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On Divisor Cordial Labeling of Certain Classes of Planar Graphs

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Abstract: A DCL of $G(V, E)$ is defined by a bijection $f : V \rightarrow \{1, 2, ..., |V|\}$ such that every line uv is given 1 if $f(u)|f(v)$ or $f(v)|f(u)$ and 0 otherwise; then the positive difference of the count of edges with labels 1 and 0 do not exceed 1. Euler's polyhedral formula, which is related to polyhedron lines, nodes & faces, serves as the foundation for planar graph theory. This paper focuses on exploring the divisor cordial labeling of certain classes of planar graphs obtained from complete graphs & complete bipartite graphs. We have explored these graphs for the graph operation, namely, vertex duplication, which is widely used in ensuring the data integrity.

Keywords: Planar graph, Duplication of vertex, Graph labeling

1.Introduction

Graph labeling is an allocation of numbers to nodes or/and lines of G (V, E), under some constraints. Graph labeling is developing at a rapid rate due to its application in diversified fields like coding theory, circuit design, database management system, X-ray crystallography, radar & missile guidance, communication networks, and network security.

All graphs demonstrated in this article are simple, finite, connected, & undirected. For number theory concepts, we refer to [4] & for terms of graph theory, we refer to [7]. For getting an in-depth detail of graph labeling techniques, results and open problems, one can refer to [6]. Cahit [5] introduced the concept of cordial labeling in an attempt to prove the famous graceful tree conjecture (GTC). By DCL and DCG, we mean divisor cordial labeling and divisor cordial graph, respectively.

After Cahit, numerous cordial variants are introduced and studied. DCL was introduced in [9] and has been explored for various families of graphs.

Definition 1.1 [9] A DCL of G is given by an invertible map $f : V(G) \rightarrow \{1, 2, 3, ..., |V(G)|\}$ such that each line $e = uw$ is given 1 if $f(u)|f(w)$ or $f(w)|f(u)$ & 0 otherwise; then $|e_f(0) - e_f(1)| \leq 1$.

Note: e_f (i) represents the number of lines having label $i \in \{0, 1\}$. The label 0 or 1 is due to the induced function $f^* : E(G) \rightarrow \{0, 1\}$ which is decided according to the constraints of the respective labelings.

The present study is focused on exploring the DCL for classes of planar graphs. Recall that, a graph is planar if it can be drawn in a way that no lines cross each other. Planar graphs are of great importance due to their variety of applications in circuit design, networking and cryptography [2]. While we plan to connect any two nodes (cities, towns, railway tracks etc.), we think of crossings carefully, especially, at the crowded places. As planarity ensures the zero crossing, most of the daily life problems can be dealt by considering the planar graphs. First we recall the definitions given in [1] and [8].

Definition 1.2 [1] The class of graph, denoted by Pl_n has $V(Pl_n) = \{v_1, v_2, ..., v_n\}$ and $E(Pl_n) = E(K_n) \setminus \{v_k v_l : 1 \leq k \leq n-4, k+2 \leq l \leq n-2\}$.

Definition 1.3 [8] Let $V_m = \{v_i : 1 \leq i \leq m\}$ and $U_n = \{u_j : 1 \leq j \leq n\}$ be the bipartition of $K_{m,n}$. The class of graph $Pl_{m,n}$ has the node set $V_m \cup U_n$ and $E(Pl_{m,n}) = E(K_{m,n}) \setminus \{v_l u_k : 3 \leq l \leq m, 2 \leq k \leq n-1\}$.

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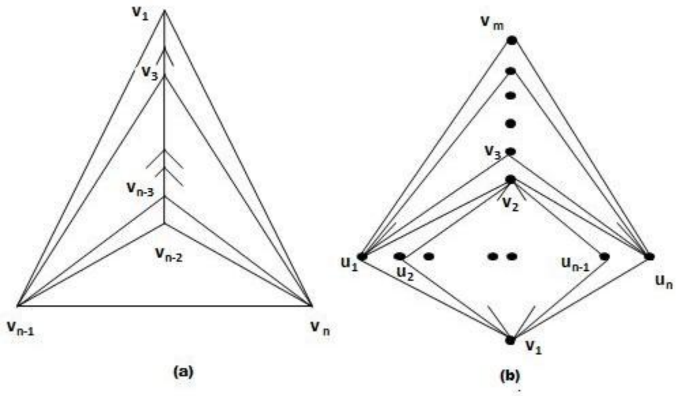

Figure 1: The class of (a) Pl_n (b) $Pl_{m,n}$

The embedding used for Pl_n is discussed as follows. Lay v_1, v_2, ... , v_{n-2} along a vertical line with v_1 at top and v_{n-2} at the bottom. Place v_{n-1} and v_n as the end points of a horizontal line perpendicular to the line having v_1, v_2, ..., v_{n-2}, at the bottom in such a fashion that v_{n-2}, v_{n-1}, v_n makes a triangular face, see Figure 1(a). Similarly, for embedding of $Pl_{m,n}$ that is going to be used for proofs is explained here.

