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Stability conditions for limit cycle of Smooth Transition Hyperbolic Tangent Autoregressive model

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ABSTRACT

In this paper, we suggest one of a discrete time non-linear time series model, known as Smooth transition Hyperbolic tangent Autoregressive model of order pSTtanhAR(p) , and finding stability condition of a limit cyclewhen the model processes a limit cycle of period $q > 1$. At first we find a stability condition of a limit cycle of STtanhAR(1) by using a local linearization method , then generalized this conditions to the p-order model with tow examples.

Keywords: Smooth Transition Hyperbolic tangent autoregressive model, Non- linear time series, Limit cycle, Stability,Local linearization method.

1.Introduction

A limit cycle is one of the non-linear natures when the system or model has a periodic solution. A local linearization technique is a useful tool used to studying the behavior of trajectoriesnear (closed) to each point of a limit cycle. In continuous time dynamical system a limit cycle is a closed curve represent a periodic solution of the system and the limit cycle is stable when the orbit of solution approaches the closed curve as $t \to \infty$. While in a discrete time dynamical systems or models a limit cycle is a finite set of points $\{x_t, x_{t+1}, \dots, x_{t+q}\}$ such that $x_{t+q} = x_t$, where q is a minimum positive integer greater than one .Many of searchers are studying the stability conditions of a limit cycle for many of non-linear time series models. In (1977) Oda and Ozaki studied exponential autoregressive model.[1] . in (1982) , Ozaki,T., Studied the Statistical Analysis of perturbed Limit cycle processes Using time series models.[2] , In (1985) Ozaki,T., Studied Nonlinear Time series models and Dynamical Systems.[3],and for the Special cases, Chan (1985)and Tong(1990) give the sufficient and necessary conditions for the geometric ergodicity of the threshold autoregressive model.[4], In (1986) Tsay R.S. studied the stability of non-linear time series.[5] ,In (1988) priestley M.B. studied the non-stability and non-linear time series.[6] .In (2010) Mohammed and Ghannam studied Cauchy autoregressive model.[7] In (2012) Salim and Youns studied the stability of a non-linear autoregressive models with trigonometric function.[8]

2. Concepts and definitions:

In this section we introduce some basic concepts and definitionsof STtanhAR(p) model and related dynamical concepts to a limit cycle and asymptotically stability of a limit cycle.

Definition 2.1

Let $\{x_t\}$ be a discrete time non-linear time series then the STtanhAR(p) model is defined by:

$$x_t = \sum_{i=1}^{p} [\alpha_i + \beta_i \, tanh \, (a(x_{t-1} - c)^2] \, x_{t-i} + Z_t \qquad \dots (2.2)$$

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Where $Z_t \sim iidN(0, \sigma_z^2)$

Where $\{Z_t\}$ be a white noise process, a and c are shape and scale parameters respectively, $\{\alpha_i\}$ and $\{\beta_i\}$ are constants $i = 1,2,3, \dots, p$

Definition 2.2: [5]

Let T be a finite positive integer. A k-dimensional vector$\mathbf{X}^*$ is called periodic point with period T if$\mathbf{X}^* = f^T(\mathbf{X}^*)$ and $\mathbf{X}^* \neq f^j(\mathbf{X}^*)$for $1 \leq \mathrm{j} < T$

Here $\mathbf{X}^*$ is a fixed point of f^T, we say that $\mathbf{X}^*$ is a periodic point with period T for some $T \geq 1$. And the ordered set $\{\mathbf{X}^*, f(\mathbf{X}^*), f^2(\mathrm{X}^*), \dots, f^{T-1}(\mathbf{X}^*)\}$ is called a T-cycle. We say that $\mathbf{X_0}$ is eventually periodic if there is a positive integer n such that $\mathbf{X}^* = f^n(\mathbf{X_0})$ is periodic. We say that $\mathbf{X_0}$ is asymptotically periodic if there exists periodic point $\mathbf{X}^*$ for which $\|f^n(\mathbf{X_0}) - f^T(\mathbf{X}^*)\| \to 0 \quad as \ n \to \infty$.

