JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 2, 2022, p. 948 - 954
https://publishoa.com
ISSN: 1309-3452

Comparing Some Estimators of a Scale Parameter for Pranav Distribution

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ABSTRACT
In this article, the problem of estimating the parameter of one of the recently introduced lifetime distributions known as the Pranav distribution is studied. The estimation of the scale parameter was addressed using the moments method, maximum likelihood method and the proposal to use the ordinary least squares method for the purpose of estimating the scale parameter of this distribution. The simulation results showed for different sample sizes and default values for the parameter, the proposed estimator was much better than the performance of other estimators, as in the most expensive cases the mean square error value of the proposed parameter using ordinary least squares method was less than its value for the parameters obtained using other methods.
Keywords: Pranav distribution, Lifetime distributions, Ordinary least squares method, Maximum likelihood method, Moments method.

Introduction

For modeling lifetime data, many one-parameter and two-parameter distributions have been proposed by several researchers. Some of them are producing helpful results for a variety of biological and engineering data sets that are deemed to be lifetime distributions. However, some of them, such as the Lindley distribution proposed by Lindley, the Akash and Shanker distributions proposed by Shanker, and the Ishita distribution proposed by Shanker, do not produce good results for all biological and engineering data sets. These distributions provide a better fit than exponential distributions, whereas Shanker's proposed Akash distribution was tested on biological data and shown to be superior to Lindley and exponential distributions.
M. E. Ghitany (2008) [3] reported that the Lindley distribution is superior to the exponential distribution, but the search's aim was to suggest a lifetime distribution that would provide a better fit than other lifetime distributions. That is the primary motive for proposing a new life distribution and used with biological data.
As a result, looking for a new lifetime distribution to might be better than Exponential, Sujatha, Ishita, Lindley, Shanker, and Akash. Therefore, Shukla (2018) proposed Pranav distribution and it is defined as a one parameter lifetime distribution with scale parameter.
Pranav distribution was obtained by mixing the exponential distribution and the gamma distribution shape parameter equal to 4 with appropriate mixing proportions, and it has real life applications. And it was given the name Pranav distribution in the name of Pranav Shukla eldest son for Kamlesh Kumar Shukla who suggested this distribution in 2018. [4] [8]
Estimating unknown parameters in the statistical distribution is one of the most problems that who are interested in the applied statistics face on a regular basis. Because many inferences are built on the basis of these estimators. Therefore, this article aims to find an estimator using the ordinary least squares method and make a comparison through simulation with the available estimators for the measurement parameter of the Pranav distribution to ensure its performance for different sample sizes and default values.

Pranav distribution

Pranav distribution is one of the lifetime distributions that was proposed in the last years and suggested by Shukla in 2018.[8]
Pranav distribution is a mixture of two–distributions, exponential distribution having scale parameter θ and gamma distribution having shape parameter equal to 4 and scale parameter θ, and their mixing proportions of $\frac{\theta^4}{(\theta^4+6)}$ and $\frac{6}{(\theta^4+6)}$ respectively, this distribution has the potential to provide a better on real lifetime data.

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The random variable t is said to be the Pranav distribution if it is p.d.f. given as below:

$$f(t;\theta)=\frac{\theta^4}{\theta^4+6}(\theta+t^3)e^{-\theta t}\,;\ \theta>0\,,t>0 \qquad (1)$$

The cumulative distribution function (Failure function) of this distribution is:

$$F(t;\theta)=1-\left[1+\frac{\theta t(\theta^2t^2+3\theta t+6)}{\theta^4+6}\right]e^{-\theta t};t>0,\theta>0 \qquad (2)$$

The most important properties for the Pranav distribution are reviewed, the mean, variance, standard deviation for the Pranav distribution are as follows:[8]

$$\mu=\frac{\theta^4+24}{\theta(\theta^4+6)}$$

$$\sigma^2=\frac{(\theta^8+84\theta^4+144)}{\theta^2(\theta^4+6)^2}$$

$$\sigma=\frac{\sqrt{(\theta^8+84\theta^4+144)}}{\theta(\theta^4+6)}$$

The moment generating function (MGF) of Pranav distribution is:

$$Mx(t)=\frac{\theta\left[\theta^4+(r+3)(r+2)(r+1)\right]}{(\theta+t)(\theta^4+6)} \qquad (3)$$

