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Self-Mappings of Banach Spaces with Unique Common Fixed Point

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ABSTRACT

Fixed point theorems for self mappings of a convex subset in Banach Spaces are considered in this manuscript. The out-turn hypothesize and enlarge the sequel due to Fisher [2] and Gregus [4]. Mapping which considershere is not necessarily commuting and given some Examplesto support the outcome of the work.

Keywords: Banach Space, Fixed point theorem, Unique fixed point, self-Mapping

Introduction:

Several authors have been carried out in analyzing the influence of fixedpoint theorems self-mappings of a closed subset of a Banach space both at single-valued and multi-valued maps. In contrast most of the applicationsdo not involve self-mapping of a closed set. A non-expansive mapping contains contraction mappings and is containedunder all continuous mappings. Browder [1], Gohde [3] and Kirk [6] have proved a fixedpoint theorem for non-expansive mappings on a closed, bounded and convex subset of a uniformly convex Banach space and in spaces with richer structure.

In this manuscript, deliberate the use of fixed point theorems for self-mappings of Banach space with unique common fixed point. Solution of Fisher [2] succeeded by Gregus [4], Hardy and Rogers [5] findings have been referred during the study.

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Prefatory:

Definition 1:

Let R and S be two self-mappings of a Banach space X. The pair {R, S} to be weakly commuting if $||RSx-SRx|| \leq ||Rx-Sx||$, for all $x \in X$.

Let X is a Banach space and C, a closed convex subset of X.

Lemma 1:
Let R, S be self-maps of C such that $||Rx-x|| \leq ||Ry-y||$ (1)
If and only if $||Sx-x|| \leq ||Sy-y||$ for all $x, y \in C$.
Then
$\inf \{\max (||Rx-x||, ||Sx-x||): x \in C\} = \max \{\inf (||Rx-x|| : x \in C), \inf (||Sx-x|| : x \in C)\}$.

Proof:

If for any $x \in C$,
Put$A(x) = \max \{||Rx-x||,||Sx-x||\}$, $m = \inf\{A(x): x \in C\}$and
$p = \inf \{||Rx-x||: x \in C\}$,
$q = \inf \{||Rx-x||: x \in C \}$.
Since $\max \{p,q\} < M(x)$, $x \in C$, $\Rightarrow$ $\max \{s,t\} \leq m$.

Suppose $\max \{p,q\} < m$.
Then there exist $x \in C$, $y \in C$ such that

$$||Rx-x|| < s + m - s = m, \quad (2)$$

and

$$||Sy - y|| < t + m - t = m. \quad (3)$$

$\Rightarrow M(x) = ||Sx-x||, M(y) = ||Ry-y||$.
$M(x) \geq m$ and $M(y) \geq m$, from (2) and (3)
$\Rightarrow ||Rx-x|| < ||Ry-y||$,and$||Sy-y|| < ||Sx-x||$,
This is a contradiction to (1)
$\Rightarrow \text{Max} \{s,t\}= m$.

Accordingly,the result follows.

Contractive condition considered here is a slight variant of that studied by Hardy and Rogers [5].

Main Results:

Theorem 1:

Let R, S be self mappings of C satisfying (1) and

$$||Rx-Sy|| \leq a\ ||x-y|| + b \max\{||Rx-x||, ||Ry-y||\} + c \max\{||Rx-x|| + ||x-y||, ||Sy-y|| + ||x-y||\} \quad (4)$$

For all x, y ∈ C, a, b, c are such that o < a < l, o < b <l,c > o,
a + b + 2c - 1 and 4c (2 - b) <a (l- a).
Then R and S have a unique common fixed point, which is also a unique fixedpoint of both R and S.