Definition 2.3: [2] and [3]

A limit cycle of a model $S_T = f(S_{T-1}, S_{T-2}, \dots, S_{T-p})$ where f is nonlinear function is defined as an closed isolated trajectory

$x_T, x_{T+1}, x_{T+2}, \dots, x_{T+q} = x_T$ … (2.3)

Where the period $q > 1$ be a smallest positive integer such that $y_{T+q} = y_T$. Closed means that if the initialvalue $(x_1, x_2, x_3, \dots, x_p)$belongs to the limit cycle, then$(x_{1+kq}, x_{2+kq}, \dots \dots, x_{p+kq}) = (x_1, x_2, x_3, \dots, x_p)$for any $k \in Z^+$. By Isolated we mean that every trajectory being sufficiently closed to the limit cycle approaches to it for $\mathrm{T} \to \infty$ or $T \to -\infty$ If it approaches to the limit cycle for $T \to \infty$, then the limit cycle is , but if it approaches to the limit cycle for $T \to -\infty$, then the limit cycle is unstable.

Definition 2.4: [5]

By an attractor for f we mean a compact set A such that the set

$B = \left\{s: \lim_{n\to\infty} \inf_{x \in A} \|f^n(s) - x\| = 0\right\}$ have a positive Lebesque measure and A is minimal with respect to this property. The set B is called the Basin of attraction for A and it is some time denoted by (A) . if the attractor is a set of T points $\{s_1, s_2, \dots, s_T\}$ such that $f(s_n) = s_{n+1}$, $n = 1,2, \dots T - 1$ and $f(s_T) = s_1$ then we call the attractor A a limit cycle and if $T = 1$ then we call it a limit point.[]

Definition 2.5: [2] and [3]

A singular point $\mu$of a (proposed) model $(X_T) = f(X_{T-1}, X_{T-2}, \dots, X_{T-p})$ where f a Non-linear function is defined to be appoint for which every trajectory of the model beginning sufficiently closed to the singular point μ approaches to it either for $T \to \infty$ or $T \to -\infty$. If it approaches to a singular point for $T \to \infty$, then it is stable singular point , and if it approaches to a singular point for $T \to -\infty$, then it is unstable singular point

3. Preliminaries

In the following proposition we find the stability condition of a limit cycle when

STtanhAR (1) has a limit cycle of period $q > 1$

PROPOSITION 3.1

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The limit cycle q (if it Exist) for model STtanhAR(1) is Orbitally Stable if the following condition is met.

$$\left|\prod_{i=1}^{q} \left[\left[\alpha_1 + \beta_1 tanh\left[a(x_{t+q-i} - c)^2\right] + 2(1-k)x_{t+q-i} \cdot \sqrt{a * tanh^{-1}(k)}\right]\right| < 1$$

… (3.1)

Proof:

Suppose that this model has a limit cycle of period$q$ and $q > 1$ and be in shape

$$x_t, x_{t+1}, x_{t+2}, \ldots, x_{t+q} = x_t$$

It is an isolated and closed trajectory and by using the local linear approximation technique and around every pointx_s and radius μ_s small enough that$\mu_s^n \to 0$ for all $n \geq 2$ and $s = t, t-1$and by using arithmetic substitution equations of the form $x_t = x_t + \mu_t$ and $x_{t-1} = x_{t-1} + \mu_{t-1}$and substitute it into the form STtanhAR(1) after canceling the white noise effect $z_t$we get

$$x_t + \mu_t = [\alpha_1 + \beta_1 tanh[a(x_{t-1} + \mu_{t-1} - c)^2]](x_{t-1} + \mu_{t-1})$$

...(3.2)

We can simplify the Expression$tanh[a(x_{t-1} + \mu_{t-1} - c)^2]$ in the form

$$tanh[a((x_{t-1} - c) + \mu_{t-1})^2] = tanh[a(x_{t-1} - c)^2 + 2a(x_{t-1} - c)\mu_{t-1} + a\mu_{t-1}^2]$$

Where $\mu_{t-1}^2 \to 0$

$$\therefore tanh[a(x_{t-1} + \mu_{t-1} - c)^2] = tanh[a(x_{t-1} - c)^2 + 2a(x_{t-1} - c)\mu_{t-1}$$

$$= \frac{tanh[a(x_{t-1} - c)^2] + tanh[2a(x_{t-1} - c)\mu_{t-1}}{1 + tanh[2a(x_{t-1} - c)\mu_{t-1}]} \qquad \ldots (3.3)$$