The characteristic function of Pranav distribution is:

$$\phi x(t)=\frac{\theta\left[\theta^4+(r+3)(r+2)(r+1)\right]}{(\theta+it)(\theta^4+6)} \qquad (4)$$

The Coefficient of variation (C.V.) is:

$$C.V.=\frac{\sigma}{\mu}=\frac{\sqrt{(\theta^8+84\theta^4+144)}}{(\theta^4+24)} \qquad (5)$$

With regard to the measure's coefficient of skewness (C.S.) and the coefficient of kurtosis (C.K.) is are defined as following:

$$C.S.=\frac{\mu_3}{\mu_2^{\frac{3}{2}}}=\frac{2(\theta^{12}+198\theta^8+324\theta^4+864)}{(\theta^8+84\theta^4+144)^{\frac{3}{2}}} \qquad (6)$$

$$C.K.=\frac{\mu_4}{\mu_2^2}=\frac{9(\theta^{16}+312\theta^{12}+2304\theta^8+10368\theta^4+10368)}{(\theta^8+84\theta^4+144)^2} \qquad (7)$$

The hazard rate function, $h(t)$ of Pranav distribution are obtained as:

$$h(t)=\frac{\theta^4(\theta+t^3)}{\theta^3t^3+3\theta^2t^2+6\theta t+\theta^4+6} \qquad (8)$$

The mean residual life function ($m(t)$) for Pranav distribution is known as:

$$m(t)=\frac{(\theta^3t^3+6\theta^2t^2+18\theta t+\theta^4+24)}{\theta(\theta^3t^3+3\theta^2t^2+6\theta t+\theta^4+6)} \qquad (9)$$

Estimating scale parameter

In this section, we'll go over some estimation methods to estimate the scale parameter of Pranav distribution which are moments method and maximum likelihood and ordinary least squares method.

1. Moments method

The moments method is one of the most well-known, the simplest and oldest methods for estimating parameters, and it gets its name from the fact that sample moments are, in some sense, estimates for population moments. Karl Pearson, a British statistician, was the first to discover the moments method in 1902. [6] [7]

The method moments estimator can be found by the solving the following equation: [8]

$$\frac{1}{n}\sum t^r=Et^r \qquad r=1,2,3,\dots,k$$

So, the r th moment about origin of the Pranav distribution can be calculated as follows:

$$Et^r=\int_0^\infty t^r\frac{\theta^4}{(\theta^4+6)}(\theta+t^3)e^{-\theta t}dt$$

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$$= \int_0^\infty \frac{t^r\theta^5}{(\theta^4+6)} e^{-\theta t} dt + \int_0^\infty \frac{t^{r+3}\theta^4}{(\theta^4+6)} e^{-\theta t} dt$$

$$Et^r = \frac{\theta^5\,\Gamma(r+1)}{(\theta^4+6)\theta^{r+1}} + \frac{\theta^4\,\Gamma(r+4)}{(\theta^4+6)\theta^{r+4}}$$

$$= \frac{(r!)\left[\theta^4+(r+3)(r+2)(r+1)\right]}{\theta^r(\theta^4+6)}; r = 1,2,3,\ldots$$

The following are the first four moments about the origin of Pranav distribution:

$$Et = \frac{\theta^4+(r+3)(r+2)(r+1)}{\theta(\theta^4+6)} = \frac{\theta^4+24}{\theta(\theta^4+6)} = \mu_1^{'} = \mu$$

$$Et^2 = \frac{(2)\left[\theta^4+(r+3)(r+2)(r+1)\right]}{\theta^2(\theta^4+6)} = \frac{2(\theta^4+60)}{\theta^2(\theta^4+6)} = \mu_2^{'}$$

$$\mu_3^{'} = \frac{6(\theta^4+120)}{\theta^3(\theta^4+6)} \quad , \quad \mu_4^{'} = \frac{24(\theta^4+210)}{\theta^4(\theta^4+6)}$$