First place u_1, u_2, ... , u_n horizontally with u_1 and u_n respectively at left and right ends. Next, place v_2, v_3, ..., v_m vertically above the segment u_1, u_2, ..., u_n, with v_2 at bottom and v_m at the top of the segment. Then place v_1 below the segment u_1, u_2, ... , u_n so that v_1, u_k, v_2, u_{k+1} forms a face of length 4 for $1 \leq k \leq n$-1. Remember, this discussion is about segment placement; no edges other than those indicated in the definitions are to be introduced, see Figure 1 (b).

Cordial labeling of Pl_n & $Pl_{m,n}$ has been established in [8]. DCL of Pl_n and $Pl_{m,n}$ in addition to exploring these two classes for graph operations are presented here.

2. Main Results

Theorem 2.1 *Pl_n admits a DCL.*

Proof. Suppose node and edge set of Pl_n are given by $\{v_i : 1 \leq i \leq n\}$, and $\{v_iv_{i+1} : 1 \leq i \leq n-3\} \cup \{v_{n-1}v_n\} \cup \{v_nv_i , v_{n-1}v_i : 1 \leq i \leq n-2\}$ respectively. One can see that $|V(Pl_n)| = n$, and $|E(Pl_n)| = 3n\text{-}6$. Consider $\psi : V(Pl_n) \to \{1, 2, ..., n\}$. Now three cases arise.

Case (i) When $n \geq 5$ is odd.

Let $\psi(v_{n-1}) = 1$, $\psi(v_n) = 2$, $\psi(v_i) = 2 + i$; $1 \leq i \leq n-2$. Clearly, $e_\psi(1) = e_\psi(0) + 1$.

Case (ii) When $n \geq 10$ is even such that $\frac{n}{2}$ is odd.

Let p be the largest prime < n. Fix $\psi(v_{n-1}) = 1$, $\psi(v_n) =$ p. Label the remaining nodes begining with v_{n-2} and proceeding to v_1 in the following fashion

$2, 2.2, 2.2^2, ..., 2.2^{k_1}$,
$3, 3.2, 3.2^2, ..., 3.2^{k_2}$,
...,..., ...,
upto, $(\frac{n}{2}-2), (\frac{n}{2}-2).2,..., (\frac{n}{2}-2).2^{k_t}$,
where $(2t-1)\,2^{k_t} \leq n$ and $t \geq 1$, $k_t \geq 0$. Assign the unutilized labels simultaneously to the remaining nodes. Observe that $e_\psi(0) = e_\psi(1) = \frac{3n}{2} - 3$.

Case (iii) When $n \geq 8$ is even, such that $\frac{n}{2}$ is even.

Let p be the largest prime < n. Fix $\psi(v_{n-1}) = 1$, $\psi(v_n) =$ p. Label the remaining nodes begining with v_{n-2} and proceeding to v_1 in the following fashion

$2, 2.2, 2.2^2, ..., 2.2^{k_1}$,
$3, 3.2, 3.2^2, ..., 3.2^{k_2}$,
...,..., ...,
upto, $(\frac{n}{2}-3), (\frac{n}{2}-3).2,..., (\frac{n}{2}-3).2^{k_t}$,
where $(2t-1)\,2^{k_t} \leq n$ and $t \geq 1$, $k_t \geq 0$. Now assigning unutilized labels simultaneously to the remaining nodes shows that $e_\psi(0) = e_\psi(1) = \frac{3n}{2} - 3$.

Hence, Pl_n is a DCG.

Theorem 2.2 *$Pl_{m,n}$ admits a DCL.*

Proof. Let $V_m = \{v_i : 1 \leq i \leq m\}$ and $U_n = \{u_j : 1 \leq j \leq n\}$. Let $V(Pl_{m,n}) = V_m \cup U_n$ and $E(Pl_{m,n}) = E(K_{m,n}) \setminus \{v_l u_k : 3 \leq l \leq m, 2 \leq k \leq n-1\}$. Clearly, $|V(Pl_{m,n})| = m + n$, and $|E(Pl_{m,n})| = 2m + 2n + 4$. Consider $\psi : V(Pl_{m,n}) \to \{1, 2, ..., m+n\}$. Now three cases arise.

