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On the Connectivity of Fiber Graphs

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Abstract. We consider the connectivity of fiber graphs with respect to Gröbner basis and Graver basis moves. First, we present a sequence of fiber graphs using moves from a Gröbner basis and prove that their edge-connectivity is lowest possible and can have an arbitrarily large distance from the minimal degree. We then show that graph-theoretic properties of fiber graphs do not depend on the size of the right-hand side. This provides a counterexample to a conjecture of Engström on the node-connectivity of fiber graphs. Our main result shows that the edge-connectivity in all fiber graphs of this counterexample is best possible if we use moves from Graver basis instead.

Keywords: Fiber connectivity, Gröbner basis, Graver basis, Fiber graph

1 Introduction

Many applications in statistics require a deeper analysis of the structure of a *fiber* of an integer matrix $A \in \mathbb{Z}^{d\times n}$ with $\ker(A) \cap \mathbb{Z}^n_{\geq 0} = \{\mathbf{0}_n\}$ and a vector $\mathbf{b} \in \mathbb{Z}^d$ defined as

$$\mathcal{F}_{A,\mathbf{b}} := \{\mathbf{u} \in \mathbb{Z}^n_{\geq 0} : A \cdot \mathbf{u} = \mathbf{b}\}. \tag{1.1}$$

Very often, one needs to sample elements of the set $\mathcal{F}_{A,\mathbf{b}}$ randomly, for example in hypothesis testing for log-linear models [6, Chapter 1]. The assumption $\ker(A) \cap \mathbb{Z}^n_{\geq 0} = \{\mathbf{0}_n\}$ makes $\mathcal{F}_{A,\mathbf{b}}$ finite for all $\mathbf{b} \in \mathbb{Z}^d$. A random sampling on $\mathcal{F}_{A,\mathbf{b}}$ can be realised by performing a random walk on a *fiber graph* $\mathrm{G}(\mathcal{F}_{A,\mathbf{b}}, \mathcal{M})$ which is defined for a set $\mathcal{M} \subseteq \ker(A) \cap \mathbb{Z}^d$ as the graph on the nodes $\mathcal{F}_{A,\mathbf{b}}$ in which two nodes $\mathbf{v}, \mathbf{u} \in \mathcal{F}_{A,\mathbf{b}}$ are adjacent if either $\mathbf{v} - \mathbf{u} \in \mathcal{M}$ or $\mathbf{u} - \mathbf{v} \in \mathcal{M}$. The set $\mathcal{M}$ can be seen as a set of directions or *moves* one is allowed to choose from during the random walk. Since random walks on graphs are essentially the same as Markov chains whose state space equals the node-set of the graph – in the context of this paper $\mathcal{F}_{A,\mathbf{b}}$ – one can ask whether this Markov chain converges against a stationary distribution. If $\mathrm{G}(\mathcal{F}_{A,\mathbf{b}}, \mathcal{M})$ is connected and non-bipartite, the Markov chain is irreducible and aperiodic and hence convergence towards

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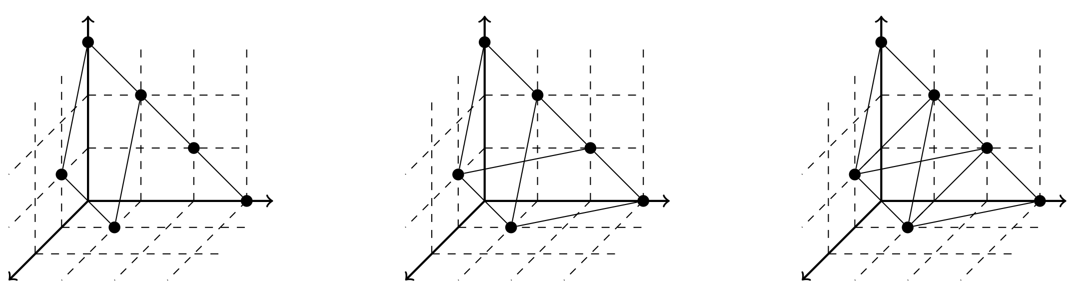
a stationary distribution is guaranteed [8, Theorem 4.9]. Thus, the study of the connectedness of fiber graphs is an important question in statistics.

The idea of sampling from fiber graphs goes back to the seminal work [4] of Diaconis and Sturmfels. They formulated the connectedness of fiber graphs equivalently in the language of commutative algebra: a set of moves $\mathcal{M} \subseteq \ker(A) \cap \mathbb{Z}^n \setminus \{\mathbf{0}_n\}$ makes the fiber graphs $\mathrm{G}(\mathcal{F}_{A,\mathbf{b}}, \mathcal{M})$ connected for all $\mathbf{b} \in \mathbb{Z}^d$ simultaneously if and only if the set of polynomials $\{\mathbf{x}^{\mathbf{m}^+} - \mathbf{x}^{\mathbf{m}^-} : \mathbf{m} \in \mathcal{M}\}$ generates the *toric ideal*

$$I_A := \langle \mathbf{x}^{\mathbf{u}^+} - \mathbf{x}^{\mathbf{u}^-} : \mathbf{u} \in \ker(A) \cap \mathbb{Z}^n \rangle.$$

The tools of commutative algebra provide a long list of moves which generate the toric ideal finitely (see [10, Chapters 3 and 10]): every reduced Gröbner bases of A with respect to a term ordering $\prec$ on $\mathbb{Z}^n_{\geq 0}$, denoted by $\mathcal{R}_\prec(A)$, the universal Gröbner basis of A, denoted by $\mathcal{U}(A)$, and the Graver basis of A, denoted by $\mathcal{G}(A)$, are Markov bases of A. We call a fiber graph using moves from a Gröbner basis a *Gröbner fiber graph* and a fiber graph using moves from the Graver basis a *Graver fiber graph*.

When working with Markov chains, it is typical to ask: What can we say about the random walk and the convergence of the corresponding Markov chain? How long do we have to run the random walk until we have a sufficiently good approximation of its stationary distribution? Is there a difference in using moves from $\mathcal{R}_\prec(A)$ rather then from $\mathcal{G}(A)$ (see Example 1.1)? In this paper we have a closer look at a more refined structural information of fiber graphs from which we think an answer to these questions can eventually be derived.



$\mathrm{G}(\mathcal{F}_{A,\mathbf{b}}, \mathcal{R}_\prec(A))$ $\quad$ $\mathrm{G}(\mathcal{F}_{A,\mathbf{b}}, \mathcal{U}(A))$ $\quad$ $\mathrm{G}(\mathcal{F}_{A,\mathbf{b}}, \mathcal{G}(A))$

Fig. 1. Different fiber graphs of the same underlying fiber.

Example 1.1. Figure 1 shows the fiber graphs for the matrix $A = (1, 1, 2) \in \mathbb{Z}^{1\times 3}$ and the right-hand side $\mathbf{b} = 3$ using different types of moves. Even if we see obvious differences in those three fiber graphs, the only statement we can make so far is that they are all connected. The mixing times of those fiber graphs with respect to the Metropolis-Hastings chain as defined in Section 7 read from left to right as follows: 5.78807, 6.32917, and 2.24376. We see that the mixing time of the Graver fiber graph surpasses the mixing time of the Gröbner fiber graphs by far.

To measure mixing we have to go beyond mere *connectedness*. One possible measurement could be the *connectivity* of the underlying fiber graph (see Section 2) which counts the number of paths between two nodes. It can be argued that the connectivity of a graph measures in some sense the possibility of 'getting stuck' in a node during a random walk and hence a small connectivity cannot lead to a good mixing time of the related Markov chain. In Section 7 we present our computational results confirming this hypothesis.

Based on the assumption that a high connectivity is a desirable property of fiber graphs, Engström conjectured in a talk at IST Austria in 2012 that the node-connectivity is best possible for Gröbner fiber graphs.

Conjecture 1 (Engström; 2012). Let $A \in \mathbb{Z}^{d\times n}$ be a matrix with $\ker(A) \cap \mathbb{Z}^n = \{\mathbf{0}_n\}$ and $\prec$ be a term ordering on $\mathbb{Z}^n$. Then for all $\mathbf{b} \in \mathbb{Z}^d$, the node-connectivity of $\mathrm{G}\left(\mathcal{F}_{A,\mathbf{b}}, \mathcal{R}_\prec(A)\right)$ equals its minimal degree.

A recent result of Potka supports Conjecture 1. He proved in [11] that the node-connectivity of certain Gröbner fiber graphs of the $n \times n$ independence-model is best possible. However, we show in Section 5 that Conjecture 1 is false in general. Let I_k be the identity matrix in $\mathbb{Z}^{k\times k}$ and let $\mathbf{1}_k$ be the k-dimensional vector having all entries equal to 1. For

$$A_k := \begin{pmatrix} I_k & I_k & \mathbf{0} & \mathbf{0} & -\mathbf{1}_k & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I_k & I_k & \mathbf{0} & -\mathbf{1}_k \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 1 & 1 \end{pmatrix} \in \mathbb{Z}^{(2k+1)\times(4k+2)} \tag{1.2}$$

the underlying fiber graph of $\mathcal{F}_{A_k,\mathbf{e}_{2k+1}}$ has node-connectivity 1 and minimal degree k when using moves from the reduced lexicographic Gröbner basis of A_k (see Corollary 5.1). Hence, we cannot expect to have a best possible connectivity in *all* Gröbner fiber graphs. Thus, [11] poses a weaker follow-up conjecture which claims that the node-connectivity in fiber graphs is best possible if the right-hand side is sufficently large.

Conjecture 2 ([11]). Let $A \in \mathbb{Z}^{d\times n}$ be a matrix and $\prec$ be a term ordering. There exists $\mathbf{N} \in \mathbb{Z}^d_{\geq 0}$ such that for all $\mathbf{b} \in \mathbb{Z}^d_{\geq 0}$ with $\mathbf{b}_i \geq \mathbf{N}_i$ for all $i \in [d]$, the node-connectivity of the fiber graph $\mathrm{G}\left(\mathcal{F}_{A,\mathbf{b}}, \mathcal{R}_\prec(A)\right)$ equals its minimal degree.