According to match $tanh(A + B) = \frac{tanh\, A + tanh\, B}{1 + tanh\, B}$

And since

$$tanh[2a(x_{t-1} - c)\mu_{t-1}] = \frac{e^{2a\mu_{t-1}(x_{t-1}-c)} - e^{-2a\mu_{t-1}(x_{t-1}-c)}}{e^{2a\mu_{t-1}(x_{t-1}-c)} + e^{-2a\mu_{t-1}(x_{t-1}-c)}}$$

By the property $tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$

By multiplying the numerator and denominator by the expression $[e^{2a\mu_{t-1}(x_{t-1}-c)}]$

We get

$$tanh[2a(x_{t-1} - c)\mu_{t-1}] = \frac{e^{4a\mu_{t-1}(x_{t-1}-c)} - 1}{e^{4a\mu_{t-1}(x_{t-1}-c)} + 1} \qquad \ldots (3.4)$$

And by using Taylor expansion of the exponential function , we get

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$$e^{4a\mu_{t-1}(x_{t-1}-c)} = 1 + 4a\mu_{t-1}(x_{t-1}-c) + \frac{16}{2!}a^2\mu_{t-1}^2(x_{t-1}-c)^2 + \cdots$$

$$\approx 1 + 4a\mu_{t-1}(x_{t-1}-c)$$

We substitute the approximation into the equation (3.4) to get

$$tanh[2a(x_{t-1}-c)\mu_{t-1}] = \frac{1 + 4a\mu_{t-1}(x_{t-1}-c) - 1}{1 + 4a\mu_{t-1}(x_{t-1}-c) + 1}$$

$$tanh[2a(x_{t-1}-c)\mu_{t-1}] = \frac{4a\mu_{t-1}(x_{t-1}-c)}{4a\mu_{t-1}(x_{t-1}-c) + 2}$$

After doing some algebraic operations, we get

$$tanh[2a(x_{t-1}-c)\mu_{t-1}]$$

$$= tanh[a(x_{t-1}-c)^2] + 2a(x_{t-1}-c)\mu_{t-1}[1 - tanh(a(x_{t-1}-c)^2)]$$

Since $tanh[a(x_{t-1}-c)^2] = k$ and $(x_{t-1}-c) = \sqrt{\frac{tanh^{-1}(k)}{a}}$

$$\therefore tanh[2a(x_{t-1}-c)\mu_{t-1}] = tanh[a(x_{t-1}-c)^2] + 2(1-k)\sqrt{a * tanh^{-1}(k)} . \mu_{t-1}$$

… (3.5)

By substituting equation(3.5) in equation(3.2) we get

$$x_t + \mu_t = \sum_{i=1}^{p}\left[\alpha_1 + \beta_1 tanh[a(x_{t-1}-c\ )^2 + 2(1-k)\sqrt{a * tanh^{-1}(k)}]\mu_{t-1}\right](x_{t-1} + \mu_{t-1})$$

$$= \alpha_1 x_{t-1} + \alpha_1\mu_{t-1} + \beta_1 x_{t-1}\, tanh[a(x_{t-1}-c)^2]$$

$$+\beta_1\, tanh[a(x_{t-1}-c)^2]\mu_{t-1\mu} + 2(1-k)\beta_1 x_{t-1}\mu_{t-1}\sqrt{a * tanh^{-1}(k)}$$

$$+2(1-k)\beta_1\mu_{t-1}^2\sqrt{a * tanh^{-1}(k)}$$

Since $\mu_{t-i}^n \to 0$ for all$i = 0,1,\dots,p$ and $n \geq 2$

Therefore

$$x_t + \mu_t = [\alpha_1 + \beta_1\, tanh(a(x_{t-1}-c)^2)]x_{t-1}$$

$$+\left[\alpha_1 + \beta_1\, tanh(a(x_{t-1}-c)^2) + 2(1-k)\beta_1 x_{t-1}\sqrt{a * tanh^{-1}(k)}\right]\mu_{t-1}$$

And since $x_t = [\alpha_1 + \beta_1\, tanh[a(x_{t-1}-c)^2]]x_{t-1}$ so

$$\mu_t = \left[\alpha_1 + \beta_1\, tanh[a(x_{t-1}-c)^2] + 2(1-k)\beta_1 x_{t-1}\sqrt{a * tanh^{-1}(k)}\right]\mu_{t-1}$$

… (3.6)

Where $k = \frac{1-\sum_{i=1}^{p}\alpha_i}{\sum_{i=1}^{p}\beta_i}$

This equation(3.6) represent afirst-order linear differential equation with unstable coefficients, it is difficult to find an exact solution to it but for theconvergence of this equation, the ratio test can be used to consider whether the difference equation converges towards zero as (t) increases without limit to infinity$t \to \infty$, if the solution is convergent, then this means that the end cycle is orbitally stable.