Case (i) When $m = n$ with $m \geq 4$, $m \neq 5$.

Let p_1, p_2 be sufficiently large primes such that $p_2 < p_1 \leq m + n$. Fix $\psi(u_1) = p_1$, $\psi(u_n) = p_2$, $\psi(v_1) = 1$ and $\psi(v_2) = 2$. Assign even labels to unlabeled u_i; $2 \leq i \leq n-1$ and remaining labels simultaneously to unlabeled nodes. Observe that $e_\psi(0) = e_\psi(1) = 2m - 2$, which ensures that $Pl_{m,m}$ is a DCG.

Case (ii) When $m > n$ where $m \geq, n \geq 2$.

Let p_1, p_2 be sufficiently large primes such that $p_2 < p_1$

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$\le m + n$. Fix $\psi(u_1) = 1$, $\psi(u_n) = p_1$, $\psi(v_1) = 2$ and $\psi(v_2) = p_2$. Assign even labels to unlabeled u_i ; $2 \le i \le n-1$ and remaining labels simultaneously to unlabeled v_j; $3 \le j \le m$. Observe that $e_\psi(0) = e_\psi(1) = m + n - 2$, which proves that $Pl_{m,n}$ is a DCG.

Case (iii) When $m < n$ where $m \ge 2, n \ge 3$.

Let p_1, p_2 be sufficiently large primes so that $p_2 < p_1 \le m + n$. Fix $\psi(u_1) = 2$, $\psi(u_n) = p_1$, $\psi(v_1) = 1$ and $\psi(v_2) = p_2$. Assign even labels to unlabeled v_j ; $3 \le j \le m$ and remaining labels to unlabeled u_i ; $2 \le i \le n-1$, simultaneously. Observe that $e_\psi(0) = e_\psi(1) = m + n - 2$ showing that $Pl_{m,n}$ is a DCG.

Thus, in all the cases, $Pl_{m,n}$ admits a DCL.

Theorem 2.3 *Duplicating a node by a node in Pl_n, $n \ge 4$ admits a DCL.*

Proof. Let $V(Pl_n) = \{v_i : 1 \le i \le n\}$ and $E(Pl_n) = \{v_iv_{i+1} : 1 \le i \le n-3\} \cup \{v_{n-1}v_n\} \cup \{v_nv_i, v_{n-1}v_i : 1 \le i \le n-2\}$. Suppose G is acquired by duplicating a node of highest degree namely, v_n, by a node, say, w. Here, $V(G) = V(Pl_n) \cup \{w\}$ and $E(G) = E(Pl_n) \cup \{v_{n-1}w, v_iw : 1 \le i \le n-2\}$. See that $|V(G)| = n+1$ whereas $|E(G)| = 4n-7$. Consider $\psi : V(G) \to \{1, 2, ..., n+1\}$ for the under mentioned cases.

Case (i) When $n \equiv 0 (\text{mod } 2)$ and $n \ge 4$.

Fix $\psi(v_{n-1}) = 1$, $\psi(w) = 2$, $\psi(v_n) = 4$. For odd values of i, $1 \le i \le \frac{n}{2} - 2$, put $\psi(v_i) = 2+i$, $\psi(v_{i+1}) = 2(\psi(v_i))$ and for the remaining nodes, assign the unused labels simultaneously. Here, $e_\psi(0) = e_\psi(1) + 1$(see Figure 2).

Case (ii) When $n \equiv 1 (\text{mod } 2)$ and $n \ge 5$.

Fix $\psi(v_{n-1}) = 1$, $\psi(w) = 2$, $\psi(v_n) = 4$. For odd values of i, $1 \le i \le \frac{n+1}{2} - 2$, put $\psi(v_i) = 2 + i$ and $\psi(v_{i+1}) = 2(\psi(v_i))$, and for remaining nodes, assign the unused labels simultaneously. Here, $e_\psi(1) = e_\psi(0) + 1$.

Thus, G is a DCG.