Proof:

Let x∈ C be arbitrary. From (4), infer that

$$||RSx-Rx|| \leq a\ ||Sx-x|| + b \max\{||RSx-Rx||, ||Rx-x||\} + c \max\{||RSx-Rx|| + ||Rx-x||, ||Rx-x|| + ||Rx-x||\},$$

$$\Rightarrow \quad ||RSx-Rx|| \leq ||Sx-x|| \quad (5)$$

Analogously, $||SRx-Rx|| \leq ||Rx-x||$ (6)

Since (5) and (6) detain for any x ∈ C.
Alsoacquire

$||RSRx-SRx|| \leq ||SRx-Rx|| \leq ||Rx-x||$ and
$||SRSx-RSx|| \leq ||RSx-Sx|| \leq ||Rx-x||$

Ago in (1), yield

$$||SSRx-SRx|| \leq ||Rx-x|| \quad (7)$$

And $||RRSx-SRx|| \leq ||Rx-x||$ (8)

Prescribe a point z as

$$z = \frac{1}{2} SRx + \frac{1}{2} SSRx. \quad (9)$$

(7) and (9),

$$\Rightarrow 2||SRx-z|| = 2\ ||RSRx-x|| = ||RRSx-RSx|| \leq ||Rx-x||. \quad (10)$$

C -Convex, z ∈C and using (4), (6), (7) and (10),

$$\Rightarrow 2||Rz-z|| \leq ||Rz-SRx|| + ||Rz-SSRx|| \quad (11)$$

$$\leq a\ ||Rx-z|| + b \max\{||Rz-z||, ||SRx-Rx||\} + c\max\{||Rz-Rx||, ||SRx-z||\} + a\{||z-SRx||\}$$

$$+ b \max\{||Rz-z||, ||SSRx-SRx||\} + c \max\{||Rz-SRx||, ||SSRx-z||\}$$

$$\leq a(||Rx-z|| + ||z-SRx||) + 2b\max\{||Rz-z||, ||Rx-x||, ||Sx-x||\} + 2c \max\{||Rz-z|| + ||Rx-z||, ||Rz-z|| + ||z-SRx||\}.$$

Further, using (4), (6) and (7),

$$\Rightarrow 2||Rx-z|| \leq ||Rx-SRx|| + ||Rx-SSRx|| \quad (12)$$

$$\leq ||Rx-x|| + a\ ||x-SRx|| + b \max\{||Rx-x||, ||SSRx-SRx||\}$$

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$+c\max\{\|SRx-Rx\|,\|x-SSRx\|\}$

$\leq (2a+l)\ \|Rx-x\|+bM(x)+c\max\{\|Rx-x\|,3M(x)\}$

$\leq (2+a+2c)\ M(x).$

$\Rightarrow$ (11) And (12) jointly $\Rightarrow$ that

$$2\|Rz-z\| \leq a(3/2+a/2+c)M(x)+2b\max\{M(x),\|Rz-z\|\}$$
$$+2c\max\{\|Rz-z\|+(l+a/2+c)M(x),\|Rz-z\|+1/2\,M(x)\}. \quad (13)$$

Then $\|Rz-z\| < M(x)$,

Otherwise (13) yield

$\|Rz-z\| < 1/2\,(3a/2 + a2/2 + 2ac + 2c2 + 2b+4c)\ \|Rz-z\|=\lambda\|Rz-z\|<\|Rz-z\|$,

Where $0 < \lambda = 1/2\,(2 + a^2/2 - a/2 + 4c - 2bc) < 1$,

By the conjecture on constants a,b,c.

$\Rightarrow \|Rz-z\| \leq \lambda M(x).$ (14)

Putting $h = \inf\{\|Rz-z\| : z = 1/2\,SR\,x + 1/2\,SSRx,\ x \in C\}$,

by virtue of the Lemma 1, and from (14),We deduce that

$h \leq \lambda.m = \lambda,.\ \mathrm{Max}\,\{p,q\}$.Thus

$h \leq \lambda.q$ (15)

Obviously $s \leq h$. (16)

Similarly, by construe $z' = 1/2\,RSx + 1/2\,RRSx$ and using (8),

$\Rightarrow 2\|RSx-z'\| = 2\,\|RRSx-z'\|$ (17)

$= \|SSRx-SRx\| < \|Rx - x\|.$

By setting:

$K = \inf\{\|Sz'-z'\|: z' = 1/2\,RS\,x + 1/2\,SSRx,\ x \in C\}$,

By handling (4), (5), (8) and (17),

We acquire the inequality:

$k \leq \lambda.p$ (18)

Resulting evidently

$k \geq q.$ (19)

Thus (15), (16), (18) and (19) , that $p \leq h \leq \lambda.q \leq \lambda.k \leq \lambda^2.p$.

p=o because $o<\lambda< 1$, and consequently $q = 0$, from (18) and (19).