This convergence towards zero occurs only if the ratio is

$$\left|\frac{\mu_{t+q}}{\mu_t}\right| < 1 \qquad \dots (3.7)$$

Where q is the number of cycles of a limit periodic

We can write the equation (3.6) in the form

$$\mu_t = T(x_{t-1})\mu_{t-1} \qquad \dots (3.8)$$

where

$$T(x_{t-1}) = \alpha_1 + \beta_1 \, tanh[a(x_{t-1} - c)^2] + 2\beta_1(1-k)x_{t-1}\sqrt{a * tanh^{-1}(k)}$$

And after q of iterations of (3.8) we get

$$\mu_{t+q} = T\left(x_{t+q-1}\right).\mu_{t+q-1} = T\left(x_{t+q-1}\right).T\left(x_{t+q-2}\right)\dots T(x_t)\mu_t$$

Continuing to repeat q times, we get

$$\mu_{t+q} = \prod_{i=1}^{q} T\left(x_{t+q-i}\right).\mu_t$$

From it we get the required percentago

$$\left|\frac{\mu_{t+q}}{\mu_t}\right| = \left|\prod_{i=1}^{q} T(x_{t+q-i})\right|$$

From the convergence condition mentioned in relation(3.7), the limit cycle of period$q$, is orbitally stable if the condition is fulfilled

$$\left|\prod_{i=1}^{q} T(x_{t+q-i})\right| < 1$$

Or otherwise

$$\left|\prod_{i=1}^{q}[\alpha_1 + \beta_1 tanh\,(a\left(x_{t+q-i} - c\right)^2\right) + 2(1-k).\,x_{t+q-i}\sqrt{a * tanh^{-1}(k)}]\right| < 1$$

Or

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$$\left|\prod_{i=1}^{q} \begin{matrix} [\alpha_1 + \beta_1 tanh\,(a(x_{t+q-i} - c)^2) \\ +2(1-k).\,x_{t+q-i}\sqrt{a * tanh^{-1}(k)}] \end{matrix}\right| < 1 \qquad \dots (*)$$

And this ends the proof.

The following proposition is a generalization of proposition(3.1) when the model has a rank of p and $p > 1$ and in this case we will have to adopt the representation of the modelSTtanhAR(p) in the state space to be of the following form .

$$X_t = \alpha + \beta\, tanh(a(\,x_{t-1} - c\,)^2)X_{t-1} + \in_t \qquad \dots (3.9)$$

Note that α , β are two matrices in the form

$$\alpha = \begin{bmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_{p-1} & \alpha_p \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \quad , \ \beta = \begin{bmatrix} \beta_1 & \beta_2 & \cdots & \beta_{p-1} & \beta_p \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Where both X_t , X_{t-1} , $\in_t$ are vectors defined by

$$X_t = (x_t, x_{t-1}, \dots, x_{t-p+1})^T \quad , \quad X_{t-1} = (x_{t-1}, x_{t-2}, \dots, x_{t-p})^T$$

$$\in_t = (z_t, 0, \dots, 0)^T$$

Since $\in_t$ is the white noise vector

And that the elements of matrix$(\alpha + \beta\, tanh[a(x_{t-1} - c)^2])$ depend on the random variable x_{t-1} and we will use for this matrix the symbol$A(\,x_{t-1})$ which is not fixed and written in the following form

$$A(x_{t-1}) = \alpha + \beta\, tanh[a(x_{t-1} - c)^2]$$

The model is expressed in equation(3.9) in the form below

$$X_t = A(x_{t-1})X_{t-1} + \in_t$$

It is also written as

$$X_{t+1} = A(x_t)X_t + \in_{t+1} \qquad \dots (3.10)$$

Proposition 3.2

The limit cycle of period q and $q > 1$, if it exist of the model STtanhAR(p) is Orbitally stable if all the eigenvalues of matrix A have absolute less than 1, where

$$A = A_q A_{q-1} \dots A_1 = \prod_{j=1}^{q} A_j$$

And

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$$A_j = \begin{bmatrix} a_{1,1}^{(j)} & a_{1,2}^{(j)} & \cdots & a_{1,p-1}^{(j)} & a_{1,p}^{(j)} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad ; j = 1,2, \dots, q$$