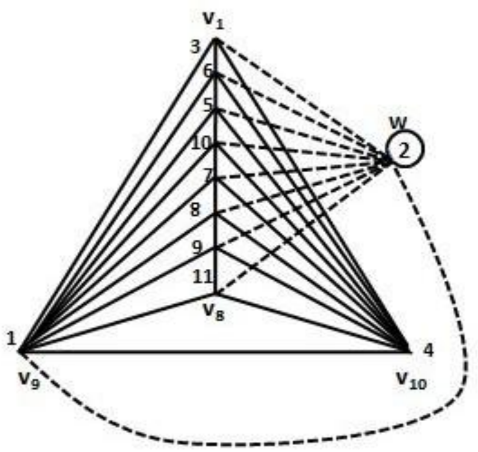

Figure 2: DCL of a graph acquired by duplication of v_{10} in Pl_{10}

Theorem 2.4 *Duplicating a node in $Pl_{m,n}$ admits a DCL $\forall$ $m, n \ge 4$.*

Proof. Let $V(Pl_{m,n}) = V_m \cup U_n$ where $V_m = \{v_i : 1 \le i \le m\}$ and $U_n = \{u_j : 1 \le j \le n\}$, and $E(Pl_{m,n}) = E(K_{m,n}) \setminus \{v_lu_k : 3 \le l \le m, 2 \le k \le n-1\}$. Let G be produced by duplicating a node namely, u_n by a node, say, w. Here, $V(G) = V(Pl_{m,n}) \cup \{w\}$ and $E(G) = E(Pl_{m,n}) \cup \{v_iw : 1 \le i \le m\}$. Clearly, $|V(G)| = m + n + 1$ and $|E(G)| = 3m + 2n + 4$. Consider $\psi : V(G) \to \{1, 2, ..., m + n + 1\}$. Now the under mentioned cases arise.

Case (i) When $n = m$.

Let k be the largest odd integer $\le \left\lfloor \frac{2n+1}{3} \right\rfloor$. Fix $\psi(u_1) = 1$, $\psi(v_1) = 2$, $\psi(v_2) = 4$, $\psi(u_n) = 8$ and $\psi(w) = k$. Assign even labels to u_i for $2 \le i \le n-2$. Next assign unused labels simultaneously to unlabeled nodes $u_{n-1}, v_3, v_4, ..., v_m$.

Case (ii) When $n > m$.

Choose p_1, p_2 sufficiently large prime such that $p_2 < p_1 \le m + n + 1$. Fix $\psi(u_1) = 2$, $\psi(u_n) = 4$, $\psi(v_1) = 1$, $\psi(v_2) = p_2$, $\psi(v_3) = 8$ and $\psi(w) = p_1$. There arise two subcases.

Subase (i) If m is odd.

Fix $\psi(v_i) = \psi(v_{i-1}) + 4$; $4 \le i \le \left\lceil \frac{m+1}{2} \right\rceil + 1$, $\psi(v_{\left\lceil \frac{m+1}{2} \right\rceil + 2}) = 6$, $\psi(v_i) = \psi(v_{i-1}) + 4$; $\left\lceil \frac{m+1}{2} \right\rceil + 3 \le i \le m$. Allot unused labels simultaneously to remaining unlabeled nodes.

Subcase (ii) If m is even.

Fix $\psi(v_i) = \psi(v_{i-1}) + 4$; $4 \le i \le \frac{m}{2} + 1$, $\psi(v_{\frac{m}{2}+2}) = 6$, $\psi(v_i) = \psi(v_{i-1}) + 4$; $\frac{m}{2} + 3 \le i \le m$. Allot unused labels

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simultaneously to unlabeled nodes. Clearly, $|e_\psi(0) - e_\psi(1)| \leq 1$.

Case (iii) When m > n.
Let p_1 and p_2 be sufficiently large primes, such that $p_2 < p_1 \leq m + n + 1$. Fix $\psi(u_1) = 1$, $\psi(v_1) = 3$, $\psi(v_2) = p_2$, $\psi(u_n) = 2$, $\psi(v_n) = 4$, $\psi(u_2) = 6$, $\psi(w) = p_1$, $\psi(u_i) = \psi(u_{i-1}) + 3$; $3 \leq i \leq k < n$ such that $\psi(u_k) \leq m+n+1$. Next assigning of unused labels simultaneously to unlabeled nodes shows that $|e_\psi(0) - e_\psi(1)| \leq 1$.
Hence, G is a DCG (see Figure 3).