So each of the sets $G\mu$ and $H\mu$ for every $\mu > o$ must be nonempty, where

$G\mu = \{x \in C: \|Rx-x\| \leq \mu\}$,

$H\mu = \{x \in C: \|Sx-x\| < \mu\}$.

Further, diam $G\mu < (4+c)\,.\mu/b$. (20)

From (4) and (6), and for any $x,y \in G\mu$,

Weacquire

$\|x-y\| \leq \|x-Rx\| + \|y-Ry\| + \|Rx-SRx\| + \|Ry-SRx\|$

$\leq 3\mu + a\,\|Rx-x\| + a\|x-y\| + b\max\{\|Ry-y\|,\|Rx-x\|\}$

$+ c\max\{\|y-Rx\| + \|Rx-SRx\|, \|Rx-y\| + \|Ry-y\|\}.$

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$\leq (3+a+b)\mu+a\|x-y\| + c\{(\|x-y\|+\|x-Rx\|+\mu)\}$.
$\leq (4+c)\mu + (a+c) \| x-y \|$.
From the last inequality, (20) follows, since $a+c = 1 - b$.
Let $\overline{H}\sigma$ denote the closure of $H\sigma$ for any $\sigma>0$, choose $x \in \overline{H}\sigma$.
Arbitrary $\in > 0$, there exists a point $y \in H\sigma$ such that $\|x-y\|\leq\in$.
Applying(4),

$$\Rightarrow \| Rx-x\| \leq \| Rx-Sy \| +\| Sy-y \| + \| x-y\| \tag{21}$$
$$\leq a\|x-y \| +b \max \{\| Rx-x\|, \| Sy-y \| \}$$
$$+ c \max \{\| x-y\| +\| y-Sy \|, \| x-y \| + \|Rx-x\| \} + \sigma + \in$$
$$\leq (l+a)\in +b \max \{\| Rx-x\| ,\sigma\} + c \max\{\in + \sigma, \in +\|Rx-x\| \}+\sigma .$$

If $\| Rx-x\| \leq \sigma$, then $x \in G\sigma\subset G\sigma/a$ since $o < a < 1$.
If $\| Rx-x\|>\sigma$, infer from (21) that
$\| Rx-x\|<(l+a+c) \in + (b+c) \| Rx-x\| +\sigma$
$\Rightarrow \| Rx-x\|\leq \sigma/a, \in$ being arbitrary and $b+c = 1-a$.
$\Rightarrow x \in G\sigma/a$, that is $\overline{H}\sigma\subset G\sigma/a$ in each case.
Let $\{\sigma_n\}$ be a decreasing sequence of reals for which $\sigma_{(n)} = \sigma_n \to 0$ as $n\to\infty$.
So $\{\overline{H}\sigma_{(n)}\}$ is a decreasing sequence of non-empty closed subsets of C suchthat, by (20),
diam $\overline{H}\sigma_{(n)}\leq$diam $G \sigma_{(n)} /a \leq (4+c) \sigma_{(n)} /ab$.
Clearly, diam $\overline{H}\sigma_{(n)} \longrightarrow 0$ as $n \longrightarrow\infty$.
As X is complete, by theCantor's Intersection Theorem,
There is a $w \in X$ such that
$\{w\} = \cap^{\infty}_{n=1}\overline{H}\sigma_{(n)}\subset\cap^{\infty}_{n=1} G \sigma_{(n)} /a$.
$\Rightarrow \| Pw-w \|\leq a. 7/a$ for every $n=l,2 ,\ldots$,and so $Pw = w$.
From (4), acquire

$$\|w-Sw\|=\|Rw-Sw\|$$
$$\leq b\max\{\|Rw-w\|,\|Sw-w\|\} + c \max \{\| w-Sw\|,\| w-Rw\| \}$$
$$= (l-a) \|Sw-w\|.$$

