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The precision space of interpolatory cubature formulæ

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Abstract. Methods from Commutative Algebra and Numerical Analysis are combined to address a problem common to many disciplines: the estimation of the expected value of a polynomial of a random vector using a linear combination of a finite number of its values. In this work we remark on the error estimation in cubature formulæ for polynomial functions and introduce the notion of a precision space for a cubature rule.

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1. Introduction

This paper deals with multidimensional integration via cubature rules and their error. Given a probability measure on the reals $\mathbb{R}^d$ and an integrable function $f : \mathbb{R}^d \longrightarrow \mathbb{R}$, a cubature rule approximates the integral $\int_{\mathbb{R}^d} f(d)\, d\lambda(x)$, namely the expected value of f with respect to λ, by a finite weighted linear combination of values of f over a finite set $\mathcal{D}$. That is, the integral of f with respect to λ is approximated by its integral with respect to a finitely supported measure over $\mathcal{D}$. There is no need for the measure to be a probability density and some of the weights can be negative. We seek cubature rules and spaces of polynomial functions over which the approximation is exact or there is no error. This gives a measure of goodness of a cubature rule. One can obtain error bounds for the integral of any sufficiently smooth function by Taylor approximation e.g. by procedures in numerical integration but we do not deal with this specifically here.

We consider interpolatory cubature rules for which the expected value of f is approximated by the expected value of its interpolatory polynomial over $\mathcal{D}$. This is the analogue of interpolatory quadrature rules in one dimension. For a revision on quadrature and orthogonal polynomials see [5]. But in d dimension the interpolating polynomial is not unique. This requires the specification of a space of polynomials $\mathcal{P}$ over which to perform interpolation. A pair $(\mathcal{D}, \mathcal{P})$ is said to be exact if for each f there is a unique $p \in \mathcal{P}$ which interpolates f over $\mathcal{D}$. The weights of a cubature rule based on an exact pair are

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the expected value of the indicator functions of the points in $\mathcal{D}$ and hence are uniquely determined. It is possible to determine $\mathcal{P}$ for any $\mathcal{D}$ e.g. via Computational Commutative Algebra techniques and the interpolating polynomials can be computed solving a (possibly large) system of linear equations based on the evaluation of f and subset of $\mathcal{P}$ at $\mathcal{D}$. There is a literature on Computational Commutative Algebra and cubature rules [6, 8, 9, 10, 13] and more recently [3, 4] The combination of Computational Commutative Algebra and numerical methods to solve the linear systems makes effective the developed theory.

Classically a cubature rule is characterized by the so-called degree of precision: the maximum degree s by which all polynomials of degree less than or equal to s are exactly integrated and a vast literature is devoted to determine cubature rules with a certain degree of precision [2, 12]. Interesting results on the degree of precision based on Computational Algebra are presented in [3, 4, 9, 13], among others. Our main result, Theorem 1, gives a formula for the error of a cubature rule when applied to any polynomial. It is not based on degree of precision but it depends only on the polynomial support. This lead us to introduce the notion of precision spaces for cubature rules, that is sets of polynomials over which the cubature rule is exact. Precision spaces generalise the notion of degree of precision [8, 5] and are almost always much bigger sets than those intervening in the definition of degree of precision. To present our results we chose to adopt the language of matrices and vectors to allow a neater presentation and slightly general theory. Our results are algorithmic in nature: Computational Commutative Algebra gives effective tools for determining $\mathcal{P}$ and Linear Algebra provides the language and the generality for describing the various results.

The error formula in Theorem 1 depends on the expected values of a polynomial basis of a vector space embedding the precision space. Such basis does not need to be made of monomials, it could e.g. made of orthogonal polynomials with respect to λ. In such a case the expected values are particularly simple to compute, being either zero or one. Term-orderings and techniques from computational algebra allows the navigation also through orthogonal polynomials or other polynomial bases. This has been largely exploited in [3] and [4] where the error is given as a function of the Fourier coefficients of the quotients of a polynomial f with respect to a Gröbner basis of the vanishing ideal of $\mathcal{D}$. Here we rather use orthogonal polynomials for the examples and express the error formula directly as function of the coefficients of f.

The paper is structured as follows. Section 2 provides the necessary background on cubature rules and ideal of points. Section 3 gives Theorem 1 on the error and in Section 4 we discuss precision spaces.

2. Set-up and remarks on interpolatory cubature rules

Let $\mathcal{D}$ be a finite set of n distinct points in $\mathbb{R}^d$. To each $d \in \mathcal{D}$ attach a weight $w_d \in \mathbb{R}$ and define the set of weights $W = \{w_d, d \in \mathcal{D}\}$. Let f be a function from $\mathbb{R}^d$ to $\mathbb{R}$ and let $X \in \mathbb{R}^d$ be a d-dimensional random vector endowed with a probability measure λ whose moments are finite (at least up to a certain degree).

For our purposes a cubature rule is a pair $(\mathcal{D}, W)$ and a rule by which the expected value of $f(X)$ with respect to λ is approximated by a weighted average of the values of f at $\mathcal{D}$ giving

$$\mathrm{E}_\lambda\left(f(X)\right) = \sum_{d\in\mathcal{D}} w_d f(d) + R_{\mathcal{D},W}(f),$$

where $w_d \in W$, $R_{\mathcal{D},W}(f)$ is the error of the cubature rule on f and $\mathrm{E}_\lambda\left(f(X)\right)$ is assumed to exist finite. The term $R_{\mathcal{D},W}(f)$ expresses a residual being the difference between the approximation and the true values of the integral. We call it error of the cubature rule following literature on integral approximation from numerical analysis [5].

Let $\mathcal{P}$ be a set of polynomial functions on $\mathbb{R}^d$, e.g. all polynomials in d variables with total degree less than a positive integer n with $n < d$, and such that for all $f : \mathbb{R}^d \longrightarrow \mathbb{R}$ there exists a unique interpolatory polynomial $p \in \mathcal{P}$ such that $f(d) = p(d)$ for all $d \in \mathcal{D}$. The pair $(\mathcal{D}, \mathcal{P})$ is said to be correct. Usually $\mathcal{P}$ is chosen to be a vector space over the reals. In this paper we consider only correct pairs with $\mathcal{P}$ a vector space.

Definition 1. *A cubature rule $(\mathcal{D}, W)$ is said to be interpolatory on $\mathcal{P}$ if for all $p \in \mathcal{P}$ $\mathrm{E}_\lambda\left(p(X)\right) = \sum_{d\in\mathcal{D}} w_d p(d)$.*

Observe that if $p \in \mathcal{P}$ is the interpolatory polynomial of f, then

$$\mathrm{E}_\lambda\left(f(X)\right) = \sum_{d\in\mathcal{D}} w_d p(d) + R_{\mathcal{D},W}(f)$$

and the cubature approximation is

$$\mathrm{E}_\lambda\left(p(X)\right) = \sum_{d\in\mathcal{D}} w_d p(d) = [w_d]^t_{d\in\mathcal{D}}[p(d)]_{d\in\mathcal{D}},$$

where vectors are column vectors and w^t indicates the transpose of w.

