{if } v \in V_1 \setminus V_2 \\ \gamma_1'(v), \text{if } v \in V_2 \setminus V_1 \\ \min\{\gamma_1(v), \gamma_1'(v)\}, \text{if } v \in V_1 \cap V_2 \end{cases}$$

and

$$(\mu_2 \cup \mu_2')(u,v) = \begin{cases} \mu_2(u,v), \text{if } (u,v) \in E_1 \setminus E_2 \\ \mu_2'(u,v), \text{if } (u,v) \in E_2 \setminus E_1 \\ \min\{\mu_2(u,v), \mu_2'(u,v)\}, \text{if } (u,v) \in E_1 \cap E_2 \end{cases}$$

$$(\gamma_2 \cup \gamma_2')(u,v) = \begin{cases} \gamma_2(u,v), \text{if } (u,v) \in E_1 \setminus E_2 \\ \gamma_2'(u,v), \text{if } (u,v) \in E_2 \setminus E_1 \\ \min\{\gamma_2(u,v), \gamma_2'(u,v)\}, \text{if } (u,v) \in E_1 \cap E_2 \end{cases}$$

Where (μ_1, γ_1) and (μ_1', γ_1') refer the vertex membership and non-membership of $\mathbf{G_{A_1}}$and $\mathbf{G_{A_2}}$ respectively, (μ_2, γ_2) and (μ_2', γ_2') refer the edge membership and non-membership of $\mathbf{G_{A_1}}$and $\mathbf{G_{A_2}}$ respectively.
By the membership and non-membership values of an arbitrary arc (u, v) in $\mathbf{G_{A1}} \cup \mathbf{G_{A2}}$, the strong arc (u, v) in $\mathbf{G_{A1}} \cup \mathbf{G_{A2}}$ will be covered by $D_1 \cup D_2$**.** Hence $D_1 \cup D_2$ will be the minimum anti-vertex covering of $\mathbf{G_{A1}} \cup \mathbf{G_{A2}}$.

Example 3.2: From the following figures: anti-vertex covering of $\mathbf{G_{A1}}$ is $\{v\}$**,** anti-vertex covering of $\mathbf{G_{A2}}$ is $\{v\}$**.** The anti-vertex covering numbers are $\beta(\mathbf{G_{A1}}) = 0.6$ and $\beta(\mathbf{G_{A2}}) = 0.5$
Anti-vertex covering of $\mathbf{G_{A1}} \cup \mathbf{G_{A2}}$ is $\{v\}$ and anti-vertex covering number of $\mathbf{G_{A1}} \cup \mathbf{G_{A2}}$ is $\beta(\mathbf{G_{A1}} \cup \mathbf{G_{A2}}) = 0.55$

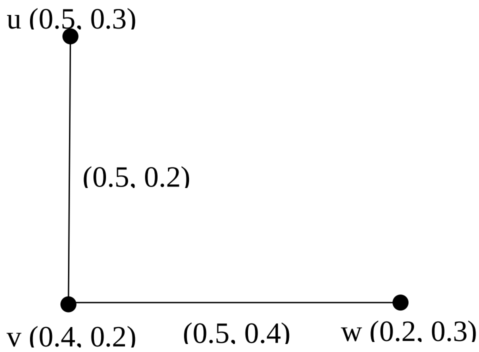

Figure 2: IAFG $\mathbf{G_{A1}}\langle V, E\rangle$

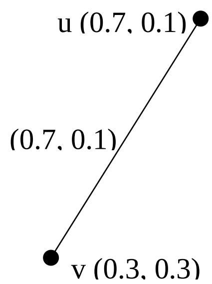

Figure 3: IAFG $\mathbf{G_{A2}}\langle V, E\rangle$

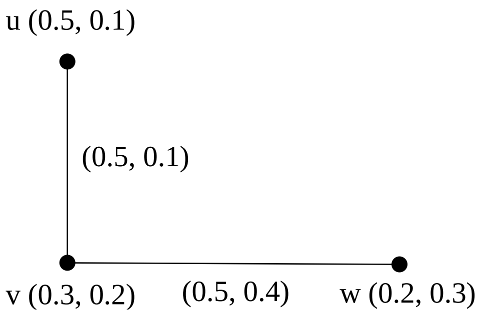

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Figure 4: IAFG $\mathbf{G_{A1}} \cup \mathbf{G_{A2}}$

Theorem 3.3: Let $\mathbf{G_{A1}} = \langle \mathbf{V_1}, \mathbf{E_1} \rangle$ and $\mathbf{G_{A2}} = \langle \mathbf{V_2}, \mathbf{E_2} \rangle$ be two intuitionistic anti-fuzzy graphs with $\mathbf{V_1} \cap \mathbf{V_2} = \varphi$. Then the anti-vertex covering number of $\mathbf{G_{A1}} \cup \mathbf{G_{A2}}$ will be

$$\beta(\mathbf{G_{A1}} \cup \mathbf{G_{A2}}) = \beta(\mathbf{G_{A1}}) + \beta(\mathbf{G_{A2}})$$