We prove in Section 2 that once we observe a bad connectivity in an arbitrary fiber of a matrix, we can construct fibers of a related matrix whose right-hand side entries exceed any given bound and whose connectivity remains bad. Thus, by modifying our original counterexample (1.2), we show in Section 5 that this gives rise to a counterexample to Conjecture 2.

Since these results diminish the hope for suitable connectivity in Gröbner fiber graphs, we consider in Section 6 a possible way out. We show that in all Graver fiber graphs of A_k, in particular even in those in which the Gröbner connectivity is lowest possible, the edge-connectivity best possible.

2 Connectivity and Fiber Graphs

In this section we recall some basic definitions from graph theory and introduce the framework of graph connectivity. We refer to [5] for a more general introduction to this field. Let $G = (V, E)$ be a simple graph in finitely many nodes V and edges E. In this notation, a fiber graph $\mathrm{G}(\mathcal{F}_{A,\mathbf{b}}, \mathcal{M})$ can be written as $(\mathcal{F}_{A,\mathbf{b}}, \{\{\mathbf{u}, \mathbf{v}\} : \mathbf{u} - \mathbf{v} \in \pm\mathcal{M}\})$. We call

$$\delta(G) := \min\{\deg(v) : v \in V\}$$

the *minimal degree* of G where $\deg(v)$ is the cardinality of the neighborhood of v in G. Let $k \in \mathbb{Z}_{\geq 0}$, then G is *k-node-connected* if $|V| > k$ and if for all $X \subseteq V$ such that $|X| < k$, the induced graph of G on the nodes $V \setminus X$ is connected. In addition, the *node-connectivity* of G is

$$\kappa(G) := \max\{k \in \mathbb{Z}_{\geq 0} : G \text{ is } k\text{-node-connected}\}.$$

Similarly, G is *k-edge-connected* if $|E| > k$ and if for all $X \subseteq E$ such that $|X| < k$ the graph $(V, E \setminus X)$ is connected. The *edge-connectivity* of G is

$$\lambda(G) := \max\{k \in \mathbb{Z}_{\geq 0} : G \text{ is } k\text{-edge-connected}\}.$$

For every graph G we have $\delta(G) \geq \lambda(G) \geq \kappa(G)$ [5, Chapter 1.4]. For example, we obtain $\delta(G) \geq \lambda(G)$ by removing all adjacent edges from a node with minimal degree in G, which isolates this node and hence gives a disconnected graph. The edge-connectivity of G is *best possible* if $\delta(G) = \lambda(G)$ and similarly the node-connectivity is *best possible* if $\delta(G) = \kappa(G)$. Even if these definitions look very convenient at a first glance, they are rather unwieldy for proving general results about fiber graphs. For our purposes, an equivalent property based on the number of paths between two nodes turns out to be more useful and enables us to put hands on the connectivity of fiber graphs (see also Menger's Theorem [5, Chapter 3.3]). To obtain a lower bound on the node-connectivity of a graph, we only have to determine the number of paths between nodes whose neighborhoods have a non-empty intersection rather than between all nodes according to Liu's criterion [9]. Since our main result concerns edge-connectivity, we modify Liu's original criterion and obtain a similar statement involving edge-connectivity (see Lemma 2.1). A proof of Liu's criterion can be found in [1] and the idea behind the proof of Lemma 2.1 is similar, which is the reason why we omit its proof here.

Lemma 2.1. *Let $k \in \mathbb{Z}_{\geq 0}$ and let $G = (V, E)$ be a connected graph with $|E| > k$. If for all $u, v \in V$ such that $\{u, v\} \in E$ there are at least k edge-disjoint paths from u to v in G, then we have $\lambda(G) \geq k$.*

Since $\delta(G) \geq \lambda(G)$, replacing k with $\delta(G)$ in Lemma 2.1 gives a sufficient condition for the edge-connectivity to be best possible. The next lemma is very useful in the proof of Proposition 2.1.

Lemma 2.2. *Let $G = (V, E)$ be a graph and $K \subseteq V$ with $|K| \leq \lambda(G)$ and $v \in V \setminus K$. Then there are $|K|$ edge-disjoint paths in G connecting all nodes of K with v.*

Proof. We extend G to a new graph G^* by adding a node u and by inserting edges between u and all nodes of K. Since $|K| \leq \lambda(G)$, G^* must have edge-connectivity at least $|K|$ due to the fact that removal of $|K|-1$ edges does not disconnect G^*. The definition of edge-connectivity gives $|K|$ many edge-disjoint paths from u to v in G^*. Clearly, those paths connect all nodes of K with v in G as well and by removing u from G^* we obtain edge-disjoint paths from all nodes of K to v in G. $\square$

The following proposition helps us out in Section 6.

Proposition 2.1. *Let $G = (V_1 \cup V_2, E)$ with $V_1 \cap V_2 = \emptyset$ and such that the induced subgraphs on V_1 and V_2 have both an edge-connectivity of at least n. Furthermore, assume that every node in V_1 has at least $m \geq n+2$ neighbors in V_2. Let $v_1 \in V_1$ and $v_2 \in V_2$ be two adjacent nodes such that there are m node-disjoint paths in G connecting v_1 and v_2 which only use edges whose end-points are in V_1 and V_2, respectively. Then there are $m+n$ edge-disjoint paths in G connecting v_1 and v_2.*

Proof. By assumption, we obtain for every $j \in [m]$ a path P_j connecting v_1 and v_2 which only uses edges between V_1 and V_2 (the edge $\{v_1, v_2\}$ is regarded as a path):

$$v_1 \xleftrightarrow{P_j} v_2. \tag{2.1}$$

Moreover, all these paths are pairwise node-disjoint and hence pairwise edge-disjoint. Denote by $N(v)$ the neighborhood of a node v in G. Since, by assumption, the induced subgraph on V_1 is n-edge-connected, we have that v_1 has at least n neighbors in V_1, i.e., $|N(v_1) \cap V_1| \geq n$. Let $w_1, \ldots, w_n$ be n arbitrary nodes in the neighborhood of v_1 in V_1. Again by assumption, we know that $|N(w_i) \cap V_2| \geq m$ for all $i \in [n]$. Our goal is to construct additional edge-disjoint paths between v_1 and v_2. Since the paths P_j are pairwise node-disjoint, every w_i could have been used by at most one path P_j. Hence, for every $i \in [n]$, up to two edges going from w_i to V_2 could have been used by the paths P_j. In particular, there are still $m-2$ edges from w_i into V_2 which have not been used by any of the paths P_j. Since we have $m-2 \geq n$ by assumption, each w_i has at least n unused neighbors in V_2 and thus we can choose for every $i \in [n]$ a node $k_i \in N(w_i) \cap V_2$ such that the edge $\{w_i, k_i\}$ is not used by any of the paths P_j and such that $k_i \neq k_{i'}$ for all $i \neq i'$. By construction, $\{v_1, w_i, k_i\}$, $i \in [n]$, give n pairwise node-disjoint paths from v_1 to k_i. Since the induced subgraph on V_2 is also n-edge-connected we can apply Lemma 2.2 on the set $\{k_i : i \in [n]\}$ and the node v_2 in the induced subgraph on V_2 and we obtain for every $i \in [n]$ a path Q_i connecting k_i with v_2 such that all of those paths are pairwise edge-disjoint (note that if there is an $i' \in [n]$ such that $k_{i'} = v_2$ we set $Q_{i'} = \emptyset$ and we still can apply the lemma on the smaller set consisting of the $w_i \neq w_{i'}$). All in all, we have for all $i \in [n]$ a path

$$v_1 \xleftrightarrow{\{v_1, w_i\}} w_i \xleftrightarrow{\{w_i, k_i\}} k_i \xleftrightarrow{Q_i} v_2 \tag{2.2}$$

and by construction these paths are pairwise edge-disjoint. Since for all $i \in [n]$ the path from v_1 to w_i stays completely in V_1, the path from Q_i only uses edges connecting nodes in V_2 and since the edges $\{w_i, k_i\}$ had not been used by the paths P_j which on the other hand only uses edges between V_1 and V_2, the paths given in (2.2) are pairwise edge-disjoint to all paths P_j, $j \in [m]$. This gives $m + n$ pairwise edge-disjoint paths in G connecting v_1 and v_2. $\square$

Our next result states that graph-theoretic properties of fiber graphs are independent of the size of the right-hand side.