$\Rightarrow Sw = w$.
So w is a common fixed point of R and S.
Let w' be another fixed point of R.
Then, applying (4) for $x = w$ and $y = w'$,

$$\Rightarrow \|w'-w\|=\|Pw'-Qw\|$$
$$\leq a \| w'-w \| +b\max \{ \| Pw ' - w' \| , \| T w - w \|\}$$
$$+c \max\{\|w'-Qw\|,\|w-Pw'\|\}-(l-b) \|w'-w\|.$$

$\Rightarrow w' = w$.
$\Rightarrow$ w is the unique fixed point of R.
Similarly, one can show that w is the unique fixed point of S.

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$\Rightarrow$Completes the proof
By Theorem1 for some iterates of S and R.
We have the following.

Theorem 2:
Let R, S: C $\rightarrow$C satisfying $\|x - P^m x\| \leq \|y - R^m y\|$ if and only if
$\|x - S^l x\| \leq \|y - S^l y\|$, and $\|R^m x - S^l y\| < a\|x-y\| + b.\max\{\|P^m x - x\|, \|S^l y - y\|\}$
+ c.max $\{\|R^m x - y\|, S^m y - x\}$ for all x,y$\in$ C, where l,m are positive integers and a,b,c are as in Theorem 1.
Then R and S have a unique common fixed point, which is also the unique fixed point of both R and S.

Proof:
By Theorem 1, the maps $R^m : C \rightarrow C$ and $S^l : C \rightarrow C$ have a unique common fixed point w.
Since $Rw = R(R^m w) = R^m (Rw)$,infer that Rw is also a fixed point of R^m.
Theorem 1, assures that w is also the unique fixed point of R^m, necessarily have Rw=w.
Similarly, one can show that Sw = w.
So w is the unique common fixed point of R and S.
If w' is another fixed point of R, we have $R^m w' = w'$, but the uniqueness of w implies w = w'.
Therefore, w is also the unique fixed point of R as well as for the map S.

Example 1:
Let X be the Banach space of reals with Euclidean norm and C = [0,2]. Define R, S: C $\rightarrow$C by putting,
$R(x) = 0$ if $0 \leq x < l$,$R(x) = 3/5$ if $1 \leq x < 2$,$S(x) = 0$ if $0 \leq x<2$,$S(2) =9/5$.
Then condition (4) of Theorem 1 does not hold.
Otherwise, taking x = 1 and y = 2,
We have:
$\|Rl - S2\| = 6/5$
$\leq a(2-l) + b \max (1-3/5), (2-9/5) + c \max (9/5-1), (2-3/5)$
$= a + 2/5b + 7/5c$
$\leq 3/5a + 2/5 + c$.
By the assumptions of Theorem 1,
$\Rightarrow 4c < a(1-a).(2-b)^{-1} < 1/2$,
$\Rightarrow 6/5 \leq 1 + 1/8 = 9/8$,
This is a contradiction.
However, Theorem 2 is trivially satisfied for l = m = 2,
Since$S^2(x) = R^2(x) = 0$ for any $x \in C$.

Remark 1: By assuming c=0 in Theorem 1, we obtain the main theorem of Fisher [2]. The proof exhibited in [2] inherently assumed the commutativity of the mappings under consideration, even though the author does not explicitly mention such hypothesis.

However, one can drop this extra requirement by modifying the arguments of Fisher [2] as indicated by the proof of our Theorem 1

Remark 2: Assuming R = S in Theorem 1, we obtain a result more general than that of Gregus [4] under a different set of conditions on the mapping S.

Conclusion:

Thus, fixed point theorem for self mapping of a convex subset in Banach Space is analyzed. Themapping considered and analyzed is not commuting and have a unique common fixed point. The example discovered from the result.

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