Proof: Let $\mathbf{G_{A1}} = \langle \mathbf{V_1}, \mathbf{E_1} \rangle$ and $\mathbf{G_{A2}} = \langle \mathbf{V_2}, \mathbf{E_2} \rangle$ be two intuitionistic anti-fuzzy graphs such that $\mathbf{V_1} \cap \mathbf{V_2} = \varphi$.
Let D_1 be the minimum anti-vertex covering set of $\mathbf{G_{A1}}$ with anti-vertex covering number β_1.
Let D_2 be the minimum anti-vertex covering set of $\mathbf{G_{A2}}$ with anti-vertex covering number β_2.
Since $\mathbf{V_1}$ and $\mathbf{V_2}$ are two disjoint vertex sets, the membership and non-membership values of vertices in $\mathbf{V_1}$ and $\mathbf{V_2}$ are preserved in $\mathbf{G_{A1}} \cup \mathbf{G_{A2}}$.
Since there is no common arc in $\mathbf{G_{A_1}}$ and $\mathbf{G_{A_2}}$, the strong arcs in these two intuitionistic anti-fuzzy graphs preserved in $\mathbf{G_{A1}} \cup \mathbf{G_{A2}}$. Hence the minimum anti-vertex covering of $\mathbf{G_{A1}} \cup \mathbf{G_{A2}}$ would be $D_1 \cup D_2$.
Since $\mathbf{V_1} \cap \mathbf{V_2} = \varphi$, there is no common vertex in D_1 and D_2. Hence the minimum anti-vertex covering number of $\mathbf{G_{A1}} \cup \mathbf{G_{A2}}$ will be the sum of anti-vertex covering numbers of $\mathbf{G_{A_1}}$ and $\mathbf{G_{A_2}}$.
Therefore, $\beta(\mathbf{G_{A1}} \cup \mathbf{G_{A2}}) = \beta(\mathbf{G_{A1}}) + \beta(\mathbf{G_{A2}})$ if $\mathbf{G_{A_1}}$ and $\mathbf{G_{A_2}}$ has no common vertices.

Theorem 3.4: Let $\mathbf{G_{A1}} = \langle \mathbf{V_1}, \mathbf{E_1} \rangle$ and $\mathbf{G_{A2}} = \langle \mathbf{V_2}, \mathbf{E_2} \rangle$ be two intuitionistic anti-fuzzy graphs with $\mathbf{V_1} \cap \mathbf{V_2} \neq \varphi$ and has at least one common strong arc. Then the anti-vertex covering number of $\mathbf{G_{A1}} \cup \mathbf{G_{A2}}$ lies between the anti-vertex covering numbers of $\mathbf{G_{A_1}}$ and $\mathbf{G_{A_2}}$, both inclusive.
Thus $\min\{\beta(\mathbf{G_{A_1}}), \beta(\mathbf{G_{A_2}})\} \leq \beta(\mathbf{G_{A1}} \cup \mathbf{G_{A2}}) \leq \max\{\beta(\mathbf{G_{A_1}}), \beta(\mathbf{G_{A_2}})\}$.

Proof: Let $\mathbf{G_{A1}} = \langle \mathbf{V_1}, \mathbf{E_1} \rangle$ and $\mathbf{G_{A2}} = \langle \mathbf{V_2}, \mathbf{E_2} \rangle$ be two intuitionistic anti-fuzzy graphs such that $\mathbf{V_1} \cap \mathbf{V_2} \neq \varphi$. Let $u, v \in \mathbf{V_1}$ and $u, v \in \mathbf{V_2}$. So, $(u, v) \in \mathbf{E_1} \cap \mathbf{E_2}$.
Suppose (u, v) is a strong arc in $\mathbf{G_{A_1}}$ and $\mathbf{G_{A_2}}$. Then one of these vertices u, v must be in D_1 and D_2. Hence the anti-vertex covering set of $\mathbf{G_{A1}} \cup \mathbf{G_{A2}}$ must contain that vertex without showing this repetition. Hence $\beta(\mathbf{G_{A1}} \cup \mathbf{G_{A2}})$ will be a value less than or equal to the bigger among $\beta(\mathbf{G_{A1}})$ and $\beta(\mathbf{G_{A2}})$.
Now, if the vertex dominating set with smaller β value has at least one strong arc, then the vertex dominating set of $\mathbf{G_{A1}} \cup \mathbf{G_{A2}}$ must contain the corresponding vertex. So $\beta(\mathbf{G_{A1}} \cup \mathbf{G_{A2}})$ will be a value greater than or equal to the smaller of $\beta(\mathbf{G_{A1}})$ and $\beta(\mathbf{G_{A2}})$.
Thus $\min\{\beta(\mathbf{G_{A_1}}), \beta(\mathbf{G_{A_2}})\} \leq \beta(\mathbf{G_{A1}} \cup \mathbf{G_{A2}}) \leq \max\{\beta(\mathbf{G_{A_1}}), \beta(\mathbf{G_{A_2}})\}$.