The central moments of the Pranav distribution are thus calculated as

$$\sigma^2 = Et^2 - (Et)^2 = \frac{2(\theta^4+60)}{\theta^2(\theta^4+6)} - \frac{(\theta^4+24)^2}{\theta^2(\theta^4+6)^2}$$

$$= \frac{(\theta^8+84\theta^4+144)}{\theta^2(\theta^4+6)^2}$$

Now by using the principle of method of moments to estimate the parameter θ of the Pranav distribution will be as follows: [8] [10]

$$\frac{1}{n}\sum t = Et \qquad r = 1$$

$$\bar{t} = \frac{\theta^4+24}{\theta(\theta^4+6)}$$

$$\hat{\theta}_{MME}^5\bar{t} - \hat{\theta}_{MME}^4 + 6\hat{\theta}_{MME}\bar{t} + 24 = 0 \qquad (10)$$

By solving the equation (10), we can get the moments method estimator $\hat{\theta}_{MME}$

2. Maximum likelihood estimation method

One of the most common methods for fitting a parametric distribution applied to a set of data that has been seen is maximum likelihood estimation. The maximum likelihood method was proposed in 1922 by Fisher (statistician/geneticist). Despite the moments method is simple and clear to use, although it rarely produces "good" estimators. Because we're aiming to determine the real parameter values that were most likely used to generate the data, we did observe, the maximum probability method is intuitively appealing. In most cases of practical interest, MLE performance is optimal for big enough data. One of the most versatile ways for fitting parametric statistical models to data is this method.[5]

Let t_1, t_2,…, t_n be a n observation from a population with p.d.f. for random sample f$(t;\theta)$. The joint p.d.f for this sample observation is:

$$L(t_1, t_2, \ldots, t_n; \theta) = f(t_1;\theta)\, f(t_2;\theta)\, \ldots\, f(t_n;\theta)$$

$$= \prod_{i=1}^{n} f(t_i;\theta)$$

The likelihood function, also known as the joint density function, is commonly denoted as L.

Let the T = h (t_1, t_2,…, t_n) be an estimator of θ. The MLE estimator states that if T is substituted for in L, the value of L will be maximum, and T is called the maximum likelihood estimator in this situation.

Because any function's natural logarithm (ln) is monotone, the estimator that maximizes L also maximizes lnL. As a result, we can use maximize lnL to find an MLE estimator.

Let the estimator of θ is $\hat{\theta}$ = T. Then to be found out the value of $\hat{\theta}$ by solving the following equation:

$$\frac{\partial ln ln\, L}{\partial\theta} = 0$$

The function lnL is will be maximum for $\hat{\theta}$ if $\frac{\partial^2 ln ln\, L}{\partial\theta^2} < 0$ .When θ in $\frac{\partial^2 ln ln\, L}{\partial\theta^2}$ is replaced by $\hat{\theta}$. However, provided the probability density function meets some regularity conditions, differentiation of lnL is achievable. We cannot solve the last equation if the regularity condition is not met; in this situation the order statistics which maximum f (t; θ) or L are known as the maximum likelihood estimator. [2]

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https://publishoa.com
ISSN: 1309-3452

Now for the Pranav distribution, the following steps will be followed to find the estimator for the scale parameter of the Pranav distribution: [8] [10] [11]
Back to the pdf of the Pranav distribution is given as follows:
$f(t;\theta) = \frac{\theta^4}{\theta^4+6}(\theta + t^3)e^{-\theta t}\,; t > 0, \theta > 0$
For the sample $(t_1, t_2, \ldots, t_n)$, the likelihood function is given as follows:
$L(\theta) = \left(\frac{\theta^4}{\theta^4+6}\right)^n \prod_{i=1}^{n}(\theta + t_i^3)\, e^{-\theta \sum_{i=1}^{n} t_i}$
Taking natural logarithm (ln) for the likelihood function:
$ln\, L(\theta) = n[4ln\theta - ln(\theta^4 + 6)] + \sum_{i=1}^{n} ln(\theta + t_i^3) - \theta \sum_{i=1}^{n} t_i$
Derivation the last equation with respect to θ:
$\frac{\partial ln\, L}{\partial \theta} = \frac{4n}{\hat{\theta}} - \frac{4n\hat{\theta}^3}{\hat{\theta}^4+6} + \sum_{i=1}^{n} \frac{1}{\hat{\theta}+t_i^3} - n\bar{t}$
Setting $\frac{\partial ln\, L}{\partial \theta} = 0$, Implies that:
$\frac{24n}{\hat{\theta}(\hat{\theta}^4+6)} + \sum_{i=1}^{n} \frac{1}{\hat{\theta}+t_i^3} - n\bar{t} = 0$ (11)
The equation (11) can be directly solved to estimate the value of $\hat{\theta}_{MLE}$.