The interpolatory property in Definition 1 provides a mean of determining the weights W when the expected value of some polynomials in $\mathcal{P}$ are known. This justifies the notation $R_{\mathcal{D},W}(f) = R_{\mathcal{D},\mathcal{P}}(f)$. Indeed if $S \subset \mathcal{P}$ is a finite set of n polynomials such that the matrix of their evaluations at $\mathcal{D}$, i.e. $[s(d)]_{d\in\mathcal{D},s\in S}$ is invertible, then

$$\left[\mathrm{E}_\lambda\left(s(X)\right)\right]^t_{s\in S} = [w_d]^t_{d\in\mathcal{D}}\left[s(d)\right]_{d\in\mathcal{D},s\in S}$$

and the weights can be determined as

$$[w_d]^t_{d\in\mathcal{D}} = \left[\mathrm{E}_\lambda\left(s(X)\right)\right]^t_{s\in S}\left[s(d)\right]^{-1}_{d\in\mathcal{D},s\in S}. \qquad (1)$$

Hence the cubature rule for the function f and its interpolating polynomial $p \in \mathcal{P}$ becomes

$$\mathrm{E}_\lambda\left(p(X)\right) = \left[\mathrm{E}_\lambda\left(s(X)\right)\right]^t_{s\in S}\left[s(d)\right]^{-1}_{d\in\mathcal{D},s\in S}\left[f(d)\right]_{d\in\mathcal{D}}. \qquad (2)$$

The evaluation matrix is often indicated with X or Z and Equation (2) is given in [11, Theorem 46] within another context. Using the notations $X_{\mathcal{D},S}$ for the evaluation matrix and the moment notation $[m_s]_{s\in S}$ for $[\mathrm{E}_\lambda\,(s(X))]_{s\in S}$, Equations (1) and (2) are written in a more compact form as

$$[w_d]_{d\in\mathcal{D}} = X^{-t}_{\mathcal{D},S}[m_s]_{s\in S} \qquad \text{and} \qquad \mathrm{E}_\lambda\,(p) = [m_s]^t_{s\in S}X^{-1}_{\mathcal{D},S}\,[f(d)]_{d\in\mathcal{D}}, \tag{3}$$

respectively. The vector of weights $[w_d]_{d\in\mathcal{D}}$ depends both on $\mathcal{D}$ and $\mathcal{P}$. Proposition 1 below shows that it does not depend on S and the weights are intrinsic to $\mathcal{P}$. Namely, given a correct pair $(\mathcal{D},\mathcal{P})$ of nodes and interpolating polynomials, the weights of the cubature rules are uniquely determined by $\mathcal{P}$ and so they will be denoted by $w_\mathcal{P}$.

Proposition 1. *Let S and R be two vector space bases of $\mathcal{P}$ and let w_S and w_R be the weight vectors for the S and R vector space bases, respectively. Then it holds $w_S = w_R$.*

Proof. As $\mathrm{Span}_\mathbb{R}(S) = \mathrm{Span}_\mathbb{R}(R)$ any $s \in S$ can be written uniquely as $s = \sum_{r\in R} c_{s,r}r$ and hence

$$m_s = \mathrm{E}_\lambda(s(X)) = \sum_{r\in R} c_{s,r}\,\mathrm{E}_\lambda(r(X)) = \sum_{r\in R} c_{s,r}m_r.$$

Let $C = [c_{s,r}]_{s\in S, r\in R}$ be the change-of-basis matrix and write

$$[s]_{s\in S} = C\,[r]_{r\in R} \qquad \text{and} \qquad [m_s]_{s\in S} = C\,[m_r]_{r\in R}\,.$$

If follows that $X^t_{\mathcal{D},S} = CX^t_{\mathcal{D},R}$. By Equation (3) we have

$$w_S = X^{-t}_{\mathcal{D},S}\,[m_s]_{s\in S} = X^{-t}_{\mathcal{D},S}C\,[m_r]_{r\in R} = X^{-t}_{\mathcal{D},R}C^{-1}C\,[m_r]_{r\in R} = w_R. \qquad \square$$

Proposition 1 is somehow obvious because cubature rules are vector space objects and not effected by a change of basis of the vector space. The next result shows how the weight vectors change by varying the interpolatory set in a exact pair and keeping the same node set $\mathcal{D}$. A priori the related cubature rules are different although supported on the same nodes. The result characterises when they are equal. See also Example 1.

Proposition 2. *Let $\mathcal{D}$ be a set of nodes and $(\mathcal{D},\mathcal{P})$ and $(\mathcal{D},\mathcal{Q})$ two correct pairs. Let R and S be vector space bases of $\mathcal{P}$ and $\mathcal{Q}$, respectively. Then*

$$w_\mathcal{P} = w_\mathcal{Q} + X^{-t}_{\mathcal{D},S}\left(X^t_{\mathcal{D},S}X^{-t}_{\mathcal{D},R}\,[m_r]_{r\in R} - [m_s]_{s\in S}\right)$$

and the weight vectors are equal if and only if

$$[m_s]_{s\in S} = X^t_{\mathcal{D},S}X^{-t}_{\mathcal{D},R}\,[m_r]_{r\in R}\,.$$

Proof. From Equation (3) the weight vectors are $w_\mathcal{P} = X^{-t}_{\mathcal{D},R}\,[m_r]_{r\in R}$ and $w_\mathcal{Q} = X^{-t}_{\mathcal{D},S}\,[m_s]_{s\in S}$. By subtracting and collecting $X^{-t}_{\mathcal{D},S}$, the thesis follows. $\square$

2.1. Cubature rules and ideal of points

Let $\mathbb{R}[x]$ be the ring of polynomials with real coefficients and in the d indeterminate $x = (x_1, \ldots, x_d)$; for $\alpha = (\alpha_1, \ldots, \alpha_d)$ a d-dimensional vector with non-negative integer entries, $x^\alpha = x_1^{\alpha_1} \ldots x_d^{\alpha_d}$ indicates a monomial. The (total) degree of x^α is $\sum_{i=1}^d \alpha_i$. Given a finite set of points $\mathcal{D} \subset \mathbb{R}^d$, $\mathcal{I}(\mathcal{D})$ denotes the ideal of all polynomials vanishing at $\mathcal{D}$ and $\mathbb{R}[x]/\mathcal{I}(\mathcal{D})$ denotes the quotient space.

The vector spaces we consider next are isomorphic to $\mathbb{R}[x]/\mathcal{I}(\mathcal{D})$ and are spanned by polynomials. The following gives two typical examples. For a term ordering τ and a τ-Gröbner basis of $\mathcal{I}(\mathcal{D})$, say G, in [13] it is considered the vector space spanned by the monomials not divisible by the leading terms of G. Let L be the set of exponents for such monomials. Let $S = \{x^\alpha\}_{\alpha \in L}$ be such monomials and $\mathcal{P} = \text{Span}_{\mathbb{R}}(S)$ be the real vector space with basis S. The pair $(\mathcal{D}, \mathcal{P})$ is correct. Proposition 2 gives the formula to express how the weights of the cubature rule change for different term orderings.

Another popular basis of $\mathcal{P} = \text{Span}_{\mathbb{R}}(S)$ is given by orthogonal polynomials w.r.t. the measure λ. Essentially for product measures it is $\{\pi_\alpha : \alpha \in L\}$ and hence still depends on a term ordering. Here the multi-dimensional orthogonal polynomials are simply product of one-dimensional orthogonal polynomials whose theory is well understood [5] namely $\pi_\alpha(x) = \pi_{\alpha_1}(x_1) \cdots \pi_{\alpha_d}(x_d)$. In [3, 4] we exploit this and discuss the error in terms of orthogonal bases for some cases.