Theorem 3.5: If D_1 and D_2 are anti-vertex covering sets of two IAF graphs $\mathbf{G_{A1}} = \langle \mathbf{V_1}, \mathbf{E_1} \rangle$ and $\mathbf{G_{A2}} = \langle \mathbf{V_2}, \mathbf{E_2} \rangle$ respectively, then the minimum anti-vertex covering set of $\mathbf{G_{A1}} \times \mathbf{G_{A2}}$ is $\{D_1 \times (\mathbf{V_2} \backslash D_2)\} \cup \{(\mathbf{V_1} \backslash D_1) \times D_2\}$.

Proof: Let $\mathbf{G_{A1}} = \langle \mathbf{V_1}, \mathbf{E_1} \rangle$ and $\mathbf{G_{A2}} = \langle \mathbf{V_2}, \mathbf{E_2} \rangle$ be two intuitionistic anti-fuzzy graphs with anti-covering set D_1 and D_2, respectively. The edge membership and non-membership values of the anti-cartesian product $\mathbf{G_{A1}} \times \mathbf{G_{A2}}$ are of the form:

$$(\mu_2 \times \mu_2')\big((\mathbf{u}, \mathbf{u_2})(\mathbf{u}, \mathbf{v_2})\big) = \mathbf{max}\{\mu_1(\mathbf{u}), \mu_2'(\mathbf{u_2}, \mathbf{v_2})\}, \forall\, \mathbf{u} \in V_1 \text{ and } (\mathbf{u_2}, \mathbf{v_2}) \in \mathbf{E_2}\,,$$
$$(\gamma_2 \times \gamma_2')\big((\mathbf{u}, \mathbf{u_2})(\mathbf{u}, \mathbf{v_2})\big) = \mathbf{min}\{\gamma_1(\mathbf{u}), \gamma_2'(\mathbf{u_2}, \mathbf{v_2})\}, \forall\, \mathbf{u} \in V_1 \text{ and } (\mathbf{u_2}, \mathbf{v_2}) \in \mathbf{E_2} \text{ and}$$
$$(\mu_2 \times \mu_2')\big((\mathbf{u_1}, \mathbf{w})(\mathbf{v_1}, \mathbf{w})\big) = \mathbf{max}\{\mu_1'(\mathbf{w}), \mu_2(\mathbf{u_1}, \mathbf{v_1})\}, \forall\, \mathbf{w} \in V_2 \text{ and } (\mathbf{u_1}, \mathbf{v_1}) \in \mathbf{E_1},$$
$$(\gamma_2 \times \gamma_2')\big((\mathbf{u_1}, \mathbf{w})(\mathbf{v_1}, \mathbf{w})\big) = \mathbf{min}\{\gamma_1'(\mathbf{w}), \gamma_2(\mathbf{u_1}, \mathbf{v_1})\}, \forall\, \mathbf{w} \in V_2 \text{ and } (\mathbf{u_1}, \mathbf{v_1}) \in \mathbf{E_1}.$$

Thus
$(\mu_2 \times \mu_2')\big((\mathbf{u}, \mathbf{u_2})(\mathbf{u}, \mathbf{v_2})\big) = \mathbf{max}\{\mu_1(\mathbf{u}), \mu_2'(\mathbf{u_2}, \mathbf{v_2})\}$ and

$$(\gamma_2 \times \gamma_2')\big((\mathbf{u}, \mathbf{u_2})(\mathbf{u}, \mathbf{v_2})\big) = \mathbf{min}\{\gamma_1(\mathbf{u}), \gamma_2'(\mathbf{u_2}, \mathbf{v_2})\},$$

$\forall\, \mathbf{u} \in V_1$ and $(\mathbf{u_2}, \mathbf{v_2}) \in \mathbf{E_2}$
$\Rightarrow \big((\mathbf{u}, \mathbf{u_2})(\mathbf{u}, \mathbf{v_2})\big)$ if $(\mathbf{u_2}, \mathbf{v_2}) \in \mathbf{E_2}$ is a strong arc in $\mathbf{G_{A2}}$.
Again
$(\mu_2 \times \mu_2')\big((\mathbf{u_1}, \mathbf{w})(\mathbf{v_1}, \mathbf{w})\big) = \mathbf{max}\{\mu_1'(\mathbf{w}), \mu_2(\mathbf{u_1}, \mathbf{v_1})\}$ and

$$(\gamma_2 \times \gamma_2')\big((\mathbf{u_1}, \mathbf{w})(\mathbf{v_1}, \mathbf{w})\big) = \mathbf{min}\{\gamma_1'(\mathbf{w}), \gamma_2(\mathbf{u_1}, \mathbf{v_1})\},$$

$\forall\, \mathbf{w} \in V_2$ and $(\mathbf{u_1}, \mathbf{v_1}) \in \mathbf{E_1}$.
$\Rightarrow \big((\mathbf{u_1}, \mathbf{w})(\mathbf{v_1}, \mathbf{w})\big)$ if $(\mathbf{u_1}, \mathbf{v_1}) \in \mathbf{E_1}$ is a strong arc in $\mathbf{G_{A1}}$.

