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Medical Service Quality in the Situation of Healthcare System

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Toric ideals to hierarchical models are invariant under the action of a product of symmetric groups. Taking the number of factors, say, m, into account, we introduce and study invariant filtrations and their equivariant Hilbert series. We present a condition that guarantees that the equivariant Hilbert series is a rational function in $m+1$ variables with rational coefficients. Furthermore we give explicit formulas for the rational functions with coefficients in a number field and an algorithm for determining the rational functions with rational coef-ficients. A key is to construct finite automata that recognize languages corresponding to invariant filtrations.

1. Introduction

Hierarchical models are used in algebraic statistics to determine dependencies among random variables; see, e.g., [17]. Such a model is determined by a simplicial complex and the number of states each random variable can take. The Markov basis to any hierarchical model corresponds to a generating set of an associated toric ideal; see [3]. This toric ideal is rather symmetric; that is, it is invariant under the action of a product of symmetric groups. The number of minimal generators of the toric ideals grows rapidly when the number of states of the considered random variables increases. However, the independent set theorem (see Theorem 2.4) shows that the symmetry can be leveraged to describe, for a fixed simplicial complex, simultaneously the generating sets and thus Markov bases for all numbers of states of the random variables. The conceptional proof of this result by Hillar and Sullivant [7] introduces the notion of an $\mathcal{S}_\infty$-invariant filtration. Informally, this is a sequence $(I_n)_{n\in\mathbb{N}}$ of compatible ideals I_n in polynomial rings R_n whose number of variables increases with n and where each I_n is invariant under the action of a symmetric group that permutes the variables of R_n. To such a filtration, the second author and Römer [14] introduced an equivariant Hilbert series in order to analyze simultaneously quantitative properties of the ideals in the filtration. It is a formal power series in two variables and they showed that it is rational with rational coefficients [14, Theorem 7.8].

The variables occurring in the elements of a toric ideal to a hierarchical model can naturally be grouped into m sets of variables, where m is the number of random variables. Permuting the variables in any one of these groups gives a group action that leaves the ideal invariant. This suggests the introduction of an $\mathcal{S}_\infty^m$-invariant filtration (see Definition 2.2). For $m = 1$ it specializes to the filtrations mentioned above. Every $\mathcal{S}_\infty^m$-invariant filtration naturally gives rise to a n equivariant Hilbert series defined as a formal power series in $m+1$ variables (see Definition 3.1). Our main result gives a condition guaranteeing

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that this power series is a rational function in $m+1$ variables with rational coefficients (see Theorem 3.5). Furthermore, we present two methods to determine this rational function. One approach is more special and produces an explicit rational function, but with coefficients in a suitable extension field of the rational numbers (see Proposition 5.4). The other approach is much more general and gives directly a formula for the rational function with rational coefficients. It determines the equivariant Hilbert series as the generating function of a regular language (see Section 5).

The remaining part of this paper is organized as follows. In Section 2, we discuss the symmetry of toric ideals to hierarchical models and introduce $\mathcal{S}_\infty^m$-invariant filtrations. Their equivariant Hilbert series in $m+1$ variables are studied in Section 3. Our main result about such Hilbert series is stated as Theorem 3.5. We reduce its proof to a special case in that section, but postpone the argument for the special case to the following section. In Section 4 we use regular languages and finite automata to establish the special case. The idea is to encode the monomials that determine the Hilbert series by a language. We then construct a deterministic finite automaton that recognizes this language. Thus, the language is regular. Using a suitable weight function we then show that the corresponding generating function of the language is essentially the desired Hilbert series. Since generating functions of regular languages are rational, this completes the argument of our main result. Furthermore, using the finite automaton that describes a regular language, there is an algorithm that determines the generating function of the language explicitly as a rational function with rational coefficients. This is explained and illustrated in Section 5. We also describe in that section a more limited direct approach that gives an explicit formula for the rational function, but with coefficients in a number field.

2. Symmetry and filtrations

After reviewing needed concepts and notation we introduce $\mathcal{S}_\infty^m$-invariant filtrations in this section.

Throughout this paper we use $\mathbb{N}$ to denote the set of positive integers and $\mathbb{N}_0$ to denote the set of nonnegative integers. For any $q \in \mathbb{N}$, we set $[q] = \{1, 2, \ldots, q\}$, and so $[0] = \varnothing$. We use $\#T$ to denote the number of elements in a finite set T.