Since

$$a_{1,1}^{(j)} = \alpha_1 + \beta_1 tanh[a(x_{t+j-1} - c)^2] + 2(1-k)x_{t+j-1}\sqrt{a * \tanh^{-1}(k)}$$

$$a_{1,k}^{(j)} = \alpha_k + \beta_k \tanh[a(x_{t+j-1} - c)^2], k = 2,3,4, \dots, p \qquad \dots (3.11)$$

Proof:

Let Model STtanhAR(p) have a representation in the state space and have a limit cycle of period q and $q > 1$ defined by the form

$$x_t, x_{t-1}, x_{t-2}, \dots, x_{t+q} = x_t$$

Which produce a closed and isolated trajectory. And by using the same assumptions used in the proof of the previous theorem and substituting in model (3.9), we get

$$X_{t+1} + \mu_{t+1} = [\alpha + \beta tanh[a(x_t + \mu_t - c)^2]] (X_t + \mu_t)$$

Where μ_t and μ_{t+1} vectors are defined as

$$\mu_T = \begin{bmatrix} \mu_t \\ \mu_{t-1} \\ \vdots \\ \mu_{t-p+1} \end{bmatrix} \qquad , \quad \mu_{t+1} = \begin{bmatrix} \mu_{t+1} \\ \mu_t \\ \vdots \\ \mu_{t-p+2} \end{bmatrix}$$

After performing some algebraic operations, as in the proof of the previous theorem, we get

$$\mu_{t+1} = \begin{bmatrix} a_{1,1}^{(1)} & a_{1,2}^{(1)} & \cdots & a_{1,p-1}^{(1)} & a_{1,p}^{(1)} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \mu_t \dots (3.12)$$

Since the matrix on the right side of equation (3.12) is denoted by the symbol

A_1 and written as

$$A_1 = \begin{bmatrix} a_{1,1}^{(1)} & a_{1,2}^{(1)} & \cdots & a_{1,p-1}^{(1)} & a_{1,p}^{(1)} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & \vdots & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \mu_t$$

$$a_{1,1}^{(1)} = \alpha_1 + \beta_1 tanh[a(x_t - c)^2] + 2(1-k)x_t\sqrt{a * \tanh^{-1}(k)}$$

$$a_{1,k}^{(1)} = \alpha_k + \beta_k tanh[a(x_t - c)^2],\ k = 2,3,\dots,p$$

And write Equation (3.12) as

$$\mu_{t+1} = A_1\mu_t$$

Therefore, it is

$$\mu_{t+2} = A_2\mu_{t+1}$$

By repeating this process for q times, taking into account the mathematical changes that occur in the equation, we get

$$\mu_{t+q} = A_q\mu_{t+q-1} = A_qA_{q-1}\dots.A_1\mu_t$$

That is

$$\mu_{t+q} = A_qA_{q-1} \dots A_1\mu_t \qquad \dots (3.13)$$

Where

$$A_j = \begin{bmatrix} a_{1,1}^{(j)} & a_{1,2}^{(j)} & \cdots & a_{1,p-1}^{(j)} & a_{1,p}^{(j)} \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & \vdots & 0 \\ \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \qquad \dots (3.14)$$

$$j = 1,2,3,\dots,q$$

and

$$a_{1,1}^{(j)} = \alpha_1 + \beta_1 tanh[a(x_t - c)^2] + 2(1-k)x_t\sqrt{a * tanh^{-1}(k)}$$

$$a_{1,k}^{(j)} = \alpha_1 + \beta_1 tanh[a(x_t - c)^2] \qquad ,\ k = 2,3,\dots,p$$

And write Equation (3.13) as follows

$$\mu_{t+q} = \prod_{j=1}^{q} A_j\, \mu_t \qquad \dots (3.15)$$

And let the product of the matrices be A_j where $j = 1,2,\dots,q$ is the matrix A and so we can write equation (3.15) in the form

$$\mu_{t+q} = A\,\mu_t \qquad \dots (3.16)$$

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And in order for the convergence to be towards zero, that is $A^j \to 0$ as it approaches $\to \infty$, the absolute values of the eigenvalues of matrix A must be less than one. In other words, if the eigenvalues of matrix A_j mentioned in equation (3.14) for $j = 1,2, \dots, q$ are less than one

Therefore, the limit cycle of period q of model STtanhAR(p) is Orbitally stable under the above condition

And this ends the proof.