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Hence $\big((\mathbf{u},\mathbf{u_2})(\mathbf{u},\mathbf{v_2})\big)$ if $(\mathbf{u_2},\mathbf{v_2}) \in \mathbf{E_2}$ and $\big((\mathbf{u_1},\mathbf{w})(\mathbf{v_1},\mathbf{w})\big)$ if $(\mathbf{u_1},\mathbf{v_1}) \in \mathbf{E_1}$ are two strong arcs in the anti-cartesian product $\mathbf{G_{A1}} \times \mathbf{G_{A2}}$.

To prove these arcs covered by the set $\{D_1 \times (\mathbf{V_2}\backslash D_2)\} \cup \{(\mathbf{V_1}\backslash D_1) \times D_2\}$, following cases are considered:

Case (i): $\big((\mathbf{u},\mathbf{u_2})(\mathbf{u},\mathbf{v_2})\big)$ if $(\mathbf{u_2},\mathbf{v_2}) \in \mathbf{E_2}$ is a strong arc in IAFG $\mathbf{G_{A2}}$.

(a). If $u \in D_1$ and assume $\mathbf{u_2} \notin D_2$, $\mathbf{v_2} \in D_2$, since D_2 is a minimum anti-vertex cover of IAFG $\mathbf{G_{A2}}$.

Thus, $(\mu_2 \times \mu_2{}')\big((\mathbf{u},\mathbf{u_2})(\mathbf{u},\mathbf{v_2})\big) = \mathbf{max}\{\mu_1(\mathbf{u}), \mu_2{}'(\mathbf{u_2},\mathbf{v_2})\}$

$$= \mathbf{max}\big\{\mu_1(\mathbf{u}), \mathbf{max}\{\mu_1{}'(\mathbf{u_2}), \mu_1{}'(\mathbf{v_2})\}\big\}$$

$$= \mathbf{max}\{\mu_1(\mathbf{u}), \mu_1{}'(\mathbf{u_2}), \mu_1{}'(\mathbf{v_2})\}$$

$$= \mathbf{max}\{\mu_1(\mathbf{u}), \mu_1{}'(\mathbf{u_2}), \mu_1(\mathbf{u}), \mu_1{}'(\mathbf{v_2})\}$$

$$= \mathbf{max}\big\{\mathbf{max}\{\mu_1(\mathbf{u}), \mu_1{}'(\mathbf{u_2})\}, \mathbf{max}\{\mu_1(\mathbf{u}), \mu_1{}'(\mathbf{v_2})\}\big\}$$

$$= \mathbf{max}\{(\mu_1 \times \mu_1{}')(\mathbf{u},\mathbf{u_2}), (\mu_1 \times \mu_1{}')(\mathbf{u},\mathbf{v_2})\}$$

also $(\gamma_2 \times \gamma_2{}')\big((\mathbf{u},\mathbf{u_2})(\mathbf{u},\mathbf{v_2})\big) = \mathbf{min}\{\gamma_1(\mathbf{u}), \gamma_2{}'(\mathbf{u_2},\mathbf{v_2})\}$

$$= \mathbf{min}\big\{\gamma_1(\mathbf{u}), \mathbf{min}\{\gamma_1{}'(\mathbf{u_2}), \gamma_1{}'(\mathbf{v_2})\}\big\}$$

$$= \mathbf{min}\{\gamma_1(\mathbf{u}), \gamma_1{}'(\mathbf{u_2}), \gamma_1{}'(\mathbf{v_2})\}$$

$$= \mathbf{min}\{\gamma_1(\mathbf{u}), \gamma_1{}'(\mathbf{u_2}), \gamma_1(\mathbf{u}), \gamma_1{}'(\mathbf{v_2})\}$$

$$= \mathbf{min}\big\{\mathbf{min}\{\gamma_1(\mathbf{u}), \gamma_1{}'(\mathbf{u_2})\}, \mathbf{min}\{\gamma_1(\mathbf{u}), \gamma_1{}'(\mathbf{v_2})\}\big\}$$

$$= \mathbf{min}\{(\gamma_1 \times \gamma_1{}')(\mathbf{u},\mathbf{u_2}), (\gamma_1 \times \gamma_1{}')(\mathbf{u},\mathbf{v_2})\}$$

It is clear that the arcs in this situation and $u \in D_1$ is covered by the vertex $(\mathbf{u},\mathbf{u_2}) \in \{D_1 \times (\mathbf{V_2}\backslash D_2)\}$.

(b). If $u \notin D_1$ and assume $\mathbf{u_2} \notin D_2$, $\mathbf{v_2} \in D_2$, since D_2 is a minimum anti-vertex cover of IAFG $\mathbf{G_{A2}}$.