A *hierarchical model* $\mathcal{M} = \mathcal{M}(\Delta, \boldsymbol{r})$ with m parameters is given by a collection $\Delta = \{F_1, F_2, \ldots, F_q\}$ of nonempty subsets $F_j \subset [m]$ with $\bigcup_{j\in[q]} F_j = [m]$ and a vector $\boldsymbol{r} = (r_1, r_2, \ldots, r_m) \in \mathbb{N}^m$. Each parameter corresponds to a random variable, and r_i denotes the number of values parameter i can take. We refer to $\boldsymbol{r}$ as the *vector of states*. Every set F_j indicates a dependency among the parameters corresponding to its vertices. Thus, we may assume that no F_j is contained in some F_i with $i \neq j$ and refer to the sets F_j as *facets*.

Diaconis and Sturmfels [3] pioneered the use of algebraic methods in order to study statistical models. We need some notation. For any subset $F = \{i_1, i_2, \ldots, i_s\} \subset [m]$, we write

$$\boldsymbol{r}_F = (r_{i_1}, r_{i_2}, \ldots, r_{i_s}) \in \mathbb{N}^s \quad \text{and} \quad [\boldsymbol{r}_F] = [r_{i_1}] \times [r_{i_2}] \times \cdots \times [r_{i_s}] \subset \mathbb{N}^s.$$

In particular, $[\boldsymbol{r}_{[m]}] = [\boldsymbol{r}] \subset \mathbb{N}^m$. Given a field $\mathbb{K}$ and a hierarchical model $\mathcal{M} = \mathcal{M}(\boldsymbol{r}, \Delta)$, consider the ring homomorphism

$$\begin{aligned} \Phi_{\mathcal{M}} : \boldsymbol{R}_{\boldsymbol{r}} = \mathbb{K}[x_{\boldsymbol{i}} \mid \boldsymbol{i} \in [\boldsymbol{r}]] &\longrightarrow \boldsymbol{S}_{\mathcal{M}} = \mathbb{K}[y_{j,\boldsymbol{i}_{F_j}} \mid F_j \in \Delta, \boldsymbol{i}_{F_j} \in [\boldsymbol{r}_{F_j}]], \\ x_{\boldsymbol{i}} &\longmapsto \prod_{F_j \in \Delta} y_{j,\boldsymbol{i}_{F_j}}. \end{aligned} \tag{2-1}$$

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The kernel of this homomorphism, denoted by $I_{\mathcal{M}}$, is called the *toric ideal* to the hierarchical model $\mathcal{M}$. We also refer to $R_{\boldsymbol{r}}/I_{\mathcal{M}}$ as the *coordinate ring* of the model $\mathcal{M}$.

In the simplest cases explicit sets of generators of such ideals are known. We use the standard partial order $\leq$ on $\mathbb{Z}^s$ given by $\boldsymbol{i} = (i_1, \ldots, i_s) \leq \boldsymbol{j} = (j_1, \ldots, j_s)$ if $i_1 \leq j_1, \ldots, i_s \leq j_s$. If $q = 1$ then $\Phi_{\mathcal{M}}$ is an isomorphism, and so $I_{\mathcal{M}}$ is zero.

Example 2.1. Let $q = 2$; i.e., $\Delta = \{F_1, F_2\}$.

(i) Suppose first that F_1 and F_2 are disjoint. Possibly permuting the positions of the entries of a vector $\boldsymbol{i} \in [\boldsymbol{r}] = [\boldsymbol{r}_{F_1 \cup F_2}]$, we write $x_{\boldsymbol{i}_{F_1}, \boldsymbol{i}_{F_2}}$ instead of $x_{\boldsymbol{i}}$. This corresponds to a bijection $[\boldsymbol{r}_{F_1 \cup F_2}] \to [\boldsymbol{r}_{F_1}] \times [\boldsymbol{r}_{F_2}]$. Using this notation, a generating set of $I_{\mathcal{M}}$ is (see, e.g., [2; 3])

$$G(\mathcal{M}(\boldsymbol{r}, \{F_1, F_2\})) = \{x_{\boldsymbol{i}_{F_1}, \boldsymbol{i}_{F_2}} x_{\boldsymbol{i}'_{F_1}, \boldsymbol{i}'_{F_2}} - x_{\boldsymbol{i}_{F_1}, \boldsymbol{i}'_{F_2}} x_{\boldsymbol{i}'_{F_1}, \boldsymbol{i}_{F_2}} \mid \boldsymbol{i}_{F_1} < \boldsymbol{i}'_{F_1} \in [\boldsymbol{r}_{F_1}],\ \boldsymbol{i}_{F_2} < \boldsymbol{i}'_{F_2} \in [\boldsymbol{r}_{F_2}]\}.$$