In particular, the basis S spanning $\mathcal{P}$ which is isomorphic to $\mathbb{R}[x]/\mathcal{I}(\mathcal{D})$ needs not depend on a term ordering. For example for $\mathcal{D} = \{(0,0), (0,-1), (1,0), (1,1), (-1,1)\}$ and $S = \{1, x_1, x_2, x_1^2, x_2^2\}$ consider $\mathcal{P} = \text{Span}_{\mathbb{R}}(S)$. Observe that $(\mathcal{D}, \mathcal{P})$ is a correct pair and although $\mathcal{P}$ and $\mathbb{R}[x]/\mathcal{I}(\mathcal{D})$ are isomorphic, S cannot be derived by any term ordering as illustrated above.

The following example shows that cubature rules associated to the same set of nodes but to different vector spaces can have the same weights, that is they give the same approximation to expectations.

Example 1. Let λ be the standard normal distribution and consider the set of nodes $\mathcal{D} = \{(-2,1), (2,-1), (0,0), (18,3)\} \subset \mathbb{R}^2$. Let $\mathcal{P}$ be the vector space isomorphic to $\mathbb{R}/\mathcal{I}(\mathcal{D})$ and spanned by $R = \{1, x_1, x_2, x_2^2\}$. We have that

$$\begin{bmatrix} \mathrm{E}_\lambda(1) \\ \mathrm{E}_\lambda(X_1) \\ \mathrm{E}_\lambda(X_2) \\ \mathrm{E}_\lambda(X_2^2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad X_{\mathcal{D},R} = \begin{bmatrix} 1 & -2 & 1 & 1 \\ 1 & 2 & -1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 18 & 3 & 9 \end{bmatrix}$$

and so the weights are $w_R = X_{\mathcal{D},R}^{-t}[1,0,0,1]^t = [1/2, 1/2, 0, 0]^t$.

Now, let $\mathcal{Q}$ be the vector space isomorphic to $\mathbb{R}/\mathcal{I}(\mathcal{D})$ and spanned by $S = \{1, x_2, x_2^2, x_2^3\}$.

Obviously, $\mathcal{P} = \mathrm{Span}_{\mathbb{R}}(R)$ and $\mathcal{Q} = \mathrm{Span}_{\mathbb{R}}(S)$ are different vector spaces and we have that

$$\begin{bmatrix} \mathrm{E}_\lambda(1) \\ \mathrm{E}_\lambda(X_2) \\ \mathrm{E}_\lambda(X_2^2) \\ \mathrm{E}_\lambda(X_2^3) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad X_{\mathcal{D},S} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 3 & 9 & 27 \end{bmatrix}.$$

By Equation (3) we have that $w_S = X_{\mathcal{D},S}^{-t}[1,0,1,0]^t = [1/2,1/2,0,0]^t = w_R$ that is the two weight vectors are equal and hence R and S with $\mathcal{D}$ define the same cubature rule.

Example 2. Let $\mathcal{D} = \{0,1,2\} \in \mathbb{R}$ and consider the following four different vector space bases $S_1 = \{1,x,x^2\}$, $S_2 = \{1,x,x^2-1\}$ or $S_3 = \{1,x^2,x^4\}$ or $S_4 = \{x+1,x^2,x^3\}$. They generated vector spaces all isomorphic to $\mathbb{R}/\mathcal{I}(\{0,1,2\})$. As S_1 and S_2 generate the same space, by Proposition 1 they give the same cubature rule. Let consider the expected value of a function f with respect to the standard normal distribution. The quadrature rule associated to S_i, $i = 1,\dots,4$, is $f(0)w_0^i + f(1)w_1^i + f(2)w_2^i$ and the weights, related to the S_i according to Equation (1), are the following:

$$w_{S_1} = w_{S_2} = [3/2,\ -1,\ 1/2]^t \qquad w_{S_3} = [1/2,\ 1/3,\ 1/6]^t \qquad w_{S_4} = [-9/4,\ 2,\ -1/4]^t.$$

Proposition 2 gives the relationship among the weights for different i's.

For $f(x) = \sin(x\pi/2)$, the interpolating polynomial based on S_1 and S_2 is $p_1 = 2x - x^2$, $p_3 = 4/3x^2 - 1/3x^4$ based on S_3 and $p_4 = 2x^2 - x^3$ on S_4. With respect to the standard normal distribution the expected value of f is zero while the values of the quadrature approximation are rather different:

$$\mathrm{E}_\lambda(p_1(X)) = -1 \qquad \mathrm{E}_\lambda(p_3(X)) = 1/3 \qquad \mathrm{E}_\lambda(p_4(X)) = 2.$$

The next section deals with the error.

3. Error of the cubature rules

Given a correct pair $(\mathcal{D},\mathcal{P})$ and a function f we consider the difference between the expected value of f and its approximation given by the cubature rule associated to $(\mathcal{D},\mathcal{P})$, that is the error $R_{\mathcal{D},\mathcal{P}}(f)$ defined as

$$R_{\mathcal{D},\mathcal{P}}(f) = \mathrm{E}_\lambda\left(f(X)\right) - \sum_{d\in\mathcal{D}} w_d f(d). \tag{4}$$

The goodness of a cubature rules is established upon its performance on sets of polynomials. In one-dimension a typical measure of goodness of quadrature rules (how cubature rules are known in one-dimension) is the degree of precision [5]. A quadrature rule is said to have degree of precision (or precise degree of exactness) n if the error is zero for all polynomials of degree not larger than n and there exists a polynomial of degree $n+1$ with nonzero error. The main results for quadrature rule supported on n distinct nodes is that the error is zero for any polynomial of degree at most $2n-1$ if and only if it is interpolatory and for all polynomials p of degree smaller than n it holds $\mathrm{E}_\lambda\left(p(X)\omega(X)\right)=0$ where $\omega(x)=\prod_{d\in\mathcal{D}}(x-d)$ is the node polynomial [5, Theorem 1.45]. In the remainder of this paper we generalize this idea by introducing the precision space of a cubature rule in Section 4. It is based on the following results.

Consider the Euclidean division of a polynomial $f=\sum_{g\in G} q_g g + r$ in d-dimensions with respect to a Gröbner basis G of $\mathcal{I}(\mathcal{D})$. In [3] a formula for the error is given as function of the Fourier coefficients of the q_g w.r.t. the orthogonal polynomial basis associated to a product measure λ. The approach is algorithmic and in [4] the error formula is studied further. Theorem 1 below gives a formula of the error $R_{\mathcal{D},\mathcal{P}}(f)$ for $f=\sum_\alpha c_\alpha x^\alpha$ as functions of the coefficients c_α. Theorem 1 improves the results in [3] because it does not require to compute the Euclidean division and the Fourier expansion of the q_g.

We present Theorem 1 in the most general form without reference to a specific vector space basis S of a finite dimensional polynomial space. That is, S is any vector space basis of $\mathcal{P}$. The error is given in terms of the coefficients of a decomposition of f over S. Note that the error is zero for all polynomials in $\mathcal{P}$ because the cubature rule is interpolatory. Hence we consider polynomials in a finite dimensional vector space $\mathcal{Q}$ subset of $\mathbb{R}[x]$ of which $\mathcal{P}$ is a sub-vector space. A vector space basis R of $\mathcal{P}$ can be completed to a vector space basis S of $\mathcal{Q}$.