3. Ordinary least squares method

The method of minimizing the error sum of squared function is called ordinary least squares (OLS) estimation. Error sum of squared is to be minimized with respect to estimators. The process of differentiation yields the estimators. Given the data the regression parameters can be estimated using the principle of least squares the estimates will be obtained by minimizing the sum of squared errors, same like before.
$\sum_{i=1}^{n} \varepsilon_i^2 = \sum_{i=1}^{n}\big(Y_i - E(Y_i)\big)^2$ (12)
A needful condition for this expression to assume a minimum value is that its partial derivatives with respect to the parameters equal to zero. [12]
Ordinary and Weighted Least-Squares Estimators. Suppose that $t_{(1)} \le t_{(2)} \le \cdots t_{(n)}$are the order statistics of a random sample from any probability distribution. The i th-order statistic has the mean and the variance as follows: [9]
$E\left(F\big(t_{(i)}\big)\right) = \frac{i}{n+1}$
And $V\left(F\big(t_{(i)}\big)\right) = \frac{i(n-i+1)}{(n+1)^2(n+2)} \quad i = 1 \ldots n$
OLS and WLS were proposed in 1988 by Swain. We can get OLS estimates for the parameters by minimizing the following function with respect to the parameters:
$R(\theta) = \sum_{i=1}^{n}(F\big(t_{(i)}\big) - \frac{i}{n+1})^2$ (13)
where $F\big(t_{(i)}\big)$ represents the theoretical c.d.f. of the observation $t_{(i)}$.
Now for the Pranav distribution in equation (2), We can determine the OLS estimator by (13) with respect to the parameter θ by solving minimizing the following equation:
$\frac{dR(\theta)}{d\theta} = 0$ where $R(\theta) = \sum_{i=1}^{n}\left(1 - \left[1 + \frac{\theta t_i(\theta^2 {t_i}^2 + 3\theta t_i + 6)}{\theta^4+6}\right] e^{-\theta t_i} - \frac{i}{n+1}\right)^2$

Therefore:
$\frac{dR}{d\theta} = 2\left(1 - \left[1 + \frac{\theta t_i(\theta^2 {t_i}^2 + 3\theta t_i + 6)}{\theta^4+6}\right] e^{-\theta t_i} - \frac{i}{n+1}\right).\left(-\left(\frac{3\theta^2 {t_i}^3 + 6\theta {t_i}^2 + 6t_i}{\theta^4+6} - \frac{4(\theta^3 {t_i}^3 + 3\theta^2 {t_i}^2 + 6\theta t_i)\theta^3}{(\theta^4+6)^2}\right) e^{-\theta t_i} + \left(1 + \frac{\theta t_i(\theta^2 {t_i}^2 + 3\theta t_i + 6)}{\theta^4+6}\right) t_i e^{-\theta t_i}\right) = 0$ (14)
The equation (14) can be directly solved to estimate the value of $\hat{\theta}_{OLS}$.

