JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 2, 2022, p. 590 - 594
https://publishoa.com
ISSN: 1309-3452

Cesaro – Like Summation of Fabulous Figurate Numbers

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ABSTRACT

The concept of Figurate numbers is very ancient but it continues to fascinate amateur and professional mathematicians to this day. The connection between the two oldest branches of mathematics namely number theory and geometry was made possible in the study of Figurate numbers. In this paper, after introducing Figurate numbers formally, we had defined a new concept which is identified as Cesaro – Like summation and apply it for the family of Figurate numbers in two kinds. These ideas will pave a new way for deep understanding and further generalizations in summability theory.

Keywords: Figurate Numbers, Generating Function, Convergent, Cesaro – Like Summation

1. Introduction

Among several interesting family of sequences of numbers, Figurate numbers continue to fascinate mathematicians and amateurs for several centuries. Several new theorems were established related to their behavior. Figurate number sequence being a quadratic sequence, have wide applications to various branches of Science and Engineering. The concept of Cesaro summation is a way of assigning specific finite real number to given divergent series in a novel way. Such ideas were developed in to a branch called Summability theory. In this paper, we will introduce Cesaro – Like summation technique for family of Figurate numbers in two kinds and try to explore some interesting properties.

2. Definitions

2.1 The *n*th Figurate number of order *m* is defined by $F_{n,m} = \dfrac{n\left[(m-2)n-(m-4)\right]}{2}$ (1)

In (1), if we take $m = 3$, then we get triangular numbers, $m = 4$, we obtain square numbers and for subsequent larger values of *m*, we obtain the corresponding *n* – gonal numbers which are special cases of Figurate numbers.

2.2 The Cesaro – Like first summation of a given infinite divergent series is defined by $(CL^1)\left(\sum_{n=1}^{\infty} a_n\right) = \int_{-1/k}^{1/k} G(x)\,dx$ (2) where $k > 1$ and CL^1 denotes Cesaro – Like summation of first kind.

2.3 The Cesaro – Like second summation of a given infinite divergent series is defined by $(CL^2)\left(\sum_{n=1}^{\infty} a_n\right) = \int_{-k}^{0} G(x)\,dx$ (3) where $k > 1$ and CL^2 denotes Cesaro – Like summation of second kind.

In (2) and (3), $G(x)$ represents the generating function of the divergent series $\sum_{n=1}^{\infty} a_n$.

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4. Generating Function of Figurate Numbers

In this section, we will try to determine the generating function for Figurate numbers, which then can be used for determining Cesaro – Like summation of first and second kinds. First we consider the function described in equation 4.

$$G_m(x)=\frac{(m-3)x+1}{(1-x)^3}\ ,m\geq 3\quad (4)$$

If m = 3, then $G_3(x)=\frac{1}{(1-x)^3}=1+3x+6x^2+10x^3+15x^4+21x^5+\cdots$ (5) is the generating function of triangular numbers and so (5) satisfies the equation $G_3(x)=\sum_{n=1}^{\infty}\frac{n(n+1)}{2}x^{n-1}$ (6). Notice that $F_{n,3}=\frac{n(n+1)}{2}$ are triangular numbers.

If m = 4, then $G_4(x)=\frac{x+1}{(1-x)^3}=1+4x+9x^2+16x^3+25x^4+\cdots$ (7) is the generating function of square numbers and (7) satisfies the equation $G_4(x)=\sum_{n=1}^{\infty}n^2x^{n-1}$ (8).

Notice that $F_{n,4}=n^2$ are square numbers. In this aspect, we prove the following theorem.

4.1 Theorem 1

The function $G_m(x)=\frac{(m-3)x+1}{(1-x)^3}$ is the generating function of Figurate numbers of order m. That is,

$$G_m(x)=\sum_{n=1}^{\infty}F_{n,m}x^{n-1}\quad (9)$$

Proof: Using Binomial expansion for general indices we obtain

$$\begin{aligned}
G_m(x)&=\frac{(m-3)x+1}{(1-x)^3}=\left[(m-3)x+1\right]\left[\sum_{n=1}^{\infty}\frac{n(n+1)}{2}x^{n-1}\right]\\
&=\sum_{n=1}^{\infty}\frac{n(n+1)}{2}x^{n-1}+\sum_{n=1}^{\infty}\frac{n(n+1)}{2}(m-3)x^{n}\\
&=1+\sum_{n=2}^{\infty}\frac{n(n+1)}{2}x^{n-1}+\sum_{n=2}^{\infty}\frac{n(n-1)}{2}(m-3)x^{n-1}\\
&=1+\sum_{n=2}^{\infty}\frac{n}{2}\left[(n+1)+(n-1)(m-3)\right]x^{n-1}=1+\sum_{n=2}^{\infty}\frac{n}{2}\left[(n+1)+(m-3)n-(m-3)\right]x^{n-1}\\
&=1+\sum_{n=2}^{\infty}\frac{n}{2}\left[(m-2)n-(m-4)\right]x^{n-1}=\sum_{n=1}^{\infty}\frac{n}{2}\left[(m-2)n-(m-4)\right]x^{n-1}\\
&=\sum_{n=1}^{\infty}F_{n,m}x^{n-1}
\end{aligned}$$

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This completes the proof.

5. Cesaro – Like Summation of First Kind

Using the definition (2) and the result (9) of theorem 1, we now prove the following theorem.