We can apply the conditions of proposition (3.1) by taking two examples with an arbitrary values of parameters such that one of them is stable and other unstable for clarification the stability of limit cycle.

Example 3.1

Let STtanhAR(1) model is given by

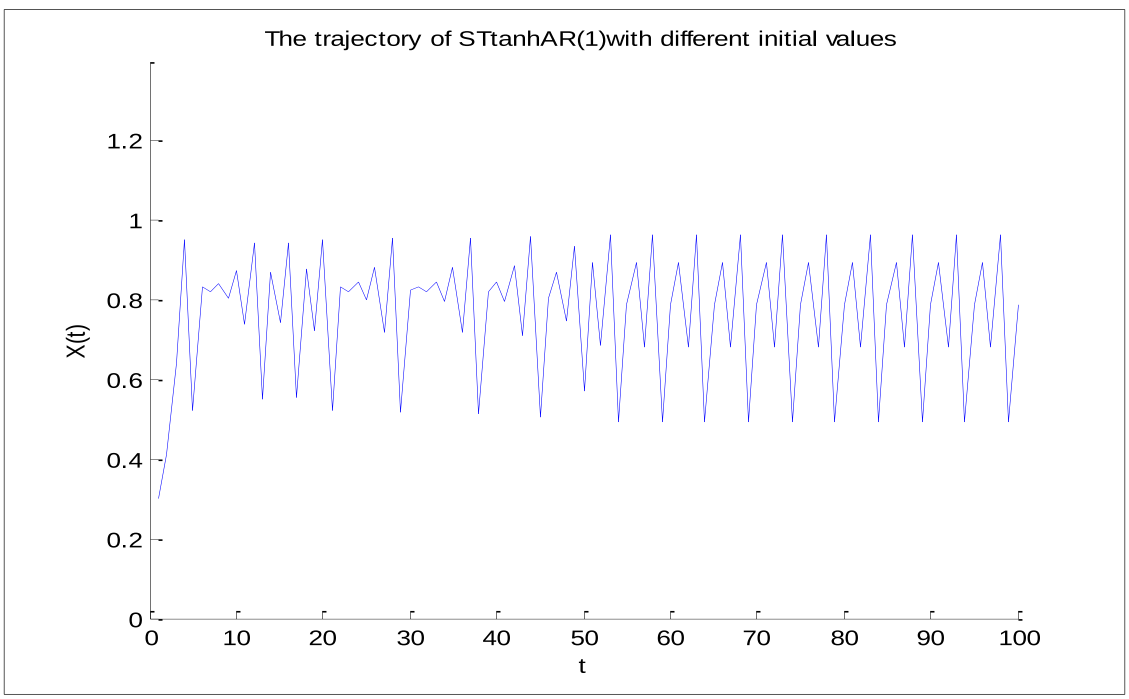
$x_t = [\alpha_1 + \beta_1 tanh\,(a(x_{t-1} - c)^2)]x_{t-1} + z_t$

$x_t = [\,1.6 - 2\ tanh(2.851883242(x_{t-1} - 0.5)^2)]\,x_{t-1} + z_t \qquad \dots (**)$

Has non-zero Singular point$\mu = 0.8294$ and a limit cycle of period 5 which is $\{0.9627,0.4919,0.7868,0.8963,0.6808,0.9627\}$. we can calculate that by using

$$\left|\prod_{i=1}^{6}\left[1.6 - 2\tanh\,(2.851883242\big(x_{t+q-i} - 0.5)^2\big) + 2(1-0.3).\,x_{t+q-i}.\sqrt{2.851883242 * \tanh^{-1}(0.3)}\right]\right|$$

$= 21.0993\ > 1$. the condition (*) does not satisfy, andthe limit cycle is orbitally unstable . note that Figure (3.1) below with different initial values.