Thus, $(\mu_2 \times \mu_2{}')\big((\mathbf{u},\mathbf{u_2})(\mathbf{u},\mathbf{v_2})\big) = \mathbf{max}\{\mu_1(\mathbf{u}), \mu_2{}'(\mathbf{u_2},\mathbf{v_2})\}$

$$= \mathbf{max}\big\{\mu_1(\mathbf{u}), \mathbf{max}\{\mu_1{}'(\mathbf{u_2}), \mu_1{}'(\mathbf{v_2})\}\big\}$$

$$= \mathbf{max}\{\mu_1(\mathbf{u}), \mu_1{}'(\mathbf{u_2}), \mu_1{}'(\mathbf{v_2})\}$$

$$= \mathbf{max}\{\mu_1(\mathbf{u}), \mu_1{}'(\mathbf{u_2}), \mu_1(\mathbf{u}), \mu_1{}'(\mathbf{v_2})\}$$

$$= \mathbf{max}\big\{\mathbf{max}\{\mu_1(\mathbf{u}), \mu_1{}'(\mathbf{u_2})\}, \mathbf{max}\{\mu_1(\mathbf{u}), \mu_1{}'(\mathbf{v_2})\}\big\}$$

$$= \mathbf{max}\{(\mu_1 \times \mu_1{}')(\mathbf{u},\mathbf{u_2}), (\mu_1 \times \mu_1{}')(\mathbf{u},\mathbf{v_2})\}$$

also $(\gamma_2 \times \gamma_2{}')\big((\mathbf{u},\mathbf{u_2})(\mathbf{u},\mathbf{v_2})\big) = \mathbf{min}\{\gamma_1(\mathbf{u}), \gamma_2{}'(\mathbf{u_2},\mathbf{v_2})\}$

$$= \mathbf{min}\big\{\gamma_1(\mathbf{u}), \mathbf{min}\{\gamma_1{}'(\mathbf{u_2}), \gamma_1{}'(\mathbf{v_2})\}\big\}$$

$$= \mathbf{min}\{\gamma_1(\mathbf{u}), \gamma_1{}'(\mathbf{u_2}), \gamma_1{}'(\mathbf{v_2})\}$$

$$= \mathbf{min}\{\gamma_1(\mathbf{u}), \gamma_1{}'(\mathbf{u_2}), \gamma_1(\mathbf{u}), \gamma_1{}'(\mathbf{v_2})\}$$

$$= \mathbf{min}\big\{\mathbf{min}\{\gamma_1(\mathbf{u}), \gamma_1{}'(\mathbf{u_2})\}, \mathbf{min}\{\gamma_1(\mathbf{u}), \gamma_1{}'(\mathbf{v_2})\}\big\}$$

$$= \mathbf{min}\{(\gamma_1 \times \gamma_1{}')(\mathbf{u},\mathbf{u_2}), (\gamma_1 \times \gamma_1{}')(\mathbf{u},\mathbf{v_2})\}$$

It is clear that the arcs in this situation and $u \notin D_1$ is covered by the vertex $(\mathbf{u},\mathbf{u_2}) \in \{(\mathbf{V_1}\backslash D_1) \times D_2\}$.
Hence the arcs in case (i) are covered by the vertex set $\{D_1 \times (\mathbf{V_2}\backslash D_2)\} \cup \{(\mathbf{V_1}\backslash D_1) \times D_2\}$.

Case (ii): $\big((\mathbf{u_1},\mathbf{w})(\mathbf{v_1},\mathbf{w})\big)$ if $(\mathbf{u_1},\mathbf{v_1}) \in \mathbf{E_1}$ is a strong arc in IAFG $\mathbf{G_{A1}}$.

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(a). If $w \in D_2$ and assume $\mathbf{u_1} \notin D_1$, $\mathbf{v_1} \in D_1$, since D_1 is a minimum anti-vertex cover of IAFG $\mathbf{G_{A1}}$.

$$
\begin{aligned}
\text{Thus, } (\mu_2 \times \mu_2')\big((\mathbf{u_1},\mathbf{w})(\mathbf{v_1},\mathbf{w})\big) &= \mathbf{max}\{\mu_2(\mathbf{u_1},\mathbf{v_1}), \mu_1'(\mathbf{w})\}\\
&= \mathbf{max}\{\mathbf{max}\{\mu_1(\mathbf{u_1}), \mu_1(\mathbf{v_1})\}, \mu_1'(\mathbf{w})\}\\
&= \mathbf{max}\{\mu_1(\mathbf{u_1}), \mu_1(\mathbf{v_1}), \mu_1'(\mathbf{w})\}\\
&= \mathbf{max}\{\mu_1(\mathbf{u_1}), \mu_1'(\mathbf{w}), \mu_1(\mathbf{v_1}), \mu_1'(\mathbf{w})\}\\
&= \mathbf{max}\big\{\mathbf{max}\{\mu_1(\mathbf{u_1}), \mu_1'(\mathbf{w})\}, \mathbf{max}\{\mu_1(\mathbf{v_1}), \mu_1'(\mathbf{w})\}\big\}\\
&= \mathbf{max}\{(\mu_1 \times \mu_1')(\mathbf{u_1},\mathbf{w}), (\mu_1 \times \mu_1')(\mathbf{v_1},\mathbf{w})\}
\end{aligned}
$$