Theorem 1 (Universality Theorem). *Every Gröbner fiber graph of a matrix $A \in \mathbb{Z}^{d\times n}$ is isomorphic to a Gröbner fiber graph of a matrix $A' \in \mathbb{Z}^{2d\times(n+d)}$ with arbitrarily large entries of its right-hand.*

Proof. Let $\mathbf{b} \in \mathbb{Z}^d$ be the right-hand side of an arbitrary fiber of A and let $\mathcal{R}$ be a Gröbner basis of A with respect to an arbitrary term ordering $\prec$ on $\mathbb{Z}^n$. Consider the following matrix:

$$A' := \begin{pmatrix} A & I_d \\ \mathbf{0} & I_d \end{pmatrix} \in \mathbb{Z}^{2d\times(n+d)}.$$

Clearly, we have $\ker(A') \cap \mathbb{Z}^{n+d} = \left\{(\mathbf{v}, \mathbf{0})^T : \mathbf{v} \in \ker(A) \cap \mathbb{Z}^n\right\}$. Thus, we obtain that $\mathcal{R}' := \{(\mathbf{v}, \mathbf{0})^\intercal : \mathbf{v} \in \mathcal{R}\}$ is a Gröbner basis of A' with respect to an arbitrary extension $\prec'$ of $\prec$ on $\mathbb{Z}^{n+d}$. We define for every bound $N \in \mathbb{Z}_{\geq 0}$ the right-hand side $\mathbf{b}'(N) := \begin{pmatrix} \mathbf{b} + \tilde{n}\cdot \mathbf{1}_d \\ \tilde{n}\cdot\mathbf{1}_d \end{pmatrix} \in \mathbb{Z}^{2d}$ where $\tilde{n} := \max\{N, N - \mathbf{b}_i : i \in [d]\} \in \mathbb{Z}_{\geq 0}$. It is easy to see that have the following correlation between fibers of A and A':

$$\mathcal{F}_{A',\mathbf{b}'(N)} = \left\{ \begin{pmatrix} \mathbf{v} \\ \tilde{n}\cdot\mathbf{1}_d \end{pmatrix} : \mathbf{v} \in \mathcal{F}_{A,\mathbf{b}} \right\}.$$

Thus, for every $N \in \mathbb{Z}_{\geq 0}$ the map $\mathcal{F}_{A,\mathbf{b}} \to \mathcal{F}_{A',\mathbf{b}'(N)}, \mathbf{v} \mapsto \begin{pmatrix} \mathbf{v} \\ \tilde{n}\cdot\mathbf{1}_d \end{pmatrix}$ gives a bijection from the nodes of $\mathcal{F}_{A',\mathbf{b}'(N)}$ to the nodes of $\mathcal{F}_{A,\mathbf{b}}$. Even more, from the relation between the Gröbner bases $\mathcal{R}$ and $\mathcal{R}'$, this map respects the set of edges and hence it gives rise to a graph isomorphism between the graphs $\mathrm{G}\left(\mathcal{F}_{A,\mathbf{b}'(N)}, \mathcal{R}'\right)$ and $\mathrm{G}\left(\mathcal{F}_{A,\mathbf{b}}, \mathcal{R}\right)$ for all $N \in \mathbb{Z}_{\geq 0}$. Since we have $\mathbf{b}'(N) \geq N \cdot \mathbf{1}_{2d}$ we found a fiber graph which is isomorphic to $\mathrm{G}\left(\mathcal{F}_{A,\mathbf{b}}, \mathcal{R}\right)$ such that the right-hand side components exceeds every given bound $N \in \mathbb{Z}_{\geq 0}$. $\square$

It is not hard to see that Theorem 1 is true if we choose Universal Gröbner bases or Graver bases as sets of allowed moves as well. With a view towards Conjecture 2: due to the isomorphism, all properties of the underlying graph carry over and hence it is enough to consider a Gröbner fiber graph of a matrix whose connectivity is strictly less than its minimal degree in a low-sized right-hand side (see Section 5). Since there are no general tools for determining the connectivity of fiber graphs available, we establish some definitions and lemmas from which our connectivity results in Section 6 benefit from. First, we slightly extend our definition of a fiber graph in Section 1 in the sense that we do not only restrict on fibers as a set of nodes but rather on arbitrary sets of integer points. For two sets $\mathcal{F} \subseteq \mathbb{Z}^k_{\geq 0}$ and $\mathcal{M} \subseteq \mathbb{Z}^k$, $\mathrm{G}(\mathcal{F},\mathcal{M})$ is the graph on $\mathcal{F}$ in which two nodes $\mathbf{v},\mathbf{u} \in \mathcal{F}$ are adjacent if either $\mathbf{v}-\mathbf{u} \in \mathcal{M}$ or $\mathbf{u}-\mathbf{v} \in \mathcal{M}$. Given an integer vector $\mathbf{w} \in \mathbb{Z}^k_{\geq 0}$, the *box* of $\mathbf{w}$ is

$$\mathfrak{B}_{\mathbf{w}} := [\mathbf{w}_1] \times [\mathbf{w}_2] \times \cdots \times [\mathbf{w}_k] \subseteq \mathbb{Z}^k_{\geq 0}$$

and the standard basis of $\mathbb{Z}^k$ is $\mathrm{E}_k := \{\mathbf{e}_i : i \in [k]\} \subseteq \mathbb{Z}^k_{\geq 0}$. The next lemma states that the node-connectivity is best possible in the graph $\mathrm{G}(\mathfrak{B}_{\mathbf{w}}, \mathrm{E}_k)$. We omit the proof since it can easily archived as a consequence of the node-version of Lemma 2.1. Recall that for $\mathbf{w} \in \mathbb{Z}^k$ the support $\mathrm{supp}(\mathbf{w}) \subseteq [k]$ is the set of indices of all non-zero coordinates of $\mathbf{w}$.

Lemma 2.3. *For $\mathbf{w} \in \mathbb{Z}^k_{\geq 0}$, the minimal degree and node-connectivity of the graph $\mathrm{G}(\mathfrak{B}_{\mathbf{w}}, \mathrm{E}_k)$ equals $|\mathrm{supp}(\mathbf{w})|$.*

In order to exploit a more refined structure of fiber graphs of A_k (see Section 4), we first have a look at sets of the following type: for a given set $\mathcal{F} \subseteq \mathbb{Z}^k$ and a vector $\mathbf{b} \in \mathbb{Z}^k$ the $\mathbf{b}$-slack of $\mathcal{F}$ is

$$\mathrm{SL}\left(\mathcal{F},\mathbf{b}\right) := \left\{ \begin{pmatrix} \mathbf{x} \\ \mathbf{b}-\mathbf{x} \end{pmatrix} : \mathbf{x} \in \mathcal{F} \right\} \subseteq \mathbb{Z}^{2k} \tag{2.3}$$

and the $\mathbf{0}_k$-slack of $\mathcal{F}$ is abbreviated as $\mathrm{SL}\left(\mathcal{F}\right) := \mathrm{SL}\left(\mathcal{F}, \mathbf{0}_k\right)$. We need in Section 3 and 4 the special case that $\mathcal{F} = \mathfrak{B}_{\mathbf{w}}$ and we denote its slack short by $\mathfrak{B}^{sl}_{\mathbf{w}} := \mathrm{SL}\left(\mathfrak{B}_{\mathbf{w}}, \mathbf{w}\right)$. In the next lemma we show that the connectivity of a graph does not change by adding slacks to the set of nodes if we slack the set of moves by $\mathbf{0}_k$, too.

Lemma 2.4. *For $\mathbf{b} \in \mathbb{Z}^k_{\geq 0}$, $\mathcal{F} \subseteq \mathfrak{B}_{\mathbf{b}}$, and a set of moves $\mathcal{M} \subseteq \mathbb{Z}^k$ we have*

$$\mathrm{G}(\mathcal{F},\mathcal{M}) \cong \mathrm{G}(\mathrm{SL}\left(\mathcal{F},\mathbf{b}\right), \mathrm{SL}\left(\mathcal{M}\right)). \tag{2.4}$$

Proof. Since $\mathcal{F} \subseteq \mathfrak{B}_{\mathbf{b}}$, we have $\mathrm{SL}\left(\mathcal{F},\mathbf{b}\right) \subseteq \mathbb{Z}^{2k}_{\geq 0}$ and hence the graph on the right-hand side of (2.4) is well-defined in the sense our definition given above. The map

$$\mathcal{F} \to \mathrm{SL}\left(\mathcal{F},\mathbf{b}\right), \mathbf{v} \mapsto \begin{pmatrix} \mathbf{v} \\ \mathbf{b}-\mathbf{v} \end{pmatrix}$$

gives a bijection between the nodes of the two graphs in (2.4) which does not only respect the set of edges, but even more induces a bijection between them, too. $\square$

3 Graver and Gröbner bases of A_k

In this section, we construct both the Graver basis and the reduced Gröbner basis with respect to a lexicographic term ordering of A_k as defined in (1.2). For this, it is necessary to recall the definition of the Graver basis of a matrix first. Let $\sqsubseteq$ be the partial ordering on $\mathbb{Z}^n$ such that for two integer vectors $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^n$ we have $\mathbf{u} \sqsubseteq \mathbf{v}$ if $\mathbf{u}_i \cdot \mathbf{v}_i \geq 0$ and $|\mathbf{u}_i| \leq |\mathbf{v}_i|$ for all $i \in [n]$. The *Graver basis* $\mathcal{G}(A)$ of a matrix $A \in \mathbb{Z}^{d\times n}$ is the set of all $\sqsubseteq$-minimal elements in $\ker(A) \cap \mathbb{Z}^n \setminus \{\mathbf{0}_n\}$. Note that $\mathcal{G}(A)$ is always a finite set [10, Chapter 3]. When it comes to calculations of Graver bases, the following definition is very helpful: for a non-negative vector $\mathbf{v} \in \mathbb{Z}^k_{\geq 0}$, let $\chi(\mathbf{v}) \in \{0,1\}^k$ be such that we have for all $i \in [k]$

$$\chi(\mathbf{v})_i = \begin{cases} 0, & \text{if } \mathbf{v}_i = 0 \\ 1, & \text{if } \mathbf{v}_i \neq 0 \end{cases}.$$

Theorem 2. *For $k > 0$, the Graver basis of A_k is the (disjoint) union of*

$$\pm\, \mathfrak{B}^{sl}_{-\mathbf{1}_k} \times \mathfrak{B}^{sl}_{\mathbf{1}_k} \times \{-1\} \times \{1\} \tag{3.1}$$

and the sets

$$\begin{aligned} &\pm\, \mathrm{SL}\,(\mathrm{E}_k) \times \{\mathbf{0}_{2k}\} \times \{0\} \times \{0\} \text{ and} \\ &\pm\, \{\mathbf{0}_{2k}\} \times \mathrm{SL}\,(\mathrm{E}_k) \times \{0\} \times \{0\}. \end{aligned} \tag{3.2}$$

Proof. Denote the union of the sets given in (3.1) and (3.2) by G. We show that for every $\mathbf{u} \in \mathbb{Z}^{4k+2}$ with $\mathbf{u} \neq \mathbf{0}_{4k+2}$ and $A_k\mathbf{u} = \mathbf{0}_{2k+1}$ there exists $\mathbf{g} \in G$ such that $\mathbf{g} \sqsubseteq \mathbf{u}$. We write $\mathbf{u} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2, s, t)^\intercal$ for vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{Z}^k$ and integers $s, t \in \mathbb{Z}$. The block structure of A_k yields the following equations:

$$\begin{aligned} \mathbf{x}_1 + \mathbf{x}_2 &= s \cdot \mathbf{1}_k \\ \mathbf{y}_1 + \mathbf{y}_2 &= t \cdot \mathbf{1}_k \\ s + t &= 0. \end{aligned} \tag{3.3}$$

We distinguish the following two cases.