Theorem 1. *Let $(\mathcal{D},\mathcal{P})$ be a correct pair and $w_{\mathcal{P}}$ its weight vector. Let R and S, $\mathcal{P}$ and $\mathcal{Q}$ be as above and $f=\sum_{s\in S} b_s s\in\mathcal{Q}$. Then the error of the cubature rule for f is a function of the coefficients of f and*

1. *of the expected values of elements in $S\setminus R$, the weights and the evaluation matrix over $\mathcal{D}$ of $S\setminus R$*

$$R_{\mathcal{D},\mathcal{P}}(f)=\left([m_s]_{s\in S\setminus R}-X^t_{\mathcal{D},S\setminus R}w_{\mathcal{P}}\right)^t[b_s]_{s\in S\setminus R}\ , \tag{5}$$

   where $X_{\mathcal{D},S\setminus R}=[s(d)]_{d\in\mathcal{D},s\in S\setminus R}$ and $m_s=\mathrm{E}_\lambda\left(s(X)\right)$,

2. *equivalently, of the error of the cubature rule on the elements of $S\setminus R$*

$$R_{\mathcal{D},\mathcal{P}}(f)=[R_{\mathcal{D},\mathcal{P}}(s)]^t_{s\in S\setminus R}[b_s]_{s\in S\setminus R}\ . \tag{6}$$

Proof. Let $f=\sum_{s\in S} b_s s$ be a polynomial in $\mathcal{Q}$; since $R\subset S$ for each $d\in\mathcal{D}$

$$f(d)=\sum_{s\in R} b_s s(d)+\sum_{s\in S\setminus R} b_s s(d)\ ,$$

that is

$$[f(d)]_{d\in\mathcal{D}}=X_{\mathcal{D},R}[b_s]_{s\in R}+X_{\mathcal{D},S\setminus R}[b_s]_{s\in S\setminus R}\ . \tag{7}$$

Furthermore

$$\mathrm{E}_\lambda\left(f(X)\right)=\sum_{s\in S} b_s m_s=[m_s]^t_{s\in R}[b_s]_{s\in R}+[m_s]^t_{s\in S\setminus R}[b_s]_{s\in S\setminus R}\ .$$

From definition of cubature error and from Equation (7) we have

$$\begin{aligned} R_{\mathcal{D},\mathcal{P}}(f) &= \mathrm{E}_\lambda\left(f(X)\right) - w_{\mathcal{P}}^t[f(d)]_{d\in\mathcal{D}} \\ &= [m_s]_{s\in R}^t[b_s]_{s\in R} + [m_s]_{s\in S\setminus R}^t[b_s]_{s\in S\setminus R} - w_{\mathcal{P}}^t\left(X_{\mathcal{D},R}[b_s]_{s\in R} + X_{\mathcal{D},S\setminus R}[b_s]_{s\in S\setminus R}\right) \\ &= \left([m_s]_{s\in R}^t - w_{\mathcal{P}}^t X_{\mathcal{D},R}\right)[b_s]_{s\in R} + \left([m_s]_{s\in S\setminus R}^t - w_{\mathcal{P}}^t X_{\mathcal{D},S\setminus R}\right)[b_s]_{s\in S\setminus R} \end{aligned}$$

and so this concludes the proof of Item 1. since, from Equation (1), $[m_s]_{s\in R}^t = w_{\mathcal{P}}^t X_{\mathcal{D},R}$. For each $s \in S \setminus R$, the s-th element of the vector $[m_s]_{s\in S\setminus R} - X_{\mathcal{D},S\setminus R}^t w_{\mathcal{P}}$ is given by

$$m_s - w_{\mathcal{P}}^t[s(d)]_{d\in\mathcal{D}} = \mathrm{E}_\lambda(s(X)) - \sum_{d\in\mathcal{D}} w_d s(d) = R_{\mathcal{D},\mathcal{P}}(s)$$

and this proves Item 2. □

When S consists of monomials x^α a typical notation is $f = \sum_\alpha b_\alpha x^\alpha$. The error itself $R_{\mathcal{D},\mathcal{P}}(f) = \mathrm{E}_\lambda\left(f(X)\right) - \sum_{d\in\mathcal{D}} w_d f(d)$ is clearly the same for all R and S because the weights do not depend on R and the expected value does not depend on the basis S upon which f is written. Note that the error formula in Theorem 1 is the scalar product of two vectors. The first vector is given by $[b_s]_{s\in S\setminus R}$, and hence depends on f and the basis in which f is written and on R. The second vector is the difference $[m_s]_{s\in S\setminus R} - X_{\mathcal{D},S\setminus R}^t w_{\mathcal{P}}$ which depends on the cubature rule $(\mathcal{D},\mathcal{P})$ and on R and S. Various authors [2, 8, 10], via the notion of exactness, study the analogue in higher dimension of the degree of exactness in one-dimension, in particular all monomials with total degree up to a given level are in a precision space. By studying the difference vector in Section 4 we investigate larger parts of precision spaces.

Corollary 1. *Let $(\mathcal{D},\mathcal{P})$ be a correct pair and $w_{\mathcal{P}}$ its weight vector and let $\mathcal{P} \subset \mathcal{Q}_1 \subset \mathcal{Q}_2$ be vector spaces and $R \subset S \subset T$ their respective vector space bases. Let $f = \sum_{s\in S} b_s s \in \mathcal{Q}_1$ and $f_1 = f + \sum_{s\in T\setminus S} b_s s \in \mathcal{Q}_2$. Then*

1. $R_{\mathcal{D},\mathcal{P}}(f_1) = R_{\mathcal{D},\mathcal{P}}(f) + \left([m_s]_{s\in T\setminus S} - X_{\mathcal{D},T\setminus S}^t w_{\mathcal{P}}\right)^t [b_s]_{s\in T\setminus S} = R_{\mathcal{D},\mathcal{P}}(f) + \sum_{s\in T\setminus S} R_{\mathcal{D},\mathcal{P}}(s) b_s$

2. *and if* $[m_s]_{s\in T\setminus S} = X_{\mathcal{D},T\setminus S}^t w_{\mathcal{P}}$, *then* $R_{\mathcal{D},\mathcal{P}}(f_1) = R_{\mathcal{D},\mathcal{P}}(f)$.

Proof. Since $R \subset S \subset T$ then, from Equation (6) we have

$$\begin{aligned} R_{\mathcal{D},\mathcal{P}}(f_1) &= [R_{\mathcal{D},\mathcal{P}}(s)]_{s\in T\setminus R}^t[b_s]_{s\in T\setminus R} = [R_{\mathcal{D},\mathcal{P}}(s)]_{s\in S\setminus R}^t[b_s]_{s\in S\setminus R} + [R_{\mathcal{D},\mathcal{P}}(s)]_{s\in T\setminus S}^t[b_s]_{s\in T\setminus S} \\ &= R_{\mathcal{D},\mathcal{P}}(f) + \sum_{s\in T\setminus S} R_{\mathcal{D},\mathcal{P}}(s) b_s \end{aligned}$$

and this concludes the proof of Item 1 since, for $s \in T \setminus S$

$$R_{\mathcal{D},\mathcal{P}}(s) = m_s - [s(d)]_{d\in\mathcal{D}}^t w_{\mathcal{P}}$$

and $[s(d)]_{d\in\mathcal{D}}$ is the s-th row of $X^t_{\mathcal{D},T\setminus S}$. Item 2 follows straightforward from Item 1. □

The first item in Corollary 1 states that if the error for f is known and f_1 is constructed from f by adding some terms, then its error can be derived from the error of the smaller function and knowledge of the expected values and the evaluation at $\mathcal{D}$ of the added terms. The second item leads into Section 4 giving a criterion by which the elements of $\mathcal{Q}_1$ can be modified without changing the error of the cubature rule over them. The following examples summarize this section.