In the special case, where $m = 2$ and, say, $F_1 = \{1\}$, $F_2 = \{2\}$, this set becomes

$$\{x_{i_1, i_2} x_{i'_1, i'_2} - x_{i_1, i'_2} x_{i'_1, i_2} \mid 1 \leq i_1 \leq i'_1 \leq r_1,\ 1 \leq i_2 \leq i'_2 \leq r_2\},$$

which is the set of 2×2 minors of a generic $r_1 \times r_2$ matrix with entries x_{i_1, i_2}. The image of the map $\Phi_{\mathcal{M}}$ in this case is known in algebraic geometry as the coordinate ring of the Segre product $\mathbb{P}^{r_1 - 1} \times \mathbb{P}^{r_2 - 1}$ whose homogeneous ideal is $I_{\mathcal{M}}$.

(ii) Consider now the general case, where F_1 and F_2 are not necessarily disjoint. Note that $[m]$ is the disjoint union of $F_1 \setminus F_2$, $F_2 \setminus F_1$ and $F_1 \cap F_2$. Thus, possibly permuting the positions of the entries of $\boldsymbol{i} \in [r]$ as above, we write $x_{\boldsymbol{i}_{F_1 \setminus F_2}, \boldsymbol{i}_{F_1 \cap F_2}, \boldsymbol{i}_{F_2 \setminus F_1}}$ for $x_{\boldsymbol{i}}$. Fixing a vector $\boldsymbol{c} \in [\boldsymbol{r}_{F_1 \cap F_2}]$, we define a set $G^{\boldsymbol{c}}(\mathcal{M}(\boldsymbol{r}_{[m] \setminus F_1 \cap F_2}, \{F_1 \setminus F_2, F_2 \setminus F_1\}))$ whose elements are

$$x_{\boldsymbol{i}_{F_1 \setminus F_2}, \boldsymbol{c}, \boldsymbol{i}_{F_2 \setminus F_1}} x_{\boldsymbol{i}'_{F_1 \setminus F_2}, \boldsymbol{c}, \boldsymbol{i}'_{F_2 \setminus F_1}} - x_{\boldsymbol{i}'_{F_1 \setminus F_2}, \boldsymbol{c}, \boldsymbol{i}_{F_2 \setminus F_1}} x_{\boldsymbol{i}_{F_1 \setminus F_2}, \boldsymbol{c}, \boldsymbol{i}'_{F_2 \setminus F_1}},$$

where

$$\boldsymbol{i}_{F_1 \setminus F_2} < \boldsymbol{i}'_{F_1 \setminus F_2} \in [\boldsymbol{r}_{F_1 \setminus F_2}] \quad \text{and} \quad \boldsymbol{i}_{F_2 \setminus F_1} < \boldsymbol{i}'_{F_2 \setminus F_1} \in [\boldsymbol{r}_{F_2 \setminus F_1}].$$

The collection

$$G(\mathcal{M}(\boldsymbol{r}, \{F_1, F_2\})) = \bigcup_{\boldsymbol{c} \in [\boldsymbol{r}_{F_1 \cap F_2}]} G^{\boldsymbol{c}}(\mathcal{M}(\boldsymbol{r}_{[m] \setminus F_1 \cap F_2}, \{F_1 \setminus F_2, F_2 \setminus F_1\}))$$

is a generating set for the ideal $I_{\mathcal{M}(\boldsymbol{r}, \{F_1, F_2\})}$; see [4; 8].

Even in the simple cases of Example 2.1, the number of minimal generators of a toric ideal $I_{\mathcal{M}}$ is large if the entries of $\boldsymbol{r}$ are large. However, many of these generators have similar shape. This can be made precise using symmetry.

Indeed, denote by $\mathcal{S}_n$ the symmetric group in n letters. Set $\mathcal{S}_{[\boldsymbol{r}]} = \mathcal{S}_{r_1} \times \mathcal{S}_{r_2} \times \cdots \times \mathcal{S}_{r_m}$. This group acts on the polynomial ring $R_{\boldsymbol{r}}$ by permuting the indices of its variables, that is,

$$(\sigma_1, \ldots, \sigma_m) \cdot x_{\boldsymbol{i}} = x_{(\sigma_1(i_1), \ldots, \sigma_m(i_m))}.$$

It is well known that toric ideals have minimal generating sets consisting of binomials. Thus, the definition of the homomorphism $\Phi_{\mathcal{M}}$ in (2-1) implies that the ideal $I_{\mathcal{M}}$ is $\mathcal{S}_{[\boldsymbol{r}]}$-invariant, that is, $\sigma \cdot f \in I_{\mathcal{M}}$ whenever