5.1 Theorem 2

The Cesaro – Like summation of first kind for the series whose terms are Figurate numbers of order *m*, is given by $(CL^1)\left(\sum_{n=1}^{\infty} F_{n,m}\right) = \frac{2k(k^2+m-3)}{(k^2-1)^2}$ (10)

Proof: From theorem 1, we know that $G_m(x)$ is the generating function for Figurate numbers of order *m*. Hence using (2) and (9), we obtain

$$(CL^1)\left(\sum_{n=1}^{\infty} F_{n,m}\right) = \int_{-1/k}^{1/k} G_m(x)\,dx = \int_{-1/k}^{1/k} \frac{(m-3)x+1}{(1-x)^3}\,dx$$

$$= (m-2)\int_{-1/k}^{1/k} \frac{1}{(1-x)^3}\,dx - (m-3)\int_{-1/k}^{1/k} \frac{1}{(1-x)^2}\,dx$$

$$= \frac{2k(k^2+m-3)}{(k^2-1)^2}$$

We note that integral considered is convergent only if $k > 1$. This completes the proof.

5.2 Corollary 1

The Cesaro – Like summation of first kind for Triangular, Square, Pentagonal and Hexagonal numbers are given by $\frac{2k^3}{(k^2-1)^2}, \frac{2k(k^2+1)}{(k^2-1)^2}, \frac{2k(k^2+2)}{(k^2-1)^2}, \frac{2k(k^2+3)}{(k^2-1)^2}$ (11) respectively.

Proof: Since Triangular, Square, Pentagonal and Hexagonal numbers are Figurate numbers of orders 3, 4, 5 and 6 respectively, (11) from (10) of theorem 2 by considering m = 3,4,5,6 respectively.

6. Cesaro – Like Summation of Second Kind

In this section, we prove a theorem similar to theorem 2 for first kind.

6.1 Theorem 3

The Cesaro – Like summation of second kind for the series whose terms are Figurate numbers of order *m*, is given by $(CL^2)\left(\sum_{n=1}^{\infty} F_{n,m}\right) = -\frac{k\left[(m-4)k-2\right]}{2(k+1)^2}$ (12)

Proof: From theorem 1, we know that $G_m(x)$ is the generating function for Figurate numbers of order *m*. Hence using (3) and (9), we obtain

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$$
\begin{aligned}
(CL^2)\left(\sum_{n=1}^{\infty} F_{n,m}\right) &= \int_{-k}^{0} G_m(x)\,dx = \int_{-k}^{0} \frac{(m-3)x+1}{(1-x)^3}\,dx \\
&= (m-2)\int_{-k}^{0} \frac{1}{(1-x)^3}\,dx - (m-3)\int_{-k}^{0} \frac{1}{(1-x)^2}\,dx \\
&= -\frac{k\left[(m-4)k-2\right]}{2(k+1)^2}
\end{aligned}
$$

We note that integral considered is convergent only if $k > 0$. This completes the proof.

6.2 Corollary 2

The Cesaro – Like summation of second kind for Triangular, Square, Pentagonal and Hexagonal numbers are given by $\frac{k(k+2)}{2(k+1)^2}, \frac{k}{(k+1)^2}, -\frac{k(k-2)}{2(k+1)^2}, -\frac{k(k-1)}{(k+1)^2}$ (13) respectively.

Proof: Since Triangular, Square, Pentagonal and Hexagonal numbers are Figurate numbers of orders 3, 4, 5 and 6 respectively, (13) from (12) of theorem 3 by considering m = 3,4,5,6 respectively.

7. Limiting Cases

We now prove two results by assuming k is very large.

7.1 Theorem 4

The Cesaro – Like summations of first and second kinds satisfy the following

$$
\lim_{k\to\infty}\left(k \times (CL^1)\left(\sum_{n=1}^{\infty} F_{n,m}\right)\right) = 2 \qquad (14)
$$

$$
\lim_{k\to\infty}\left((CL^2)\left(\sum_{n=1}^{\infty} F_{n,m}\right)\right) = -\frac{m-4}{2} \qquad (15)
$$

Proof: From (10), we have

$$
\lim_{k\to\infty}\left(k \times (CL^1)\left(\sum_{n=1}^{\infty} F_{n,m}\right)\right) = \lim_{k\to\infty}\left(\frac{2k^2(k^2+m-3)}{(k^2-1)^2}\right) = 2
$$

Similarly, from (12), we have

$$
\lim_{k\to\infty}\left((CL^2)\left(\sum_{n=1}^{\infty} F_{n,m}\right)\right) = \lim_{k\to\infty}\left(-\frac{k\left[(m-4)k-2\right]}{2(k+1)^2}\right) = -\frac{m-4}{2}
$$

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This completes the proof.

8. Conclusion

By considering the divergent series whose terms are Figurate numbers of order *m*, we had defined Cesaro – Like summation in two kinds and try to explore their properties. Theorem 1 provides the proof for determining the generating function of Figurate numbers of order *m*. In theorem 2, a nice closed expression is obtained for the Cesaro – Like summation of first kind and in theorem 3, a compact formula for Cesaro – Like summation of second kind is obtained. Some special cases are discussed in the two corollaries following theorems 2 and 3. Finally, the behavior of Cesaro – Like summation of Figurate numbers of order *m* when is obtained in theorem 4. These results are relatively new and provide scope for obtaining similar results in future.

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