Case 1: $s = -t = 0$. Clearly, we have $\mathbf{x}_1 = -\mathbf{x}_2$ and $\mathbf{y}_1 = -\mathbf{y}_2$. As $\mathbf{u} \neq \mathbf{0}_{4k+2}$ we can assume without loss of generality that $\mathbf{x}_1 \neq \mathbf{0}_k$. Thus, there is $i \in [k]$ and $\lambda \in \{-1, 1\}$ such that

$$\lambda \cdot \begin{pmatrix} \mathbf{e}_i \\ -\mathbf{e}_i \end{pmatrix} \sqsubseteq \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}$$

which gives rise to an element in $\mathrm{SL}\,(\mathrm{E}_k) \times \{\mathbf{0}_{2k}\} \times \{0\} \times \{0\}$ which is less than $\mathbf{u}$ with respect to $\sqsubseteq$.

Case 2: $s = -t \neq 0$. Without restricting generality (since G is symmetric we can multiply $\mathbf{u}$ by -1 if necessary) we can assume that $t > 0$ and as t is an integer we have $t \geq 1$ and thus $s = -t \leq -1$. Clearly, we have $-\mathbf{x}_1^- \sqsubseteq \mathbf{x}_1$ and $\mathbf{x}_1^+ \sqsubseteq \mathbf{x}_1$ and hence we have $-\chi(\mathbf{x}_1^-) \sqsubseteq \mathbf{x}_1$. As $s \leq -1$, equation (3.3) gives $-\mathbf{1}_k + \chi(\mathbf{x}_1^-) \sqsubseteq \mathbf{x}_2$ which implies

$$\begin{pmatrix} -\chi(\mathbf{x}_1^-) \\ -\mathbf{1}_k + \chi(\mathbf{x}_1^-) \end{pmatrix} \sqsubseteq \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix}.$$

Similarly, one can show that

$$\begin{pmatrix} \chi(\mathbf{y}_1^+) \\ \mathbf{1}_k - \chi(\mathbf{y}_1^+) \end{pmatrix} \sqsubseteq \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}.$$

Since $-\chi(\mathbf{x}_i^-) \in \mathfrak{B}_{-\mathbf{1}_k}$ and $\chi(\mathbf{y}_1^+) \in \mathfrak{B}_{\mathbf{1}_k}$ and due to $s \leq -1$ and $t \geq 1$ we found an element in $\mathfrak{B}^{sl}_{-\mathbf{1}_k} \times \mathfrak{B}^{sl}_{\mathbf{1}_k} \times \{-1\} \times \{1\} \subseteq G$ which is less than $\mathbf{u}$ with respect to the partial ordering $\sqsubseteq$. $\square$

In the following, we consider a Gröbner basis with respect to the *lexicographic ordering* $\prec_{\text{LEX}}$ on $\mathbb{Z}^n_{\geq 0}$ where for two integer vectors $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^n_{\geq 0}$ with $\mathbf{u} \neq \mathbf{v}$ we have $\mathbf{u} \prec_{\text{LEX}} \mathbf{v}$ if $\mathbf{u}_i < \mathbf{v}_i$ for the smallest $i \in [n]$ such that $\mathbf{u}_i \neq \mathbf{v}_i$. The next theorem extracts the reduced Gröbner basis of A_k with respect to $\prec_{\text{LEX}}$ from its Graver basis.

Theorem 3. *For $k > 0$, the reduced Gröbner basis of A_k with respect to $\prec_{\text{LEX}}$ consists of the vector*

$$(\mathbf{0}_k, \mathbf{1}_k, \mathbf{0}_k, -\mathbf{1}_k, 1, -1)^\intercal$$

and the vectors of the sets

$$\begin{aligned} &\text{SL}\,(\text{E}_k) \times \{\mathbf{0}_{2k}\} \times \{0\} \times \{0\} \text{ and} \\ &\{\mathbf{0}_{2k}\} \times \text{SL}\,(\text{E}_k) \times \{0\} \times \{0\}. \end{aligned} \tag{3.4}$$

Proof. As any reduced Gröbner basis of A_k is contained in the Graver basis of A_k [12, Proposition 4.11], the result follows immediately by extracting those elements from the Graver basis $\mathcal{G}(A_k)$, given in Theorem 2, that cannot be reduced by other elements of $\mathcal{G}(A_k)$ with respect to $\prec_{\text{LEX}}$. $\square$

4 The Fiber-Structure of A_k

Equipped with explicit descriptions of both the Graver basis and the reduced $\prec_{\text{LEX}}$-Gröbner basis of A_k, we discover in this section the structure of $\mathcal{F}_{A_k,\mathbf{b}}$ for any given right-hand side vector $\mathbf{b} \in \mathbb{Z}^{2k+1}$. We write $\mathbf{b} = (\mathbf{w}_1, \mathbf{w}_2, c)^\intercal \in \mathbb{Z}^{2k+1}$ with vectors $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{Z}^k$ and $c \in \mathbb{Z}$. We assume that $\mathcal{F}_{A_k,\mathbf{b}} \neq \emptyset$ and hence we can choose an arbitrary element $\mathbf{u} \in \mathcal{F}_{A_k,\mathbf{b}}$ and write $\mathbf{u} = (\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2, s, t)^\intercal \in \mathbb{Z}^{4k+2}_{\geq 0}$ with vectors $\mathbf{x}_1, \mathbf{x}_2, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{Z}^k_{\geq 0}$ and $s, t \in \mathbb{Z}_{\geq 0}$. Since we have $A_k\mathbf{u} = \mathbf{b}$, we obtain the following relations:

$$\begin{aligned} \mathbf{x}_1 + \mathbf{x}_2 &= \mathbf{w}_1 + s \cdot \mathbf{1}_k \\ \mathbf{y}_1 + \mathbf{y}_2 &= \mathbf{w}_2 + t \cdot \mathbf{1}_k \\ s + t &= c. \end{aligned} \tag{4.1}$$

We see immediately that we must have $\mathbf{w}_1 + s \cdot \mathbf{1}_k \geq \mathbf{0}_k$, $\mathbf{w}_2 + t \cdot \mathbf{1}_k \geq \mathbf{0}_k$ and $c \geq 0$, since otherwise $\mathcal{F}_{A_k,\mathbf{b}} = \emptyset$. As t is uniquely determined by $t = c - s$, those inequalities give

$$\underbrace{\max\{(\mathbf{w}_1^-)_i : i \in [k]\}}_{=\|\mathbf{w}_1^-\|_\infty} \leq s \leq c - \underbrace{\max\{(\mathbf{w}_2^-)_i : i \in [k]\}}_{=\|\mathbf{w}_2^-\|_\infty}. \tag{4.2}$$

So we can define both a lower and an upper bound on s by

$$l(\mathbf{b}) := \|\mathbf{w}_1^-\|_\infty \text{ and } u(\mathbf{b}) := c - \|\mathbf{w}_2^-\|_\infty.$$

If $l(\mathbf{b}) > u(\mathbf{b})$, we certainly have $\mathcal{F}_{A_k,\mathbf{b}} = \emptyset$ and hence we can assume that $l(\mathbf{b}) \leq u(\mathbf{b})$. The equations in (4.1) suggest that we can regard $\mathbf{x}_2$ and $\mathbf{y}_2$ as slack variables since they are already uniquely determined by the choices of $\mathbf{x}_1 \in \mathfrak{B}_{\mathbf{w}_1+s\cdot\mathbf{1}_k}$ and $\mathbf{y}_1 \in \mathfrak{B}_{\mathbf{w}_2+(c-s)\cdot\mathbf{1}_k}$. Hence, any element of the fiber looks like

$$\mathbf{v}_\mathbf{b}(\mathbf{x},\mathbf{y},s) := \begin{pmatrix} \mathbf{x} \\ \mathbf{w}_1 + s\cdot\mathbf{1}_k - \mathbf{x} \\ \mathbf{y} \\ \mathbf{w}_2 + (c-s)\cdot\mathbf{1}_k - \mathbf{y} \\ s \\ c-s \end{pmatrix} \tag{4.3}$$

for $\mathbf{x} \in \mathfrak{B}_{\mathbf{w}_1+s\cdot\mathbf{1}_k}$ and $\mathbf{y} \in \mathfrak{B}_{\mathbf{w}_2+(c-s)\cdot\mathbf{1}_k}$. Using our definition of slacked boxes as defined in (2.3), we obtain an explicit description of elements in $\mathcal{F}_{A_k,\mathbf{b}}$ which have their $(4k+1)$th coordinate equal to s:

$$\mathcal{B}_\mathbf{b}(s) := \mathfrak{B}^{sl}_{\mathbf{w}_1+s\cdot\mathbf{1}_k} \times \mathfrak{B}^{sl}_{\mathbf{w}_2+(c-s)\cdot\mathbf{1}_k} \times \{s\} \times \{c-s\} \subseteq \mathbb{Z}^{2k+2k+2}_{\geq 0}. \tag{4.4}$$

This gives us a very convenient partition of the fiber into $u(\mathbf{b}) - l(\mathbf{b}) + 1$ disjoint sets:

$$\mathcal{F}_{A_k,\mathbf{b}} = \bigcup_{s=l(\mathbf{b})}^{u(\mathbf{b})} \mathcal{B}_\mathbf{b}(s). \tag{4.5}$$

We see that the Graver moves from the sets defined in (3.1) connect nodes from two adjacent boxes $\mathcal{B}_\mathbf{b}(s_1)$ and $\mathcal{B}_\mathbf{b}(s_2)$ with $|s_1 - s_2| = 1$, whereas Graver moves from (3.2) connect nodes within the same box $\mathcal{B}_\mathbf{b}(s)$ (see Figure 2).