$$
\begin{aligned}
\text{also } (\gamma_2 \times \gamma_2')\big((\mathbf{u_1},\mathbf{w})(\mathbf{v_1},\mathbf{w})\big) &= \mathbf{min}\{\gamma_2(\mathbf{u_1},\mathbf{v_1}), \gamma_1'(\mathbf{w})\}\\
&= \mathbf{min}\{\mathbf{min}\{\gamma_1(\mathbf{u_1}), \gamma_1(\mathbf{v_1})\}, \gamma_1'(\mathbf{w})\}\\
&= \mathbf{min}\{\gamma_1(\mathbf{u_1}), \gamma_1(\mathbf{v_1}), \gamma_1'(\mathbf{w})\}\\
&= \mathbf{min}\{\gamma_1(\mathbf{u_1}), \gamma_1'(\mathbf{w}), \gamma_1(\mathbf{v_1}), \gamma_1'(\mathbf{w})\}\\
&= \mathbf{min}\big\{\mathbf{min}\{\gamma_1(\mathbf{u_1}), \gamma_1'(\mathbf{w})\}, \mathbf{min}\{\gamma_1(\mathbf{v_1}), \gamma_1'(\mathbf{w})\}\big\}\\
&= \mathbf{min}\{(\gamma_1 \times \gamma_1')(\mathbf{u_1},\mathbf{w}), (\gamma_1 \times \gamma_1')(\mathbf{v_1},\mathbf{w})\}
\end{aligned}
$$

Hence it is clear that the arcs in this situation and $\mathbf{w} \in D_2$ is covered by the vertex $(\mathbf{u_1},\mathbf{w}) \in \{(\mathbf{V_1} \backslash D_1) \times D_2\}$.

(b). If $\mathbf{w} \notin D_2$ and assume $\mathbf{u_1} \notin D_1$, $\mathbf{v_1} \in D_1$, since D_1 is a minimum anti-vertex cover of IAFG $\mathbf{G_{A1}}$.

$$
\begin{aligned}
\text{Thus, } (\mu_2 \times \mu_2')\big((\mathbf{u_1},\mathbf{w})(\mathbf{v_1},\mathbf{w})\big) &= \mathbf{max}\{\mu_2(\mathbf{u_1},\mathbf{v_1}), \mu_1'(\mathbf{w})\}\\
&= \mathbf{max}\{\mathbf{max}\{\mu_1(\mathbf{u_1}), \mu_1(\mathbf{v_1})\}, \mu_1'(\mathbf{w})\}\\
&= \mathbf{max}\{\mu_1(\mathbf{u_1}), \mu_1(\mathbf{v_1}), \mu_1'(\mathbf{w})\}\\
&= \mathbf{max}\{\mu_1(\mathbf{u_1}), \mu_1'(\mathbf{w}), \mu_1(\mathbf{v_1}), \mu_1'(\mathbf{w})\}\\
&= \mathbf{max}\big\{\mathbf{max}\{\mu_1(\mathbf{u_1}), \mu_1'(\mathbf{w})\}, \mathbf{max}\{\mu_1(\mathbf{v_1}), \mu_1'(\mathbf{w})\}\big\}\\
&= \mathbf{max}\{(\mu_1 \times \mu_1')(\mathbf{u_1},\mathbf{w}), (\mu_1 \times \mu_1')(\mathbf{v_1},\mathbf{w})\}
\end{aligned}
$$

$$
\begin{aligned}
\text{also } (\gamma_2 \times \gamma_2')\big((\mathbf{u_1},\mathbf{w})(\mathbf{v_1},\mathbf{w})\big) &= \mathbf{min}\{\gamma_2(\mathbf{u_1},\mathbf{v_1}), \gamma_1'(\mathbf{w})\}\\
&= \mathbf{min}\{\mathbf{min}\{\gamma_1(\mathbf{u_1}), \gamma_1(\mathbf{v_1})\}, \gamma_1'(\mathbf{w})\}\\
&= \mathbf{min}\{\gamma_1(\mathbf{u_1}), \gamma_1(\mathbf{v_1}), \gamma_1'(\mathbf{w})\}\\
&= \mathbf{min}\{\gamma_1(\mathbf{u_1}), \gamma_1'(\mathbf{w}), \gamma_1(\mathbf{v_1}), \gamma_1'(\mathbf{w})\}\\
&= \mathbf{min}\big\{\mathbf{min}\{\gamma_1(\mathbf{u_1}), \gamma_1'(\mathbf{w})\}, \mathbf{min}\{\gamma_1(\mathbf{v_1}), \gamma_1'(\mathbf{w})\}\big\}\\
&= \mathbf{min}\{(\gamma_1 \times \gamma_1')(\mathbf{u_1},\mathbf{w}), (\gamma_1 \times \gamma_1')(\mathbf{v_1},\mathbf{w})\}
\end{aligned}
$$

Hence it is clear that the arcs in this situation and $\mathbf{w} \notin D_2$ is covered by the vertex $(\mathbf{v_1},\mathbf{w}) \in \{D_1 \times (\mathbf{V_2} \backslash D_2)\}$.
Hence the arcs in case (ii) are covered by the vertex set $\{D_1 \times (\mathbf{V_2} \backslash D_2)\} \cup \{(\mathbf{V_1} \backslash D_1) \times D_2\}$.
Therefore $\{D_1 \times (\mathbf{V_2} \backslash D_2)\} \cup \{(\mathbf{V_1} \backslash D_1) \times D_2\}$ is the minimum anti-vertex covering and which covered all strong arcs of the anti-cartesian product $\mathbf{G_{A1}} \times \mathbf{G_{A2}}$.