The Simulation:

The simulation experiments used to obtain estimates for scale parameter of Pranav distribution, and compare the performance of the estimators with the estimators of maximum likelihood method and moments method. The steps below have been taken:

1) Specify default values for the scale parameter of the Pranav distribution (θ=0.5,1.5,2.5 and 3.5), respectively.
2) The sample sizes taken were (n = 20,30,50,90,120) in order to assess the influence of sample size on estimators derived from estimation methods.
3) 1000 times (R = 1000) replicate each experiment.
4) Estimate scale parameter of Pranav distribution by three estimation methods have moments method according to the equations (10), the maximum likelihood method according to the equations (11), and the ordinary least squares method according to the equations (14).
5) Make comparison among estimators using mean square error (MSE). It calculates the average squared difference between the estimated value and the true value of the parameter. And for a good estimator we need to have the small value of MSE. [1]

In the context of this study, the formula for this criterion will be as follows:

$$MSE(\hat{\theta}) = \frac{\sum_{i=1}^{R}(\hat{\theta}_i - \theta)^2}{R}$$

Where: $\hat{\theta}$: Estimated value of the scale parameter
θ: True value of the scale parameter
R: Number of the replications

Note that all-computing operations were performed through a computer program designed by the researcher using the Maple 18 program.

Results and discussion

The results of simulation are summarized in the table (1) we noticed that, by increasing sample sizes, estimated values of the scale parameter were near to default values for this parameter in all estimation methods and for all instances of other parameters, we found that the ordinary least squares estimate method was the best for estimating the scale parameter in most cases with default parameters since it had the smallest MSE among the other estimators, mentioned in bold font, when the default parameter (θ = 0.5) and sample sizes (20, 30, 50) we found that the maximum likelihood method was the best for estimating the scale parameter (θ), MSE values decrease with increasing sample size for all parameter values and sample sizes, in general, the OLS method performed better than the other methods in these experiments because most of the results support the OLS method, followed by the moments method and finally the maximum likelihood method.

Table (1) Estimation of scale parameters and the MSE of Pranav distribution using methods of estimation and various sample sizes

θ	n	MME		MLE		OLS	
		$\hat{\theta}_{MME}$	MSE	$\hat{\theta}_{MLE}$	MSE	$\hat{\theta}_{OLS}$	MSE
$\theta = 0.5$	20	0.4508	**0.002412**	0.5498	0.002457	0.4333	0.004446
	30	0.4568	**0.001857**	0.5619	0.001858	0.4459	0.002926
	50	0.4846	**0.000236**	0.4834	0.000248	0.4780	0.000480
	90	0.5068	0.000046	0.4920	0.000040	0.5019	**0.000036**

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Volume 13, No. 2, 2022, p. 948 - 954
https://publishoa.com
ISSN: 1309-3452

	120	0.4848	0.000022	0.4979	**0.000020**	0.4786	0.000245
	20	1.6151	**0.132060**	1.7300	0.141789	1.6267	0.160507
	30	1.7204	**0.048588**	1.5319	0.052250	1.7637	0.069579
$\theta = 1.5$	50	1.6030	0.001061	1.4751	**0.001010**	1.5835	0.006986
	90	1.4876	0.000015	1.4199	**0.000056**	1.5092	0.000086
	120	1.4862	0.000007	1.4507	0.000008	1.5026	**0.000006**
	20	2.6917	0.036760	2.6988	0.039544	2.5505	**0.02558**
	30	2.7505	0.006275	2.7600	0.006763	2.5746	**0.005572**
$\theta = 2.5$	50	2.6144	0.001310	2.6174	0.001379	2.5476	**0.001269**
	90	2.6508	**0.000227**	2.5309	0.000230	2.6674	0.000280
	120	2.6322	0.000174	2.4327	**0.000171**	2.6553	0.000241
	20	3.6721	0.029621	3.6650	0.027244	3.6452	**0.021091**
	30	3.9819	0.023232	3.9810	0.023144	3.8137	**0.020847**
$\theta = 3.5$	50	3.7037	0.006151	3.7054	0.006219	3.5743	**0.005531**
	90	3.6791	0.005211	3.6810	0.005279	3.5717	**0.005150**
	120	3.5791	0.005159	3.5770	0.005138	3.5694	**0.004821**

Conclusions

The ordinary least squares method was used to propose a scale parameter estimator for the Pranav distribution. Through simulation, it was proved that the performance of the proposed estimator was better than the performance of the estimators of the maximum likelihood method and the method of moments, where the results showed that for different sample sizes and different default values for a parameter that value of the mean squares error for the ordinary least squares estimator in most cases was in favor of the estimator proposed in this article, when the sample size increased, followed by the moments method and finally the maximum likelihood method.

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