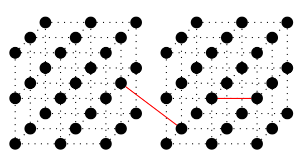


Fig. 2. Different types of Graver moves of A_k.

Even more, since the $(4k+1)$th and $(4k+2)$th coordinates coincide for all elements in $\mathcal{B}_\mathbf{b}(s)$, the next lemma follows immediately.

Lemma 4.1. *For $\mathbf{b} \in \mathbb{Z}^{2k+1}$ and $s \in [l(\mathbf{b}), u(\mathbf{b})]$, the following equality holds:*

$$\mathrm{G}(\mathcal{B}_\mathbf{b}(s), \mathcal{G}(A_k)) = \mathrm{G}(\mathcal{B}_\mathbf{b}(s), \mathcal{R}_{\prec_{\text{LEX}}}(A_k)).$$

Based on our observations in Section 2, we know that the node-connectivity in those induced subgraphs is best possible as the next lemma shows.

Lemma 4.2. *Let* $\mathbf{b} \in \mathbb{Z}^{2k+1}$ *such that* $\mathcal{F}_{A_k,\mathbf{b}} \neq \emptyset$. *For all* $s \in [l(\mathbf{b}), u(\mathbf{b})]$, *the minimal degree and the node-connectivity of the graph* $\mathrm{G}(\mathcal{B}_s(\mathbf{b}), \mathcal{G}(A_k))$ *equal*

$$|\operatorname{supp}(\mathbf{w}_1 + s \cdot \mathbf{1}_k)| + |\operatorname{supp}(\mathbf{w}_2 + (c-s) \cdot \mathbf{1}_k)|.$$

Proof. Using the representation of $\mathcal{B}_{\mathbf{b}}(s)$ in (4.4) and a projection onto the first $4k$ coordinates, we obtain that the induced subgraph of $\mathrm{G}(\mathcal{F}_{A_k,\mathbf{b}}, \mathcal{G}(A_k))$ on the nodes $\mathcal{B}_{\mathbf{b}}(s)$ is isomorphic to the graph

$$\mathrm{G}(\mathfrak{B}^{sl}_{\mathbf{w}_1+s\cdot\mathbf{1}_k} \times \mathfrak{B}^{sl}_{\mathbf{w}_2+(c-s)\cdot\mathbf{1}_k}, \mathrm{SL}(\mathrm{E}_k) \times \{\mathbf{0}_{2k}\} \cup \{\mathbf{0}_{2k}\} \times \mathrm{SL}(\mathrm{E}_k)). \tag{4.6}$$

Graphs of this particular structure can be interpreted as the *Cartesian product* of two related graphs, in our case here, $\mathrm{G}(\mathfrak{B}^{sl}_{\mathbf{w}_1+s\cdot\mathbf{1}_k}, \mathrm{SL}(\mathrm{E}_k))$ and $\mathrm{G}(\mathfrak{B}^{sl}_{\mathbf{w}_2+(c-s)\cdot\mathbf{1}_k}, \mathrm{SL}(\mathrm{E}_k))$ (we refer to [3] for a definition). This gives that the minimal degree of this graph is the sum of the minimal degrees of $\mathrm{G}(\mathfrak{B}^{sl}_{\mathbf{w}_1+s\cdot\mathbf{1}_k}, \mathrm{SL}(\mathrm{E}_k))$ and of $\mathrm{G}(\mathfrak{B}^{sl}_{\mathbf{w}_2+(c-s)\cdot\mathbf{1}_k}, \mathrm{SL}(\mathrm{E}_k))$. Using the isomorphism given in Lemma 2.4, their minimal degrees coincide with the minimal degrees of the graphs $\mathrm{G}(\mathfrak{B}_{\mathbf{w}_1+s\cdot\mathbf{1}_k}, \mathrm{E}_k)$ and $\mathrm{G}(\mathfrak{B}_{\mathbf{w}_2+(c-s)\cdot\mathbf{1}_k}, \mathrm{E}_k)$, respectively. Applying the formula of Lemma 2.3, the minimal degree of the graph given in (4.6) equals

$$|\operatorname{supp}(\mathbf{w}_1 + s \cdot \mathbf{1}_k)| + |\operatorname{supp}(\mathbf{w}_2 + (c-s) \cdot \mathbf{1}_k)|.$$

as claimed. □

Whereas Lemma 4.1 states that the Gröbner and Graver fiber graphs coincide on the subgraph induced by $\mathcal{B}_{\mathbf{b}}(s)$, Lemma 4.2 says that the node-connectivity in those subgraphs is best possible. But what about moves between two neighbouring boxes of $\mathcal{F}_{A_k,\mathbf{b}}$? Let us now determine under which conditions nodes of neighboring boxes are adjacent to each other. For that it is necessary that $\mathcal{F}_{A_k,\mathbf{b}}$ has at least two boxes, which is precisely the case if $l(\mathbf{b}) < u(\mathbf{b})$. To simplify our proofs it is reasonable to define for all choices $\mathbf{v}_1, \mathbf{v}_2 \in \mathfrak{B}_{\mathbf{1}_k}$ the following move from the Graver basis of A_k:

$$\mathfrak{g}^k(\mathbf{v}_1, \mathbf{v}_2) := \begin{pmatrix} -\mathbf{v}_1 \\ -\mathbf{1}_k + \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{1}_k - \mathbf{v}_2 \\ -1 \\ 1 \end{pmatrix} \in \mathfrak{B}^{sl}_{-\mathbf{1}_k} \times \mathfrak{B}^{sl}_{\mathbf{1}_k} \times \{-1\} \times \{1\}.$$

Choose $s \in [l(\mathbf{b}), u(\mathbf{b})]$ and let $(\mathbf{x}, \mathbf{y})^\intercal \in \mathfrak{B}_{\mathbf{w}_1+\mathbf{1}_k\cdot s} \times \mathfrak{B}_{\mathbf{w}_2+(c-s)\cdot\mathbf{1}_k}$ and $\mathbf{v}_1, \mathbf{v}_2 \in \mathfrak{B}_{\mathbf{1}_k}$. The following conditions on $\mathbf{v}_1$ and $\mathbf{v}_2$

$$\operatorname{supp}(\mathbf{v}_1) \subseteq \operatorname{supp}(\mathbf{x}) \text{ and } [k] \setminus \operatorname{supp}(\mathbf{v}_1) \subseteq \operatorname{supp}(\mathbf{w}_1 + s \cdot \mathbf{1}_k - \mathbf{x}) \tag{4.7a}$$

$$\operatorname{supp}(\mathbf{v}_2) \subseteq \operatorname{supp}(\mathbf{y}) \text{ and } [k] \setminus \operatorname{supp}(\mathbf{v}_2) \subseteq \operatorname{supp}(\mathbf{w}_2 + (c-s) \cdot \mathbf{1}_k - \mathbf{y}) \tag{4.7b}$$

lead to a technical characterization for a Graver move to be applicable at $\mathbf{v_b}(\mathbf{x},\mathbf{y},s)$:

$$\mathbf{v_b}(\mathbf{x},\mathbf{y},s) \xleftrightarrow{\mathfrak{g}^k(\mathbf{v}_1,\mathbf{v}_2)} \mathbf{v_b}(\mathbf{x}-\mathbf{v}_1,\mathbf{y}+\mathbf{v}_2,s-1) \iff \text{(4.7a) and } s > l(\mathbf{b})$$
$$\mathbf{v_b}(\mathbf{x},\mathbf{y},s) \xleftrightarrow{-\mathfrak{g}^k(\mathbf{v}_1,\mathbf{v}_2)} \mathbf{v_b}(\mathbf{x}+\mathbf{v}_1,\mathbf{y}-\mathbf{v}_2,s+1) \iff \text{(4.7b) and } s < u(\mathbf{b}).$$

In particular, we see that only a fraction of moves between two adjacent boxes of $\mathcal{F}_{A_k,\mathbf{b}}$ are actually moves from the lexicographic Gröbner basis of A_k. So the main difference of the fiber graphs of A_k with respect to Graver and Gröbner moves results from how the boxes $\mathcal{B}_\mathbf{b}(s)$ are connected among each other. From our observations in this section, we obtain that there is a large number of Graver moves between two neighboring boxes and we summarize this results in the following proposition.