Example 3. Let $\mathcal{D} = \{(1,2),(-1,3),(4,0),(1,1)\} \subset \mathbb{R}^2$ be a set of nodes and let $\mathcal{P}$ be the vector space spanned by $R = \{1, x_1, x_2, x_2^2\}$ isomorphic to $\mathbb{R}[x_1,x_2]/\mathcal{I}(\mathcal{D})$. R can be obtained via Gröbner methods with the deg-lex term ordering. Hence by construction the pair $(\mathcal{D},\mathcal{P})$ is correct. We consider the standard normal distribution λ. Since

$$[m_s]_{s\in R} = \begin{bmatrix} \mathrm{E}_\lambda(1) \\ \mathrm{E}_\lambda(X_1) \\ \mathrm{E}_\lambda(X_2) \\ \mathrm{E}_\lambda\left(X_2^2\right) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad X_{\mathcal{D},R} = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & -1 & 3 & 9 \\ 1 & 4 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

the weight vector is $w^t_{\mathcal{P}} = [m_s]^t_{s\in R} X^{-1}_{\mathcal{D},R} = [-14/5,\ 11/10,\ 2/5,\ 23/10]$. Let $\mathcal{Q}$ be the vector space of the polynomials in $\mathbb{R}[x_1,x_2]$ of degree strictly less than 4: it is spanned by the set $S = R \cup \{x_1x_2, x_1^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3\}$. For a polynomial $f = \sum_{s\in S} b_s s \in \mathcal{Q}$ we have that

$$[m_s]_{s\in S\setminus R} = \begin{bmatrix} \mathrm{E}_\lambda(X_1X_2) \\ \mathrm{E}_\lambda\left(X_1^2\right) \\ \mathrm{E}_\lambda\left(X_1^3\right) \\ \mathrm{E}_\lambda\left(X_1^2X_2\right) \\ \mathrm{E}_\lambda\left(X_1X_2^2\right) \\ \mathrm{E}_\lambda\left(X_2^3\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad X_{\mathcal{D},S\setminus R} = \begin{bmatrix} 2 & 1 & 1 & 2 & 4 & 8 \\ -3 & 1 & -1 & 3 & -9 & 27 \\ 0 & 16 & 64 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}$$

and so $[m_s]^t_{s\in S\setminus R} - w^t_{\mathcal{P}} X_{\mathcal{D},S\setminus R} = [33/5,\ -6,\ -24,\ 0,\ 94/5,\ -48/5]$. From Theorem 1

$$R_{\mathcal{D},\mathcal{P}}(f) \;=\; [33/5,\ -6,\ -24,\ 0,\ 94/5,\ -48/5][b_s]_{s\in S\setminus R}\,. \tag{8}$$

Example 4 (Example 3 contd.)**.** Let $\mathcal{T}$ be the vector space spanned by $T = S \cup \{x_1^4, x_2^4\}$ and let $f_1 \in \mathcal{R}$ be written as $f_1 = f + \sum\limits_{s\in\{x_1^4,x_2^4\}} b_s s$ with $f = \sum_{s\in S} b_s s \in \mathcal{Q}$. Since $\mathcal{Q} \subset \mathcal{T}$, from Corollary 1 and Formula (8), the error $R_{\mathcal{D},\mathcal{P}}(f_1)$ can be written from the error $R_{\mathcal{D},\mathcal{P}}(f)$ knowing

$$[m_s]_{s\in T\setminus S} = \left[\mathrm{E}_\lambda\left(X_1^4\right), \mathrm{E}_\lambda\left(X_2^4\right)\right]^t = [3,3]^t \quad \text{and} \quad X_{\mathcal{D},T\setminus S} = \begin{bmatrix} 1 & 16 \\ 1 & 81 \\ 256 & 0 \\ 1 & 1 \end{bmatrix}$$

giving

$$R_{\mathcal{D},\mathcal{P}}(f_1) \;=\; R_{\mathcal{D},\mathcal{P}}(f) + \left([m_s]_{s\in T\setminus S} - X^t_{\mathcal{D},T\setminus S} w_{\mathcal{P}}\right)^t [b_s]_{s\in T\setminus S}$$

$$= [33/5,\ -6,\ -24,\ 0,\ 94/5,\ -48/5][b_s]_{s\in S\setminus R} - [100, 218/5][b_s]_{s\in T\setminus S}\ . \quad (9)$$

Example 5. The cubature rule of Example 3 can be described using a vector space basis of $\mathcal{P}$ made of bi-variate Hermite polynomials. Thus for α a non negative integer number let $H_\alpha(x_1)$ be the univariate Hermite polynomial of degree α. The set of the H_α's as α varies is known to be an orthogonal set w.r.t. the standard normal distribution. For $L = \{(0,0),(1,0),(0,1),(0,2)\}$ the set $H = \{H_\alpha(x) = H_{\alpha_1}(x_1)H_{\alpha_2}(x_2) : \alpha = (\alpha_1,\alpha_2) \in L\}$ is a vector space basis of $\mathcal{P}$ and it holds

$$[m_s]_{s\in H} = \begin{bmatrix} \mathrm{E}_\lambda(1) \\ \mathrm{E}_\lambda(H_{1,0}(X)) \\ \mathrm{E}_\lambda(H_{0,1}(X)) \\ \mathrm{E}_\lambda(H_{0,2}(X)) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad X_{\mathcal{D},H} = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & -1 & 3 & 8 \\ 1 & 4 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{bmatrix}.$$

By Proposition 1 there is no need to recompute the weight vector. As before consider $f_1 \in \mathcal{T}$ and evaluate $R_{\mathcal{D},\mathcal{P}}(f_1)$ w.r.t. the basis K of $\mathcal{T}$ consisting of product Hermite polynomials $K = \{H_\alpha \mid \alpha \in L_1\}$ where $L_1 = L \cup \{(1,1),(2,0),(3,0),(2,1),(1,2),(0,3),(4,0),(0,4)\}$. For $f_1 = \sum_{\alpha\in L_1} b_\alpha H_\alpha \in \mathcal{T}$, from Equation (5) we have

$$R_{\mathcal{D},\mathcal{P}}(f_1) = \left([m_\alpha]_{\alpha\in L_1\setminus L} - X^t_{\mathcal{D},K\setminus H} w_{\mathcal{P}}\right)^t [b_\alpha]_{\alpha\in L_1\setminus L}\ ,$$

where

$$X_{\mathcal{D},K\setminus H} = \begin{bmatrix} 2 & 0 & -2 & 0 & 3 & 2 & -2 & -5 \\ -3 & 0 & 2 & 0 & -8 & 18 & -2 & 30 \\ 0 & 15 & 52 & 0 & -4 & 0 & 163 & 3 \\ 1 & 0 & -2 & 0 & 0 & -2 & -2 & -2 \end{bmatrix}.$$

Since $w^t_{\mathcal{P}} X_{\mathcal{D},K\setminus H} = [-33/5, 6, 24, 0, -94/5, 48/5, 64, 218/5]$ and $[m_\alpha]_{\alpha\in L_1\setminus L}$ is the zero vector, we conclude that

$$R_{\mathcal{D},\mathcal{P}}(f) = \left[\frac{33}{5},\ -6,\ -24,\ 0,\ \frac{94}{5},\ -\frac{48}{5},\ -64,\ -\frac{218}{5}\right] \begin{bmatrix} b_{(1,1)} \\ b_{(2,0)} \\ b_{(0,3)} \\ b_{(1,2)} \\ b_{(3,0)} \\ b_{(4,0)} \\ b_{(0,4)} \end{bmatrix}. \quad (10)$$

In conclusion although $R_{\mathcal{D},\mathcal{P}}(f)$ does not depend on $\mathcal{T}$, its expression changes when the basis of $\mathcal{T}$ varies as is evident comparing Equations (9) and (10).