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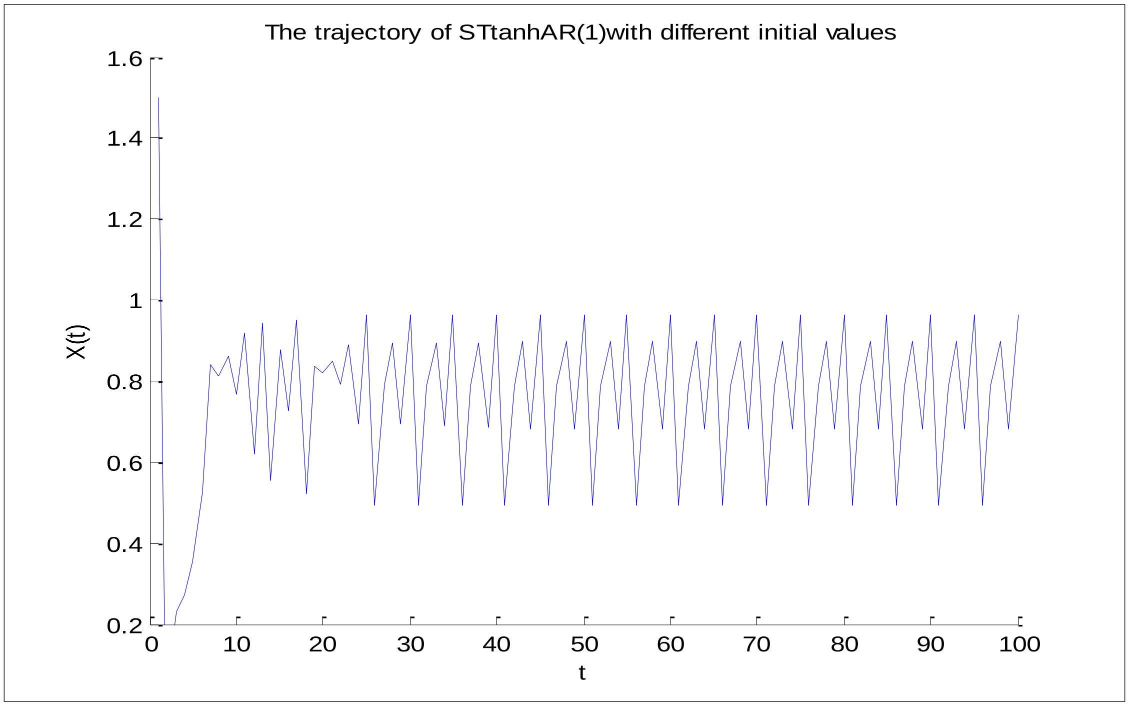


Figure (3.1) plotting the limit cycle orbitally unstable of STtanhAR(1) with different initial values