Proposition 4.1. *Let $\mathbf{b} = (\mathbf{w}_1,\mathbf{w}_2,c)^\intercal \in \mathbb{Z}^{2k+1}$ with $\mathbf{w}_1,\mathbf{w}_2 \in \mathbb{Z}^k$ and $c \in \mathbb{Z}$ such that $\mathcal{F}_{A_k,\mathbf{b}} \neq \emptyset$ and consider the fiber graph $\mathrm{G}(\mathcal{F}_{A_k,\mathbf{b}},\mathcal{G}(A_k))$. For $s \in [l(\mathbf{b}),u(\mathbf{b})]$, a node $\mathbf{v} \in \mathcal{B}_\mathbf{b}(s)$ has neighbors in $\mathcal{B}_\mathbf{b}(s-1)$ if and only if $s > l(\mathbf{b})$ and in this case that are at least 2^k many. In the same way, $\mathbf{v}$ has neighbors in $\mathcal{B}_\mathbf{b}(s+1)$ if and only if $s < u(\mathbf{b})$ and that are at least 2^k many in this case.*

Proof. The statement of the proposition follows immediately from the fact that moves of the form $\mathfrak{g}^k(\chi(\mathbf{x}),\mathbf{v}_2)$ are applicable at $\mathbf{v_b}(\mathbf{x},\mathbf{y},s)$ for all $\mathbf{v}_2 \in \mathfrak{B}_{\mathbf{1}_k}$ if $s > l(\mathbf{b})$ and in the same way we see that moves of the form $-\mathfrak{g}^k(\mathbf{v}_1,\chi(\mathbf{y}))$ are applicable at $\mathbf{v_b}(\mathbf{x},\mathbf{y},s)$ if $s < u(\mathbf{b})$ for all $\mathbf{v}_1 \in \mathfrak{B}_{\mathbf{1}_k}$. □

5 Gröbner Fiber Graphs of A_k

As mentioned in the previous section, the number of edges between two boxes of a fiber is significantly higher under the Graver basis than the reduced lexicographic Gröbner basis and our hope is that this affects the connectivity of the fiber graphs. Indeed, considering the fiber of $\mathbf{e}_{2k+1} \in \mathbb{Z}^{2k+1}$, we have that $l(\mathbf{e}_{2k+1}) = 0$ and $u(\mathbf{e}_{2k+1}) = 1$. Thus, (4.5) gives

$$\begin{aligned}\mathcal{F}_{A_k,\mathbf{e}_{2k+1}} &= \mathcal{B}_{\mathbf{e}_{2k+1}}(0) \cup \mathcal{B}_{\mathbf{e}_{2k+1}}(1)\\ &= \mathfrak{B}^{sl}_{\mathbf{0}_k} \times \mathfrak{B}^{sl}_{\mathbf{1}_k} \times \{0\} \times \{1\} \cup \mathfrak{B}^{sl}_{\mathbf{1}_k} \times \mathfrak{B}^{sl}_{\mathbf{0}_k} \times \{1\} \times \{0\}\\ &= \{\mathbf{0}_k\} \times \mathfrak{B}^{sl}_{\mathbf{1}_k} \times \{0\} \times \{1\} \cup \mathfrak{B}^{sl}_{\mathbf{1}_k} \times \{\mathbf{0}_k\} \times \{1\} \times \{0\}.\end{aligned}$$

This combined with Lemma 4.2 implies that the minimal degree of $\mathrm{G}\left(\mathcal{F}_{A_k,\mathbf{e}_{2k+1}},\mathcal{R}_{\prec_{\mathrm{LEX}}}(A_k)\right)$ is at least k. Due to the connection to slacked boxes, Lemma 4.2 explains the structure of the fiber within a box very well. But what about edges between two boxes with respect to Gröbner moves? According to Theorem 3, the only move available is

$$\mathfrak{g}^k(\mathbf{0}_k,\mathbf{0}_k) = (\mathbf{0}_k,-\mathbf{1}_k,\mathbf{0}_k,\mathbf{1}_k,\mathbf{0}_k,-1,1)^\intercal$$

and according to Section 4, this move can be applied only once in the fiber $\mathcal{F}_{A_K,\mathbf{e}_{2k+1}}$, namely as move between the following nodes:

$$(\mathbf{0}_k, \mathbf{0}_k, \mathbf{0}_k, \mathbf{1}_k, 0, 1)^\intercal \xleftrightarrow{\mathfrak{g}^k(\mathbf{0}_k,\mathbf{0}_k)} (\mathbf{0}_k, \mathbf{1}_k, \mathbf{0}_k, \mathbf{0}_k, 1, 0)^\intercal.$$

This means there is only a single edge connecting $\mathcal{B}_{\mathbf{b}}(0)$ and $\mathcal{B}_{\mathbf{b}}(1)$ (see Figure 3) and hence the minimal degree of $\mathrm{G}\left(\mathcal{F}_{A_k,\mathbf{e}_{2k+1}}, \mathcal{R}_{\prec_{\text{LEX}}}(A_k)\right)$ equals k. Thus, removing this edge gives a non-

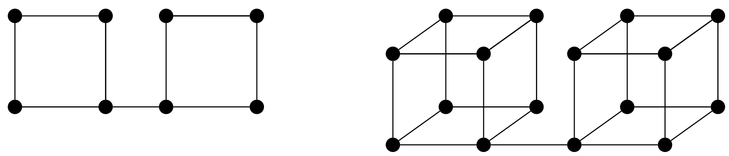


Fig. 3. A sketch of $\mathcal{B}_{\mathbf{e}_{2k+1}}(0)$ and $\mathcal{B}_{\mathbf{e}_{2k+1}}(1)$ for $k = 2$ and $k = 3$ with respect to Gröbner moves.

connected graph, i.e., the edge-connectivity of the fiber graph equals 1. Since in all graphs the node-connectivity is always less than the edge-connectivity, we obtain the following corollary.

Corollary 5.1. *For $k > 0$, the edge-connectivity of the fiber graph*

$$\mathrm{G}\left(\mathcal{F}_{A_k,\mathbf{e}_{2k+1}}, \mathcal{R}_{\prec_{\text{LEX}}}(A_k)\right)$$

equals 1, whereas its minimal degree equals k. In particular, A_k gives a counterexample to Conjecture 1 for $k \geq 2$.

However, a priori A_k does not provide evidence against Conjecture 2 since the conjecture claims that the node-connectivity equals the minimal degree only for sufficiently large right-hand sides. But Theorem 1 gives us an instruction how to modify A_k such that it becomes a counterexample to Conjecture 2 as well:

$$B_k := \begin{pmatrix} A_{k+1} & I_{2k+1} \\ \mathbf{0} & I_{2k+1} \end{pmatrix} \in \mathbb{Z}^{(6k+3)\times(4k+2)}.$$

Corollary 5.2. *For $k > 0$, there exists a term ordering $\prec_k$ on $\mathbb{Z}^{6k+3}_{\geq 0}$ such that for all $N \in \mathbb{Z}_{\geq 0}$ there exists $\mathbf{b} \in \mathbb{Z}^{4k+2}$ with $\mathbf{b} \geq N \cdot \mathbf{1}_{4k+2}$ such that the edge-connectivity of $\mathrm{G}\left(\mathcal{F}_{B_k,\mathbf{b}}, \mathcal{R}_{\prec_k}(B_k)\right)$ equals 1 whereas its minimal degree equals k. In particular, B_k gives a counterexample to Conjecture 2 for $k \geq 2$.*

6 Graver Fiber Graphs of A_k

As shown in the last section, node-connectivity and even edge-connectivity fail to be best possible in general in Gröbner fiber graphs. As the number of moves in the Graver basis enlarge the number of moves in a Gröbner basis by far, we hope that this circumstance reflects positively onto the connectivity of those fiber graphs. So let us now investigate how the situation looks like if we replace Gröbner moves with Graver moves. We prove that even if the edge-connectivity of some Gröbner fiber graphs of A_k is rather bad, the edge-connectivity of its Graver fiber graphs is best possible. With Proposition 4.1 in mind, let us first determine the minimal degree of the Graver fiber graphs.

Proposition 6.1 (Minimal degree). *Let* $\mathbf{b} = (\mathbf{w}_1, \mathbf{w}_2, c)^\intercal \in \mathbb{Z}^{2k+1}$ *with* $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{Z}^k$ *and* $c \in \mathbb{Z}$. *If* $l(\mathbf{b}) = u(\mathbf{b})$, *then we have*

$$\delta(\mathrm{G}\left(\mathcal{F}_{A_k,\mathbf{b}}, \mathcal{G}(A_k)\right)) = |\operatorname{supp}(\mathbf{w}_1 + \|\mathbf{w}_1^-\|_\infty \cdot \mathbf{1}_k)| + |\operatorname{supp}(\mathbf{w}_2 + \|\mathbf{w}_2^-\|_\infty \cdot \mathbf{1}_k)|. \tag{6.1}$$

Otherwise, if $l(\mathbf{b}) < u(\mathbf{b})$, *then we have*

$$\delta(\mathrm{G}\left(\mathcal{F}_{A_k,\mathbf{b}}, \mathcal{G}(A_k)\right)) = \min_{j\in\{1,2\}}\{|\operatorname{supp}(\mathbf{w}_j + \|\mathbf{w}_j^-\|_\infty \cdot \mathbf{1}_k)|\} + k + 2^k. \tag{6.2}$$