4. Precision space

Definition 2. *1. A nonnegative integer s is called degree of exactness or degree of precision or simply degree for a cubature rule, if $R_{\mathcal{D},W}(f) = 0$ for all polynomials f with total degree less than s and if there is a polynomial g of total degree $s+1$ such that $R_{\mathcal{D},W}(g) \neq 0$.*

2. *A set of polynomials $\mathcal{S}$ is called a precision space for a cubature rule $(\mathcal{D}, W)$ if the error $R_{\mathcal{D},W}(f) = 0$ for all $f \in \mathcal{S}$.*

Item 1 of this definition is standard, see e.g. [2, 8] while Item 2 is of interest here. Clearly a good precision space includes all polynomials with degree at most the degree of precision and possibly many more. In this paper we present progress towards characterizing them. An important result in d-dimensional space states that the lower bound on the number of nodes in a cubature rule of degree s, is $\binom{d+\lfloor s/2 \rfloor}{\lfloor s/2 \rfloor}$ nodes where $\lfloor a \rfloor$ is the integer part of a, see [8, 12]. Most research has been in the direction of determining cubature rules of a given degree. This paper discusses a method for deciding whether a space of polynomials is a precision space for an interpolatory cubature with assigned nodes. The method is constructive in that it can be turned into an algorithm for constructing a precision space by iteratively analyzing polynomials degree by degree.

Theorem 2 below gives necessary and sufficient conditions for a polynomial vector space to be a precision space. Theorem 2 generalises [13, Theorem 2.5] where $S = \left\{ x^\alpha = x_1^{\alpha_1} \cdots x_d^{\alpha_d} : \sum_{i=1}^d \alpha_i \leq s \right\}$ for a positive integer s and the terminology 'degree of precision' is used.

Theorem 2. *Let $\mathcal{D}$ be a finite set of points in $\mathbb{R}^d$ and let S be a set of polynomials such that $X_{\mathcal{D},S}$ has rank equal to the number of points in $\mathcal{D}$. Then $\mathcal{D}$ is the set of nodes of a cubature rule with precision space $\mathcal{S} = \mathrm{Span}_{\mathbb{R}}(S)$ if and only if $\mathrm{E}_\lambda(f) = 0$ for all $f \in \mathcal{I}(\mathcal{D}) \cap \mathcal{S}$.*

Proof. Assume that the cubature rule is exact for all $f \in \mathcal{S}$. Then for all $f \in \mathcal{I}(\mathcal{D}) \cap \mathcal{S}$ it holds $\mathrm{E}_\lambda(f) = \sum_{d \in \mathcal{D}} f(d) w_d$ and thus $\mathrm{E}_\lambda(f) = 0$ as $f(d) = 0$ for all $d \in \mathcal{D}$. Vice-versa, let $f \in \mathcal{S}$ and let R be a subset of S with as many elements as points in $\mathcal{D}$ and such that $X_{\mathcal{D},R}$ is invertible. This defines a unique polynomial $r \in \mathrm{Span}_{\mathbb{R}}(R)$ interpolating f at $\mathcal{D}$. We can write $f = g + r$ with $g \in \mathcal{I}(\mathcal{D})$. As both f and r belong to $\mathcal{S}$, then also $g = f - r \in \mathcal{S}$ and thus $g \in \mathcal{I}(\mathcal{D}) \cap \mathcal{S}$. From the assumption that $\mathrm{E}_\lambda(g) = 0$ it follows that $\mathrm{E}(g) = \mathrm{E}(f - r) = 0$, that is $\mathrm{E}(f) = \mathrm{E}(r)$. □

The hypothesis on the rank of $X_{\mathcal{D},S}$ is not restrictive, because if $(\mathcal{D}, \mathcal{P})$ is a correct pair and $\mathcal{S}$ a precision space for it, then also $\mathcal{P} \cup \mathcal{S}$ is a precision space for it. Thus there is no loss of generality in considering $\mathcal{S} \supset \mathcal{P}$. The cubature rule in Theorem 2 is not necessarily unique; it depends on which independent columns of $X_{\mathcal{D},S}$ are chosen, say they correspond to a set of polynomials R in S; it is interpolatory for $(\mathcal{D}, \mathrm{Span}_{\mathbb{R}}(R))$. Theorem 2 is mainly used to check whether a space of polynomials is a precision space.

Example 6. In one dimension consider $\mathcal{D} = \{1, -1\}$, $S = \{x, x^2 - 1, x^4\}$ and $\mathcal{S} = \mathrm{Span}_{\mathbb{R}}(S)$. A polynomial $f = c_1 x + c_2(x^2 - 1) + c_3 x^4 \in \mathcal{S}$ is also in $\mathcal{I}(\mathcal{D})$ if and only if $f(1) = f(-1) = 0$, that is $c_1 = c_3 = 0$ and the only polynomials in $\mathcal{I}(\mathcal{D}) \cap \mathcal{S}$ are of the form $c(x^2 - 1)$ where c is a constant (because $\mathcal{S}$ is a vector space). For any probability measure λ for which $\mathrm{E}_\lambda(X^2 - 1) = 0$ (e.g. the standard normal), all polynomials in $\mathcal{I}(\mathcal{D}) \cap \mathcal{S}$ have zero expected value. Moreover, for $R = \{x, x^4\}$ it turns out that $X_{\mathcal{D},R}$ is invertible. Consider the interpolatory cubature rule $(\mathcal{D}, \mathrm{Span}_{\mathbb{R}}(R))$ whose weights are $w^t_{\mathrm{Span}_{\mathbb{R}}(R)} = [1.5, 1.5]$. For $f \in \mathcal{S}$, $f = c_1x + c_2(x^2-1) + c_3x^4$ it holds

$$\mathrm{E}(f) = 3c_3 \qquad \text{and} \qquad \sum_{d\in\mathcal{D}} w_d f(d) = 1.5(c_1+c_3) + 1.5(-c_1+c_3) = 3c_3$$

and thus $\mathcal{S}$ is a precision space for $(\mathcal{D}, \mathrm{Span}_{\mathbb{R}}(R))$.

Theorem 2, given $\mathcal{D}$ and a polynomial space $\mathcal{S}$, gives a condition to determine whether there exists a cubature rule for which $\mathcal{S}$ is a precision space. Theorem 3 gives conditions to determine a precision space $\mathcal{S}$ for an exact pair $(\mathcal{D}, \mathcal{P})$. It follows from Theorem 1.