Theorem 3.6: Let $\mathbf{G_{A1}} = \langle \mathbf{V_1},\ \mathbf{E_1} \rangle$ and $\mathbf{G_{A2}} = \langle \mathbf{V_2},\ \mathbf{E_2} \rangle$ be two intuitionistic anti-fuzzy graphs with anti-vertex covering D_1 and D_2, respectively. The minimum anti-vertex covering of the anti-join $\mathbf{G_{A1}} + \mathbf{G_{A2}}$ is either $\mathbf{V_1} \cup D_2$ or $D_1 \cup \mathbf{V_2}$.

Proof: Let $\mathbf{G_{A1}} = \langle \mathbf{V_1},\ \mathbf{E_1} \rangle$ and $\mathbf{G_{A2}} = \langle \mathbf{V_2},\ \mathbf{E_2} \rangle$ be two intuitionistic anti-fuzzy graphs with anti-vertex covering D_1 and D_2, respectively. The arc membership values in $\mathbf{G_{A1}}+\mathbf{G_{A2}}$ are of the form

$$(\mu_2 + \mu_2')(u,v) = \begin{cases} (\mu_2 \cup \mu_2')(u,v), \text{if } (u,v) \in E_1 \cup E_2 \\ \max\{\mu_1(u), \mu_1'(v)\}, \text{if } (u,v) \in E' \end{cases}$$

and the arc non-membership values in $\mathbf{G_{A1}}+\mathbf{G_{A2}}$ are of the form

$$(\gamma_2 + \gamma_2')(u,v) = \begin{cases} (\gamma_2 \cup \gamma_2')(u,v), \text{if } (u,v) \in E_1 \cup E_2 \\ \min\{\gamma_1(u), \gamma_1'(v)\}, \text{if } (u,v) \in E' \end{cases}$$

The strong arcs in $\mathbf{G_{A1}}$ and $\mathbf{G_{A2}}$ are covered by D_1 and D_2, respectively. Now we need to prove that the arcs of the form (u,v), where $u \in \mathbf{V_1}$ and $v \in \mathbf{V_2}$ are covered by one of the vertex set $\mathbf{V_1}, \mathbf{V_2}$.
Consider the arcs of the form (u,v), where $u \in \mathbf{V_1}$ and $v \in \mathbf{V_2}$ are strong arcs and which are incident with vertices in $\mathbf{V_1}$ and vertices in $\mathbf{V_2}$ by the definition of $\mathbf{G_{A1}}+\mathbf{G_{A2}}$. Hence $\mathbf{V_1}$ and $\mathbf{V_2}$ both the sets covered all the arcs of the form (u,v), where $u \in \mathbf{V_1}$ and $v \in \mathbf{V_2}$. Thus the minimum anti-vertex covering of $\mathbf{G_{A1}}+\mathbf{G_{A2}}$ is either the anti-union $\mathbf{V_1} \cup D_2$ or $D_1 \cup \mathbf{V_2}$.

Theorem 3.7: If $\mathbf{G_A} = \langle \mathbf{V}, \mathbf{E} \rangle$ is an IAFG with minimum anti-vertex cover $D \subseteq \mathbf{V}$, then $\mathbf{V} \setminus D$ is an anti-independent set of $\mathbf{G_A}$.

Proof: Let $\mathbf{G_A} = \langle \mathbf{V}, \mathbf{E} \rangle$ be an IAFG with minimum anti-vertex cover D. By the definition of anti-vertex covering of IAFG, evry strong arc in $\mathbf{G_A}$ is incident with at least one vertex of D. So each vertex v in $\mathbf{V} \setminus D$ is adjacent to at least one vertex in D.

Suppose $\mathbf{V} \setminus D$ is not an anti-independent set in IAFG $\mathbf{G_A}$. Therefore u, v in $\mathbf{V} \setminus D$ is adjacent in $\mathbf{G_A}$ and there exist a strong arc between u and v. Here the strong arc (u,v) is not covered by any of the nodes in D. This is a contradiction to $\mathbf{V} \setminus D$ is not an anti-independent set of $\mathbf{G_A}$. Since D is the minimum anti-vertex covering set of $\mathbf{G_A}$, there does not exist any strong arc between two vertices of $\mathbf{V} \setminus D$. Hence $\mathbf{V} \setminus D$ will be an anti-independent set of IAFG $\mathbf{G_A} = \langle \mathbf{V}, \mathbf{E} \rangle$.

Theorem 3.8: If $\mathbf{G_A} = \langle \mathbf{V}, \mathbf{E} \rangle$ is a complete IAFG, then the anti-vertex covering number, $\beta(\mathbf{G_A}) = \delta_N(\mathbf{G_A})$, the minimum neighbourhood degree of $\mathbf{G_A}$.