Proof. If $l(\mathbf{b}) = s = u(\mathbf{b})$, the first statement is a reformulation of Lemma 4.2 due to $\mathcal{F}_{A_k,\mathbf{b}} = \mathcal{B}_{\mathbf{b}}(s)$. So assume that we have $l(\mathbf{b}) < u(\mathbf{b})$. Since $s \in [l(\mathbf{b}), u(\mathbf{b})]$, we must have either $s > l(\mathbf{b})$ or $s < u(\mathbf{b})$ and hence we have either $s > \|\mathbf{w}_1^-\|_\infty$ of $c - s > \|\mathbf{w}_2^-\|_\infty$. Putting those inequalities into the equation for the minimal degree in Lemma 4.2, we obtain that a node in $\mathcal{B}_{\mathbf{b}}(s)$ has at least

$$\min_{j\in\{1,2\}}\{|\operatorname{supp}(\mathbf{w}_j + \|\mathbf{w}_j^-\|_\infty \cdot \mathbf{1}_k)|\} + k$$

neighbors in his own box $\mathcal{B}_{\mathbf{b}}(s)$. Furthermore, due to Proposition 4.1 and since either $s > l(\mathbf{b})$ or $s < u(\mathbf{b})$, a node in $\mathcal{B}_{\mathbf{b}}(s)$ has either at least 2^k neighbors in $\mathcal{B}_{\mathbf{b}}(s-1)$ or at least 2^k neighbors in $\mathcal{B}_{\mathbf{b}}(s+1)$. This shows that the minimal degree of $\mathrm{G}\left(\mathcal{F}_{A_k,\mathbf{b}}, \mathcal{G}(A_k)\right)$ is greater or equal than the right-hand side of the term given in (6.2). Clearly, the node with minimal degree has to be either in $\mathcal{B}_{\mathbf{b}}(l(\mathbf{b}))$ or in $\mathcal{B}_{\mathbf{b}}(u(\mathbf{b}))$. Thus, either

$$\begin{pmatrix} \mathbf{w}_1 + l(\mathbf{b}) \cdot \mathbf{1}_k \\ \mathbf{0}_k \\ \mathbf{w}_2 + (c - l(\mathbf{b})) \cdot \mathbf{1}_k \\ \mathbf{0}_k \\ l(\mathbf{b}) \\ c - l(\mathbf{b}) \end{pmatrix} \in \mathcal{B}_{\mathbf{b}}(l(\mathbf{b})) \text{ or } \begin{pmatrix} \mathbf{w}_1 + u(\mathbf{b}) \cdot \mathbf{1}_k \\ \mathbf{0}_k \\ \mathbf{w}_2 + (c - u(\mathbf{b})) \cdot \mathbf{1}_k \\ \mathbf{0}_k \\ u(\mathbf{b}) \\ c - u(\mathbf{b}) \end{pmatrix} \in \mathcal{B}_{\mathbf{b}}(u(\mathbf{b}))$$

has the smallest degree in $\mathrm{G}\left(\mathcal{F}_{A_k,\mathbf{b}}, \mathcal{G}(A_k)\right)$. $\square$

With an explicit formula for the minimal degree of $\mathrm{G}\left(\mathcal{F}_{A_k,\mathbf{b}}, \mathcal{G}(A_k)\right)$ in mind we can determine the edge-connectivity of those fiber graphs explicitly. First, we consider edges between two neighboring boxes and we show that we find a suitable number of disjoint paths connecting their end-points. Please note that we make these paths even node-disjoint in this case.

Lemma 6.1 (Edges within Box). *Let* $\mathbf{b} \in \mathbb{Z}^{2k+1}$ *and* $s \in [l(\mathbf{b}), u(\mathbf{b})]$. *Then for any two adjacent nodes in* $\mathcal{B}_{\mathbf{b}}(s)$ *there exist* $\delta(\mathrm{G}\left(\mathcal{F}_{A_k,\mathbf{b}}, \mathcal{G}(A_k)\right))$ *many node-disjoint paths in* $\mathrm{G}\left(\mathcal{F}_{A_k,\mathbf{b}}, \mathcal{G}(A_k)\right)$ *connecting them.*

Proof. We write $\mathbf{b} = (\mathbf{w}_1, \mathbf{w}_2, c)^\intercal \in \mathbb{Z}^{2k+1}$ with $\mathbf{w}_1, \mathbf{w}_2 \in \mathbb{Z}^k$ and $c \in \mathbb{Z}$. Since we have $\mathcal{F}_{A_k,\mathbf{b}} \neq \emptyset$ by assumption, we must have $l(\mathbf{b}) \leq u(\mathbf{b})$. Due to Lemma 4.2 and Proposition 6.1 there is nothing to show for $l(\mathbf{b}) = u(\mathbf{b})$ and hence we assume that $l(\mathbf{b}) < u(\mathbf{b})$. Without restricting generality, the two adjacent nodes we need to connect with a sufficient number of node-disjoint paths look like

$$\mathbf{v}_{\mathbf{b}}(\mathbf{x}, \mathbf{y}, s) \longleftrightarrow \mathbf{v}_{\mathbf{b}}(\mathbf{x} + \mathbf{e}_j, \mathbf{y}, s) \tag{6.3}$$

with $j \in [n]$, $s \in [l(\mathbf{b}), u(\mathbf{b})]$, $\mathbf{x} \in \mathfrak{B}_{\mathbf{w}_1 + s \cdot \mathbf{1}_k}$, and $\mathbf{y} \in \mathfrak{B}_{\mathbf{w}_2 + (c-s) \cdot \mathbf{1}_k}$. By Lemma 4.2 we find $|\operatorname{supp}(\mathbf{w}_1 + s \cdot \mathbf{1}_k)| + |\operatorname{supp}(\mathbf{w}_2 + (c-s) \cdot \mathbf{1}_k)|$ node-disjoint paths connecting $\mathbf{v}_{\mathbf{b}}(\mathbf{x}, \mathbf{y}, s)$ and $\mathbf{v}_{\mathbf{b}}(\mathbf{x} + \mathbf{e}_j, \mathbf{y}, s)$ which only use nodes in $\mathcal{B}_{\mathbf{b}}(s)$. If we have $s > l(\mathbf{b})$, we define

$$\mathbf{v}_1 := \begin{cases} \chi(\mathbf{x}) - \mathbf{e}_j, & \text{if } \mathbf{x}_j > 0 \\ \chi(\mathbf{x}), & \text{if } \mathbf{x}_j = 0 \end{cases} \text{ and } \mathbf{v}_1' := \begin{cases} \chi(\mathbf{x}), & \text{if } \mathbf{x}_j > 0 \\ \chi(\mathbf{x}) + \mathbf{e}_j, & \text{if } \mathbf{x}_j = 0 \end{cases}.$$

Then we have $\mathbf{v}_1 \in \mathfrak{B}_{\mathbf{1}_k}$ and $\mathbf{v}_1' \in \mathfrak{B}_{\mathbf{1}_k}$. Since we have by (6.3) that $\mathbf{x} + \mathbf{e}_j \leq \mathbf{w}_1 + s \cdot \mathbf{1}_k$, it is easy to see that $\mathbf{v}_1$ fulfills (4.7a) and hence the Graver move $\mathfrak{g}^k(\mathbf{v}_1, \mathbf{v})$ is applicable at $\mathbf{v}_{\mathbf{b}}(\mathbf{x}, \mathbf{y}, s)$ for every $\mathbf{v} \in \mathfrak{B}_{\mathbf{1}_k}$. As $\mathbf{x} - \mathbf{v}_1 + \mathbf{v}_1' = \mathbf{x} + \mathbf{e}_j$ by construction, this gives for every $\mathbf{v} \in \mathfrak{B}_{\mathbf{1}_k}$ a path

$$\begin{aligned} &\mathbf{v}_{\mathbf{b}}(\mathbf{x}, \mathbf{y}, s) \\ \longleftrightarrow &\mathbf{v}_{\mathbf{b}}(\mathbf{x} - \mathbf{v}_1, \mathbf{y} + \mathbf{v}, s - 1) \in \mathcal{B}_{\mathbf{b}}(s-1) \\ \longleftrightarrow &\mathbf{v}_{\mathbf{b}}(\mathbf{x} - \mathbf{v}_1 + \mathbf{v}_1', \mathbf{y} + \mathbf{v}_2 - \mathbf{v}, s - 1 + 1) = \mathbf{v}_{\mathbf{b}}(\mathbf{x} + \mathbf{e}_j, \mathbf{y}, s) \end{aligned}$$

which only uses edges with end-points $\mathcal{B}_{\mathbf{b}}(s)$ and $\mathcal{B}_{\mathbf{b}}(s-1)$. On the other hand, if we have $s < u(\mathbf{b})$, we have for every $\mathbf{v} \in \mathfrak{B}_{\mathbf{1}_k}$ a path

$$\begin{aligned} &\mathbf{v}_{\mathbf{b}}(\mathbf{x}, \mathbf{y}, s) \\ \longleftrightarrow &\mathbf{v}_{\mathbf{b}}(\mathbf{x} + \mathbf{v}, \mathbf{y} - \chi(\mathbf{y}), s + 1) \in \mathcal{B}_{\mathbf{b}}(s+1) \\ \longleftrightarrow &\mathbf{v}_{\mathbf{b}}(\mathbf{x} + \mathbf{v} + \mathbf{e}_j, \mathbf{y} - \chi(\mathbf{y}), s + 1) \in \mathcal{B}_{\mathbf{b}}(s+1) \\ \longleftrightarrow &\mathbf{v}_{\mathbf{b}}(\mathbf{x} + \mathbf{v} + \mathbf{e}_j - \mathbf{v}, \mathbf{y} - \chi(\mathbf{y}) + \chi(\mathbf{y}), s + 1 - 1) = \mathbf{v}_{\mathbf{b}}(\mathbf{x} + \mathbf{e}_j, \mathbf{y}, s). \end{aligned}$$

Here, the second edge is feasible since $j \in \operatorname{supp}(\mathbf{w}_1 + s \cdot \mathbf{1}_k - \mathbf{x})$ by assumption (6.3) and hence we have for the slack variable of $\mathbf{x} + \mathbf{v}$ that

$$j \in \operatorname{supp}\left(\mathbf{w}_1 + (s+1) \cdot \mathbf{1}_k - (\mathbf{x} + \mathbf{v})\right).$$

All in all, we get in any case 2^k many edge-disjoint paths which only use edges outside of $\mathcal{B}_{\mathbf{b}}(s)$ and hence these paths are node-disjoint to those walking within $\mathcal{B}_{\mathbf{b}}(s)$. Thus, there are

$$|\operatorname{supp}(\mathbf{w}_1 + s \cdot \mathbf{1}_k)| + |\operatorname{supp}(\mathbf{w}_2 + (c-s) \cdot \mathbf{1}_k)| + 2^k \geq \delta(\mathrm{G}(\mathcal{F}_{A_k,\mathbf{b}}, \mathcal{G}(A_k)))$$

node-disjoint paths between the end-points of the edge given in (6.3). □

In the next lemma we prove that we can find a suitable number of paths even for end-points of edges in neighbouring boxes of $\mathcal{F}_{A_k,\mathbf{b}}$ as well. Here, Proposition 2.1 plays an important role and hence we shortly recall its statement: given two subgraphs with a certain connectivity yield a lower bound on the connectivity of the induced graph on the union of those subgraphs if we can prove the existence of a suitable number of paths walking between them. In the situation of Proposition 6.2, the subgraphs whose connectivity is already known are the induced subgraphs on the boxes $\mathcal{B}_{\mathbf{b}}(s)$. So the idea behind the proof of Lemma 6.2 is to find a sufficient number of edges between two neighbouring boxes.