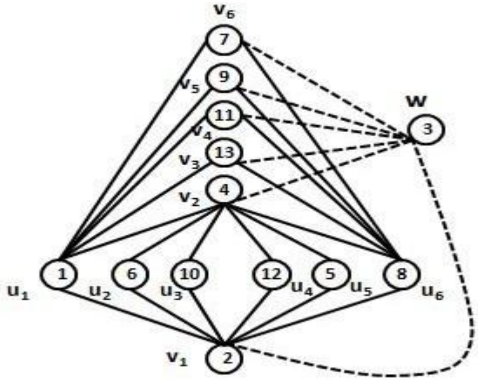

Figure 3: DCL of a graph obtained by duplicating u_6 in $Pl_{6,6}$

Theorem 2.5 *$Pl_n \odot K_1$ admits a DCL.*
Proof. Let $G = Pl_n \odot K_1$ with $V(G) = V(Pl_n) \cup \{u_i : 1 \leq i \leq n\}$, and $E(G) = E(Pl_n) \cup \{v_i u_i : 1 \leq i \leq n\}$. Clearly, $|V(G)| = 2n$, and $|E(G)| = 4n - 6$. Consider $\psi : V(G) \rightarrow \{1, 2, ..., 2n\}$ defined by fixing $\psi(v_{n-1}) = 1$, $\psi(v_n) = 2$, $\psi(u_{n-1}) = 4$, $\psi(u_n) = 3$, $\psi(v_1) = 6$, $\psi(v_i) = \psi(v_{i-1}) + 2$; $2 \leq i \leq n - 3$, $\psi(u_i) = \psi(v_i) - 1$; $1 \leq i \leq n - 3$, $\psi(v_{n-2}) = \psi(v_{n-3}) + 1$ and $\psi(u_{n-2}) = \psi(v_{n-2}) + 1$. Observe that $e_\psi(0) = e_\psi(1) = 2n - 3$, proving that G is a DCG (see Figure 4).

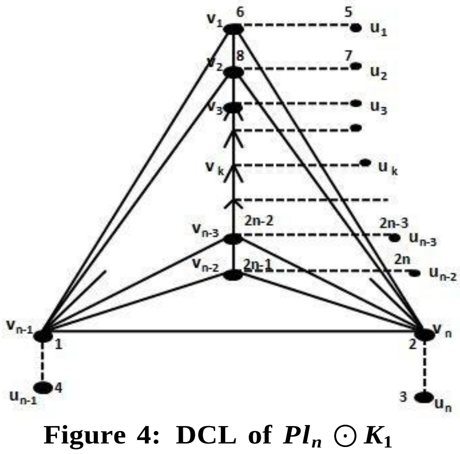

Figure 4: DCL of $Pl_n \odot K_1$

Theorem 2.6 *$Pl_{m,m} \odot K_1$ admits a DCL.*
Proof. Let $V(Pl_{m,n}) = V_m \cup U_n$ where $V_m = \{v_i : 1 \leq i \leq m\}$, and $U_n = \{u_j : 1 \leq j \leq n\}$ and, $E(Pl_{m,n}) = E(K_{m,n}) \setminus \{v_l u_k : 3 \leq l \leq m, 2 \leq k \leq n - 1\}$. Let $G = Pl_{m,m} \odot K_1$ with $V(G) = V(Pl_{m,n}) \cup \{v_i', u_j', : 1 \leq i \leq m, 1 \leq j \leq n\}$, and $E(G) = E(Pl_{m,n}) \cup \{v_i v_i', u_i u_j': 1 \leq i \leq m, 1 \leq j \leq n\}$. The cardinality of node and edge set of G is respectively 4m and 6m − 4. Consider $\psi : V(G) \rightarrow \{1, 2, ..., 4m\}$ given by fixing $\psi(u_1) = 1$, $\psi(u_2) = 8$, $\psi(u_i) = \psi(u_{i-1}) + 4$; $3 \leq i \leq n - 1$, $\psi(u_n) = 6$, $\psi(v_1) = 2$, $\psi(v_2) = 4$, $\psi(v_1') = 4m - 1$, $\psi(u_1') = 4m$, $\psi(v_2') = \psi(v_2) - 1$, $\psi(u_i') = \psi(u_i) - 1$; $2 \leq i \leq n$. Alloting available odd labels to v_i ; $3 \leq i \leq m$ and $\psi(v_i') = \psi(v_i) + 1$; $3 \leq i \leq m$ implies that $e_\psi(0) = e_\psi(1) = 3m - 2$, which shows that G is a DCG.

3. Conclusion

In the present study we have established that some classes of planar graphs admit a DCL. In addition to this, we have explored these graphs for graph operations of high interest and utility. DCL has been established for many graph families, still a lot has to be accomplished, thus an open area of research.

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