Example 3.2

Consider the following 0f STtanhAR(1) model

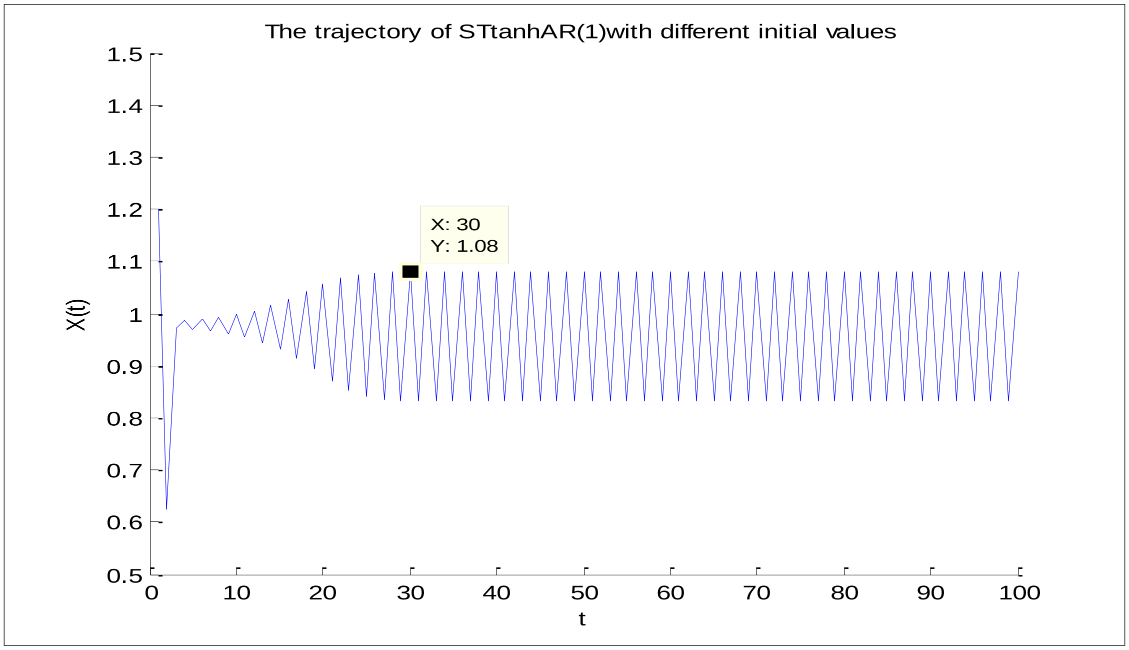
$x_t = [\ 1.6 - 1.5\ \ tanh(1.851883242(x_{t-1} - 0.5)^2)]\ x_{t-1} + z_t$ ... $(***)$

Has a non-zero singular point $\mu = 0.9783$and a limit cycle of period2 which is {1.08,0.8319,1.08} we can calculate that by using

$$\left|\prod_{i=1}^{6}\left[1.6 - 1.5\ tanh\ (1.851883242\left(x_{t+q-i} - 0.5\right)^2\right) + 2(1 - 0.4).\ x_{t+q-i}.\sqrt{1.851883242 * tanh^{-1}(0.4)}\right]\right|$$

= 0.3767 < 1

the condition (*) is satisfying therefore the limit cycle is Orbitally stable . Note thatFigure(3.2) below shows the stability of limit cycle with different initial values.



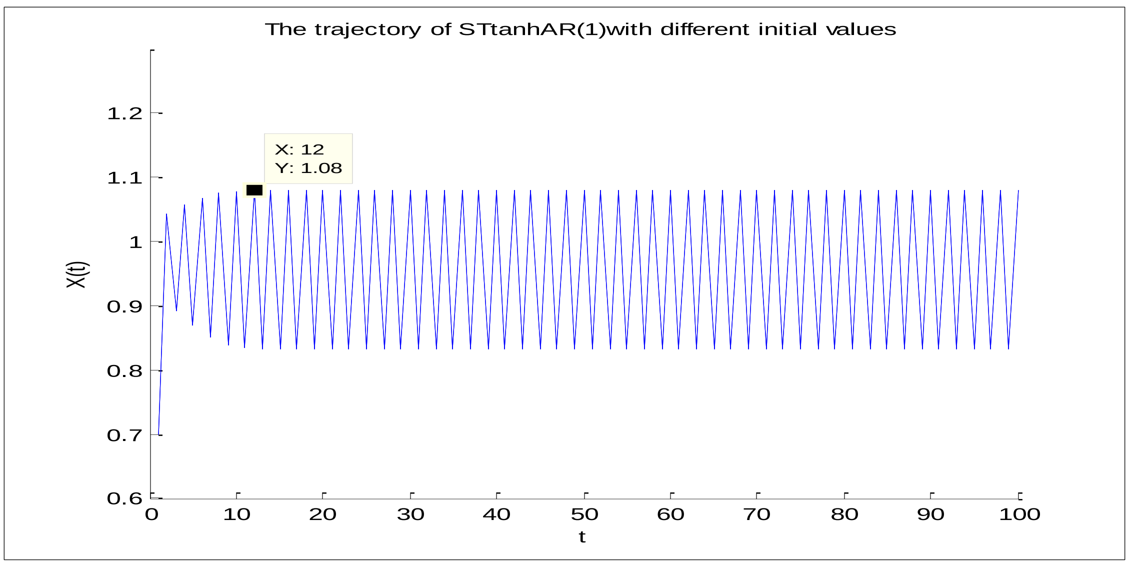


Figure (3.2) plotting the limit cycle orbitally stable with different initial values

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