Theorem 3. *Let $(\mathcal{D}, \mathcal{P})$ be a correct pair and R be a vector space basis of $\mathcal{P}$. A polynomial vector space $\mathcal{S} \supset \mathcal{P}$ with basis S is a precision space for $(\mathcal{D}, \mathcal{P})$ if and only if*

$$[m_s]_{s\in\mathcal{S}\setminus R} = X^t_{\mathcal{D},\mathcal{S}\setminus R} w_{\mathcal{P}} \tag{11}$$

equivalently $R_{\mathcal{D},\mathcal{P}}(s) = 0$ for all $s \in S \setminus R$.

Proof. For $f \in \mathcal{S}$, by Theorem 1 we have

$$R_{\mathcal{D},\mathcal{P}}(f) = \left([m_s]_{s\in\mathcal{S}\setminus T} - X^t_{\mathcal{D},\mathcal{S}\setminus T} w_{\mathcal{P}}\right)^t [b_s]_{s\in\mathcal{S}\setminus T} \ ,$$

If $\mathcal{S}$ is precision space, then $R_{\mathcal{D},\mathcal{P}}(f) = 0$ for all $f \in \mathcal{S}$ and by the generality of f, Equation (11) holds. Vice-versa if Equation (11) holds, then for all $f \in \mathcal{S}$ we have $R_{\mathcal{D},\mathcal{P}}(f) = 0$ and thus $\mathcal{S}$ is precision space. The last assertion follows from the equivalence of Items 1 and 2 in Theorem 1. □

The results of this paper can be put in practice as follows. Start with a correct pair $(\mathcal{D}, \mathcal{P})$ and a vector space basis R of $\mathcal{P}$, typical examples of R are 1) monomials spanning the quotient space $\mathbb{R}[x]/\mathcal{I}(\mathcal{D})$ or 2) corresponding orthogonal polynomials. Consider a superset S of R. Evaluate $c = [m_s]_{s\in\mathcal{S}\setminus R} - X^t_{\mathcal{D},\mathcal{S}\setminus R} w_{\mathcal{P}}$. The elements $s \in S\setminus R$ corresponding to zero-entries of c can be added to R and the space spanned by their union is a precision space for $(\mathcal{D}, \mathcal{P})$. Clearly for a single element s this reduces to add s to R if $R_{\mathcal{D},\mathcal{P}}(s) = 0$. In the case that R is an order ideal and one monomial at a time is included in R, then algorithms such as those in [1] and [7, Theorem 6.4.36] can provide the order in which to take the single monomials. In the following examples λ is the d-dimensional standard normal distribution.

Example 7. Let $\mathcal{D} = \{(1,2), (-1,3), (4,0), (1,1)\} \subset \mathbb{R}^2$ be as in Example 3, $\mathcal{P} = \mathrm{Span}_{\mathbb{R}}(R)$ with $R = \{1, x_1, x_2, x_2^2\}$ and the cubature rule $(\mathcal{D}, \mathcal{P})$. In Example 4 we showed that for $T = R \cup \{x_1x_2, x_1^2, x_1^3, x_1^2x_2, x_1x_2^2, x_2^3, x_1^4, x_2^4\}$ it holds

$$[m_s]^t_{s\in T\setminus R} - w^t_{\mathcal{P}} X_{\mathcal{D},T\setminus R} = [33/5,\ -6,\ -24,\ 0,\ 94/5,\ -48/5, -100, -218/5].$$

The zero entry in fourth position shows that a precision space for $(\mathcal{D}, \mathcal{P})$ is

$$\mathcal{S} = \mathrm{Span}_{\mathbb{R}}(R \cup \{x_1^2x_2\}) = \mathrm{Span}_{\mathbb{R}}(\{1, x_2, x_1, x_2^2, x_1^2x_2\}).$$

There are three more monomials of degree four: $x_1^3x_2$, $x_1^2x_2^2$ and $x_1x_2^3$. As

$$s = x_1^3x_2 \quad \Rightarrow \quad R_{\mathcal{D},\mathcal{P}}(s) = m_s - \sum_{d\in\mathcal{D}} w_d s(d) = 0 - (-2.8*2 - 1.1*3 + 0.4*0 + 2.3*1) \neq 0$$

$$s = x_1^2x_2^2 \quad \Rightarrow \quad R_{\mathcal{D},\mathcal{P}}(s) = m_s - \sum_{d\in\mathcal{D}} w_d s(d) = 1 - (-2.8*4 + 1.1*9 + 0.4*0 + 2.3*1) = 0$$

$$s = x_1x_2^3 \quad \Rightarrow \quad R_{\mathcal{D},\mathcal{P}}(s) = m_s - \sum_{d\in\mathcal{D}} w_d s(d) = 0 - (-2.8*8 - 1.1*27 + 0.4*0 + 2.3*1) \neq 0$$

only $x_1^2x_2^2$ can be included into a precision space. This shows that the set of polynomials $\mathrm{Span}_{\mathbb{R}}(\{1, x_2, x_1, x_2^2, x_1^2x_2, x_1^2x_2^2\})$ is the intersection of the largest possible precision space with the polynomials of degree at most four and that the degree of precision is one.

Example 8. Consider $\mathcal{D} = \{x \in \mathbb{R}^5 \mid x_j^2 - 1 = 0, j = 1, \ldots 5\}$ the full factorial design 2^5 and the sets $T_n = \{x_1^{\alpha_1} \ldots x_5^{\alpha_5} \mid \sum_{i=1}^5 \alpha_i = n\}$ for $n = 0, 1, 2, 3, 4, 5$. Classical theory shows that $\mathbb{R}[x]/\mathcal{I}(\mathcal{D})$ is spanned by

$$\begin{aligned} R = \{\, & 1, x_5, x_4, x_3, x_2, x_1, x_4x_5, x_3x_5, x_2x_5, x_1x_5, x_3x_4, x_2x_4, x_1x_4, x_2x_3, x_1x_3, x_1x_2, \\ & x_3x_4x_5, x_2x_4x_5, x_1x_4x_5, x_2x_3x_5, x_1x_3x_5, x_1x_2x_5, x_2x_3x_4, x_1x_3x_4, x_1x_2x_4, x_1x_2x_3, \\ & x_2x_3x_4x_5, x_1x_3x_4x_5, x_1x_2x_4x_5, x_1x_2x_3x_5, x_1x_2x_3x_4, x_1x_2x_3x_4x_5 \,\} \end{aligned}$$

and that there is no other order ideal spanning the quotient space. For $\mathcal{P} = \mathrm{Span}_{\mathbb{R}}(R)$ application of Theorem 3 shows that T_0, T_1, T_2, T_3, T_5 and their union span precision spaces, while $\{x_5^4, x_4^4, x_3^4, x_2^4, x_1^4\} \subset T_4$ cannot belong to any precision space. Thus the degree of the cubature rule is three.

From Theorem 2 if $\mathcal{D}_1 \subset \mathcal{D}$ and $\mathcal{S}$ is a precision space for a cubature rule with nodes in $\mathcal{D}$, then there exists $\mathcal{S}_1 \subset \mathcal{S}$ which is a precision space for a cubature rule with nodes $\mathcal{D}_1$. This follows from the fact that $\mathcal{I}(\mathcal{D}_1) \supset \mathcal{I}(\mathcal{D})$. The difference between $\mathcal{S}$ and $\mathcal{S}_1$ is given by $\mathcal{S} \cap (\mathcal{I}(\mathcal{D}_1) \setminus \mathcal{I}(\mathcal{D}))$.