Lemma 6.2 (Edges between adjacent Boxes). *Let $k > 0$ and $\mathbf{b} \in \mathbb{Z}^{2k+1}$. Then for any adjacent nodes in different boxes there are $\delta(\mathrm{G}(\mathcal{F}_{A_k,\mathbf{b}}, \mathcal{G}(A_k)))$ many edge-disjoint paths connecting them.*

Proof. By assumption, there exist at least two boxes in $\mathrm{G}(\mathcal{F}_{A_k,\mathbf{b}}, \mathcal{G}(A_k))$ and hence we must have $l(\mathbf{b}) < u(\mathbf{b})$. Without restricting generality, we can assume that the edge between the two adjacent nodes looks like:

$$\mathbf{u}_1 := \mathbf{v}_{\mathbf{b}}(\mathbf{x}, \mathbf{y}, s) \xleftrightarrow{\mathfrak{g}^k(\mathbf{v}_1, \mathbf{v}_2)} \mathbf{v}_{\mathbf{b}}(\mathbf{x} - \mathbf{v}_1, \mathbf{y} + \mathbf{v}_2, s-1) := \mathbf{u}_2 \tag{6.4}$$

with $s > l(\mathbf{b})$ and $\mathbf{v}_1, \mathbf{v}_2 \in \mathfrak{B}_{\mathbf{1}_k}$. Let us verify the assumptions of Proposition 2.1. As already shown in Lemma 4.2, the edge-connectivity in the two graphs $\mathrm{G}(\mathcal{B}_{\mathbf{b}}(s-1), \mathcal{G}(A_k))$ and $\mathrm{G}(\mathcal{B}_{\mathbf{b}}(s), \mathcal{G}(A_k))$ is at least

$$n := \min_{j \in \{1,2\}} \{|\operatorname{supp}(\mathbf{w}_j + \|\mathbf{w}_j^-\|_\infty \cdot \mathbf{1}_k)|\} + k.$$

Since we have $m := 2^k \geq 2k - 2 \geq n - 2$ it is left to prove that there are 2^k node-disjoint paths connecting $\mathbf{u}_1$ with $\mathbf{u}_2$ and which only use edges between $\mathcal{B}_{\mathbf{b}}(s-1)$ and $\mathcal{B}_{\mathbf{b}}(s)$. For this, we define the sets

$$\begin{aligned} W_s &:= \{\mathbf{v}_{\mathbf{b}}(\mathbf{x} - \mathbf{v}_1 + \mathbf{z}, \mathbf{y}, s) : \mathbf{z} \in \mathfrak{B}_{\mathbf{1}_k}\} \subseteq \mathcal{B}_{\mathbf{b}}(s) \\ W_{s-1} &:= \{\mathbf{v}_{\mathbf{b}}(\mathbf{x} - \mathbf{v}_1, \mathbf{y} + \mathbf{z}, s-1) : \mathbf{z} \in \mathfrak{B}_{\mathbf{1}_k}\} \subseteq \mathcal{B}_{\mathbf{b}}(s-1). \end{aligned} \tag{6.5}$$

It is easy to see that W_s is completely contained in the neighborhood of every node in W_{s-1} and vice versa. This means that $\mathrm{G}(\mathcal{F}_{A_k,\mathbf{b}}, \mathcal{G}(A_k))$ has a complete bipartite graph on the node sets W_s and W_{s-1} as subgraph including our original edge (6.4). This gives 2^k many node-disjoint paths between $\mathbf{u}_1$ and $\mathbf{u}_2$ only using edges between $\mathcal{B}_{\mathbf{b}}(s)$ and $\mathcal{B}_{\mathbf{b}}(s-1)$. Applying Proposition 2.1, we obtain $m + n = \delta(\mathrm{G}(\mathcal{F}_{A_k,\mathbf{b}}, \mathcal{G}(A_k)))$ edge-disjoint connecting paths connecting $\mathbf{u}_1$ and $\mathbf{u}_2$. □

Combining all the results of this section, we obtain our main theorem.

Theorem 4. *For $k > 0$, the edge-connectivity in all Graver fiber graphs of A_k equals its minimal degree.*

Proof. From Lemma 2.1 we know that we only have to consider paths between adjacent nodes. From the decomposition of the fiber $\mathcal{F}_{A,\mathbf{b}}$ given in (4.5) we obtain that there are only two kinds of edges: edges within boxes and edges connecting two neighboring boxes. Lemma 6.1 and Lemma 6.2 state that we found in both cases $\delta(\mathrm{G}\left(\mathcal{F}_{A_k,\mathbf{b}}, \mathcal{G}(A_k)\right))$ many edge-disjoint paths connecting the adjacent nodes of that edge. □

Unfortunately, Theorem 4 says nothing about the node-connectivity of the fiber graphs and we do not know whether it is best possible or not. Nevertheless, the results of this section make us suggest that requiring the Graver basis as set of edges should suffice that the edge-connectivity (not the node-connectivity!) equals the minimal degree in all fiber graphs of arbitrary integer matrices.

Conjecture 3. Let $A \in \mathbb{Z}^{d\times n}$ be an integer matrix with $\ker(A) \cap \mathbb{Z}^n_{\geq 0} = \{\mathbf{0}_n\}$. Then in all Graver fiber graphs of A, the edge-connectivity equals its minimal degree.

7 Computational Results

In this section, we present how random walks on fiber graphs of A_k behave. Therefore, let us first introduce briefly the framework. Let $G = (\{v_1, \ldots, v_n\}, E)$ be a simple graph. Consider the random walk which has for $i, j \in [n]$ the probability

$$p_G(v_i, v_j) = \begin{cases} \min\{1/\deg(v_i), 1/\deg(v_j)\}, & \text{if } \{v_i, v_j\} \in E \text{ and } i \neq j \\ \sum_{\{v_i,v_k\}\in E} \max\{0, 1/\deg(v_i) - 1/\deg(v_k)\}, & \text{if } i = j \\ 0, & \text{if } \{v_i, v_j\} \notin E \end{cases}$$

to traverse from v_i to v_j. The matrix $P_G = (p_G(v_i, v_j))_{i,j\in[n]}$ is precisely the transition probability matrix of the Metropolis-Hastings chain on G whose stationary distribution is the uniform distribution on $\{v_1, \ldots, v_n\}$ [2, Section 1.2.2]. Given a vertex v_i and a time step $t \in \mathbb{N}$, the jth-entry of the vector $P_G^t \cdot \mathbf{e}_i \in [0,1]^n$ is the probability that a random walk starting at v_i is at v_j in time step t. Let $\mu(P_G) \in [0,1]$ be the *second largest eigenvalue modulus* (SLEM) of P_G. Since $(P_G^t \cdot \mathbf{e}_i)_{t\in\mathbb{N}}$ converges to uniform $\frac{1}{n} \cdot \mathbf{1}_n$ asymptotically with $\mu(P_G)^t$ [2, Section 1.1.2], $\mu(P_G)$ is an indicator of how fast the convergence of the corresponding Markov chain towards its stationary distribution is.

In our experiments with *Macaulay2* [7] we considered this random walk on the fiber graphs $\mathrm{G}\left(\mathcal{F}_{A_k,\mathbf{e}_{2k+1}}, \mathcal{G}(A_k)\right)$ and $\mathrm{G}\left(\mathcal{F}_{A_k,\mathbf{e}_{2k+1}}, \mathcal{R}_{\prec_{\text{LEX}}}(A_k)\right)$, respectively. The left plot of Figure 4 shows how the SLEM of those chains behaves if k rises. It seems that both the SLEM of the Gröbner chain

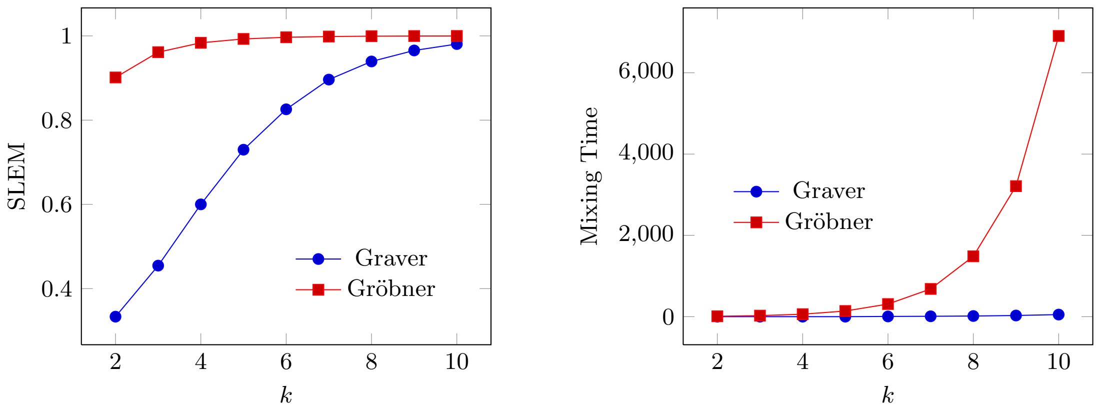


Fig. 4. Plots of SLEM and mixing time of $\mathcal{F}_{A_k,\mathbf{e}_{2k+1}}$ with respect to Graver and Gröbner moves.

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