Proof: Let $\mathbf{G_A} = \langle \mathbf{V}, \mathbf{E} \rangle$ be a complete IAFG. To find the anti-vertex covering number, we can use the following process:
Let $\mathbf{V} = \{v_1, v_2, v_3, \dots \dots, v_n\}$ be the vertex set of complete IAFG $\mathbf{G_A}$.
Let v_1 be the vertex having minimum degree in $\mathbf{G_A}$. Since $\mathbf{G_A}$ is a complete IAFG, there exists strong arcs between v_1 and other vertices of $\mathbf{G_A}$. These strong arcs are covered by v_1 and the sub graph induced by $\mathbf{V} \setminus \{v_1\}$ is also a complete IAFG, denoted by $\mathbf{G_{A1}}$. Choose a vertex v_2 having minimum degree in $\mathbf{G_{A1}}$. Since $\mathbf{G_{A1}}$ is complete, there exist strong arcs between v_2 and other vertices of $\mathbf{G_{A1}}$. These strong arcs are covered by v_2 and the sub graph induced by $\mathbf{V} \setminus \{v_1, v_2\}$ is also a complete IAFG, denoted by $\mathbf{G_{A2}}$. Continue this process till we get the vertex v_n having maximum degree in $\mathbf{G_A}$.
From this procedure, it is clear that all strong arcs are covered by the vertex set $\mathbf{V} \setminus \{v_n\}$ and which will be the minimum anti-vertex cover of $\mathbf{G_A}$. Since $\mathbf{G_A}$ is a complete IAFG, $\mathbf{V} \setminus \{v_n\}$ is the neighbourhood of vertex v_n.
So the anti-vertex covering number, $\beta(\mathbf{G_A}) = |\,\mathbf{V} \setminus \{v_n\}\,|$

$$= |\text{ Neighbourhood of vertex } v_n\,|$$

$$= |\,N(v_n)\,|$$

$$= \delta_N(\mathbf{G_A})$$

Theorem 3.9: If the IAFG $\mathbf{G_A} = \langle \mathbf{V}, \mathbf{E} \rangle$ is complete and bipartite, then the anti-vertex covering number of $\mathbf{G_A}$ is the minimum of the cardinalities of the partition vertices of $\mathbf{V}$ in $\mathbf{G_A}$.

Proof: Let $\mathbf{G_A} = \langle \mathbf{V}, \mathbf{E} \rangle$ be a complete, bipartite IAFG. Suppose the vertex set $\mathbf{V}$ is partitioned into $\mathbf{V_1}$ and $\mathbf{V_2}$. Since $\mathbf{G_A}$ is complete, each arc is a strong arc. Since $\mathbf{G_A}$ is complete bipartite with vertex partitions as $\mathbf{V_1}$ and $\mathbf{V_2}$, every strong arc has one end vertex in $\mathbf{V_1}$ and other end vertex in $\mathbf{V_2}$. So $\mathbf{V_1}$ and $\mathbf{V_2}$ will cover all the arcs of $\mathbf{G_A}$.
Thus, the anti-vertex covering number of $\mathbf{G_A}$ = minimum of cardinalities of $\mathbf{V_1}$ and $\mathbf{V_2}$.
That is, $\beta(\mathbf{G_A}) = \mathbf{min}\{|\,\mathbf{V_1}|, |\,\mathbf{V_2}|\}$.

Theorem 3.10: In an IAFG $\mathbf{G_A} = \langle \mathbf{V}, \mathbf{E} \rangle$, if a vertex $u \in \mathbf{V}$ satisfy $d_N(u) = \triangle_N(\mathbf{G_A})$, then u exist in the minimum anti-vertex covering set.

Proof: Let $\mathbf{G_A} = \langle \mathbf{V}, \mathbf{E} \rangle$ be an IAFG with minimum anti-vertex covering D. Let $u \in \mathbf{V}$ with neighbourhood degree equals maximum neighbourhood degree of $\mathbf{G_A}$. That is, $d_N(u) = \triangle_N(\mathbf{G_A})$. Assume that $u \notin D$, the minimum anti-vertex cover of $\mathbf{G_A}$. Since $d_N(u) = \triangle_N(\mathbf{G_A})$, u has maximum neighbourhood degree in $\mathbf{G_A}$ and there exist a strong arc uv in $\mathbf{G_A}$ such that $v \in D$.
Since u has maximum neighbourhood degree in $\mathbf{G_A}$, $d_{\mathbf{G_A}}(u) < d_{\mathbf{G_A}}(v)$.
Thus D is not a minimum anti-vertex cover of $\mathbf{G_A}$ and which is a contradiction to the assumption that $u \notin D$. So $u \in D$ is the vertex having maximum neighbourhood degree in $\mathbf{G_A}$ and a vertex in minimum anti-vertex cover of $\mathbf{G_A}$.

V. CONCLUSION

Anti-fuzziness has an indispensable role in the real world problems. In this paper, we defined and presented some new parameters of intuitionistic anti-fuzzy graphs. The anti-vertex covering is defined and derived some theorems on anti-vertex covering in terms of other intuitionistic anti-fuzzy graph parameters and operations. Intuitionistic anti-fuzzy graph has distinct applications in the field of information technology, artificial intelligence, efficiency management etc. Further, we proposed to extend the study of intuitionistic anti-fuzzy graphs on its new dimensions.

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