Example 9. Example 8 shows that a precision space $\mathcal{S}$ for $\mathcal{D}$ is the vector space generated by the monomials in T_0, T_1, T_2, T_3, T_5 and $T_4 \setminus \{x_5^4, x_4^4, x_3^4, x_2^4, x_1^4\}$. Consider the fraction of the 2^5 full factorial generated by $x_1x_2x_3x_4x_5 = 1$, namely $\mathcal{D}_1 = \{x \in \mathbb{R}^5 \mid x_j^2 - 1 = 0, j = 1, \ldots, 5, x_1 = x_2x_3x_4x_5\}$. Again from Theorem 3 it follows that the precision space $\mathcal{S}_1$ for $\mathcal{D}_1$ differs from $\mathcal{S}$ by the polynomials whose support contains $x_1x_2x_3x_4x_5$. For the fraction $\mathcal{D}_1 = \{x \in \mathbb{R}^5 \mid x_j^2 - 1 = 0, j = 1, \ldots 5, x_1x_2x_3x_4 = 1\}$ the precision space $\mathcal{S}_1$ differs from $\mathcal{S}$ by the polynomials whose support contains $x_1x_2x_3x_4x_5$ and $x_1x_2x_3x_4$. There is a clear pattern between the generators of the fractional factorial designs and the generators of the precision space which we are investigating in another work.

Example 10. Let λ be the standard normal distribution and let the set of nodes $\mathcal{D}$ be the star composite design with central point in n dimension. Without loss of generality assume the central point in $O = (0, \ldots, 0)$, the levels of the 2^n full factorial part $\mathcal{F}$ at ± 1 and the levels of arms $\mathcal{A}$ at ± 2, as in Example 47 of [11].

Firstly consider $n = 3$ and let $\mathcal{P}$ be the vector space isomorphic to $\mathbb{R}/\mathcal{I}(\mathcal{D})$ spanned by

$$R = \{1, x_1, x_2, x_3, x_1^2, x_2^2, x_3^2, x_1^4, x_1^3, x_2x_1^2, x_3x_1^2, x_2x_1, x_3x_1, x_3x_2, x_3x_2x_1\}.$$

Since $[m_s]_{s\in R} = [1, 0, 0, 0, 1, 1, 1, 3, 0, 0, 0, 0, 0, 0, 0]^t$, from Equation (1) the weights of the cubature rule $(\mathcal{D}, \mathcal{P})$ are

$$w_O = \frac{1}{6}, \quad w_d = \frac{1}{24} \text{ for all } d \in \mathcal{F}, \quad w_d = \frac{1}{12} \text{ for all } d \in \mathcal{A}.$$

Since $\mathcal{P}$ is a precision space, the degree of precision of the cubature rule is at least 2. In order to detect the degree of precision and a precision space larger than $\mathcal{P}$, we consider the following partition of the set T_7 of all terms with total degree less than or equal to 7:

$$T_7 = R \bigcup_{k=3}^{7} O_k \bigcup S_1 \bigcup S_2,$$

where $O_k = \{x_1^{m_1}x_2^{m_2}x_3^{m_3} \mid m_1 + m_2 + m_3 = k, \text{ and at least one } m_j \text{ is odd}\}$ for $k = 3, \ldots, 7$, $S_1 = \{x_i^4,\ x_i^6 \mid i = 1, 2, 3\}$ and $S_2 = \{x_i^2x_j^2,\ x_i^4x_j^2 \mid i, j = 1, 2, 3, i \neq j\} \cup \{x_1^2x_2^2x_3^2\}$. By Theorem 3 each O_k, $k = 3, \ldots, 7$, spans a precision space, since $[m_s]_{s\in O_k} - w_{\mathcal{P}}^t X_{O_k}$ is the zero vector. Furthermore, the terms in $\{x_1^6, x_2^6, x_3^6\} \subset S_1$ and the elements of S_2 cannot belong to any precision space. We conclude that the largest precision space containing terms of total degree less than or equal to 7 is spanned by $R \cup_{k=3}^{7} O_k \cup \{x_2^4, x_3^4\}$ and that the degree of the cubature rule is three.

Next, for any positive integer n, let $\mathcal{P}$ be the vector space isomorphic to $\mathbb{R}/\mathcal{I}(\mathcal{D})$ spanned by

$$R = \{1,\ x_1^2, \ldots, x_n^2,\ x_1^4,\ x_1x_1^2, \ldots, x_nx_1^2,\ \Pi_{i\in I}x_i, \text{ for all } I \in P_n\},$$

with P_n consisting of all the subsets of $\{1, \ldots, n\}$ (see [11]). Because of the special structure of the set of nodes $\mathcal{D}$, straightforward computations show that

$$\sum_{d\in\mathcal{F}} t(d) = \begin{cases} 2^n & \text{if } t = x_{i_1}^{2k_1} \ldots x_{i_h}^{2k_h} \\ 0 & \text{otherwise,} \end{cases} \qquad (12)$$

$$\sum_{d\in\mathcal{A}} t(d) = \begin{cases} 2n & \text{if } t = 1 \\ 2^{2k+1} & \text{if } t = x_i^{2k},\ k > 0 \\ 0 & \text{otherwise.} \end{cases}$$

This implies that the weights $w_{\mathcal{P}} = [w_d]_{d\in\mathcal{D}}$ are such that

$$w_O = \frac{4-n}{6}, \quad w_d = \frac{2^{-n}}{3} \text{ for all } d \in \mathcal{F}, \quad w_d = \frac{1}{12} \text{ for all } d \in \mathcal{A}.$$

In order to detect the degree of precision of the cubature rule and a precision space larger than $\mathcal{P}$, we consider, for any integer $k \geq 3$, the set O_k of the terms with total degree k and at least an odd exponent

$$O_k = \{x_1^{m_1} \ldots x_n^{m_n} \mid \text{there exists } m_i = 2p + 1,\ p \in \mathbb{Z}_{\geq 0}\}.$$

Since $[m_s]_{s\in O_k}$ is the zero vector and since Equation (12) implies that X_{O_k} is the zero matrix, also $[m_s]_{s\in O_k} - w^t_{\mathcal{P}} X_{O_k}$ is the zero vector and thus, by Theorem 3, O_k spans a precision space.

For the set $S_1 = \{x_i^{2p} \mid p \in \mathbb{Z}_{\geq 2}\}$, the generic coordinate of $w^t_{\mathcal{P}} X_{S_1}$ is $1/3 + 2^{2p}/6$ and the corresponding coordinate of $[m_s]_{s\in S_1}$ is $(2p-1)!!$, thus the error is zero if and only if $1/3 + 2^{2p}/6 = (2p-1)!!$ that is when $p = 2$. Finally, for the set $S_2 = \{x_1^{2p_1} \dots x_n^{2p_n} \mid p_i \in \mathbb{Z}_{\geq 0}$ and there exist $p_j, p_k > 0\}$, $w^t_{\mathcal{P}} X_{S_2}$ has all the coordinates equal to $1/3$ while $[m_s]_{s\in S_2}$ is a vector with integer coordinate, thus the set S_2 is not contained into a precision space.

In conclusion the precision space of the cubature rule is spanned by

$$R \bigcup_{k\geq 3} O_k \bigcup \{x_2^4, \dots, x_n^4\}$$

and, as for the case $n = 3$, the degree of precision of the cubature rule is three, independently of the space dimension n.

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