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$\sigma \in S_{[\boldsymbol{r}]}$ and $f \in I_{\mathcal{M}}$. In some cases, this invariance can be used to obtain all minimal generators of $I_{\mathcal{M}}$ from a subset by using symmetry. For example, in the special case $m = q = 2$, $F_1 = \{1\}$, $F_2 = \{2\}$, with $r_1, r_2 \geq 2$, the set $G(\mathcal{M}(\boldsymbol{r}, \{F_1, F_2\}))$ can be obtained from

$$x_{1,1}x_{2,2} - x_{1,2}x_{2,1}$$

using the action of $\mathcal{S}_{r_1} \times \mathcal{S}_{r_2}$. Note that this is true for every vector $\boldsymbol{r} = (r_1, r_2)$. There is a vast generalization of this observation using the concept of an invariant filtration.

The symmetric group $\mathcal{S}_n$ is naturally embedded into $\mathcal{S}_{n+1}$ as the stabilizer of $\{n+1\}$. Using this construction componentwise, we get an embedding of $\mathcal{S}_{[\boldsymbol{r}]}$ into $\mathcal{S}_{[\boldsymbol{r}']}$ if $\boldsymbol{r} \leq \boldsymbol{r}'$. Set

$$\mathcal{S}_\infty^m = \bigcup_{\boldsymbol{r} \in \mathbb{N}^m} \mathcal{S}_{[\boldsymbol{r}]}.$$

Definition 2.2. An $\mathcal{S}_\infty^m$*-invariant filtration* is a family $(I_{\boldsymbol{r}})_{\boldsymbol{r} \in \mathbb{N}^m}$ of ideals $I_{\boldsymbol{r}} \subset R_{\boldsymbol{r}}$ such that every ideal $I_{\boldsymbol{r}}$ is $\mathcal{S}_{[\boldsymbol{r}]}$-invariant and, as subsets of $R_{\boldsymbol{r}'}$,

$$\mathcal{S}_{[\boldsymbol{r}']} \cdot I_{\boldsymbol{r}} \subset I_{\boldsymbol{r}'} \quad \text{whenever } \boldsymbol{r} \leq \boldsymbol{r}'.$$

Note that fixing Δ, the ideals $(I_{\mathcal{M}(\Delta,\boldsymbol{r})})_{\boldsymbol{r} \in \mathbb{N}_0^m}$ form an $\mathcal{S}_\infty^m$-invariant filtration. It is useful to extend these ideas.

Remark 2.3. Let T be any nonempty subset of $[m]$. For vectors $\boldsymbol{r} \in \mathbb{N}^m$, we want to fix the entries in positions supported at T, but vary the other entries. To this end write $(\boldsymbol{r}_{[m]\setminus T}, \boldsymbol{r}_T)$ instead of $\boldsymbol{r}$.

Fix a vector $\boldsymbol{c} \in \mathbb{N}^{m \setminus \#T}$. Let $I_{\boldsymbol{r}} \subset R_{\boldsymbol{r}}$ be an $\mathcal{S}_\infty^m$-invariant filtration. Restricting $\mathcal{S}_{[\boldsymbol{r}]}$ and its action to components supported at T, we get an $\mathcal{S}_\infty^{\#T}$-invariant filtration of ideals $I_{\boldsymbol{r}_T} = I_{\boldsymbol{c},\boldsymbol{r}_T} \subset R_{\boldsymbol{c},\boldsymbol{r}_T}$ with $\boldsymbol{r}_T \in \mathbb{N}^{\#T}$.

Note that this idea applies to the ideals $I_{\mathcal{M}(\Delta,\boldsymbol{r})}$ with fixed Δ. We can now state the mentioned extension of the example given above Definition 2.2. It is called independent set theorem and has been established by Hillar and Sullivant in [7, Theorem 4.7]; see also [5].

Theorem 2.4. *Fix Δ and consider a subset $T \subset [m]$ such that $\#(F_j \cap T) \leq 1$ for every $j \in [q]$. Assume the number of states of every parameter $j \in [m] \setminus T$ is fixed, and consider the hierarchical models $\mathcal{M}(\Delta, \boldsymbol{r}_T) = \mathcal{M}(\Delta, (\boldsymbol{c}, \boldsymbol{r}_T))$, where $\boldsymbol{c} \in \mathbb{N}^{m-\#T}$. Then the ideals $I_{\mathcal{M}(\Delta,\boldsymbol{r}_T)}$ form an $\mathcal{S}_\infty^{\#T}$-invariant filtration $\mathscr{I}_{\Delta,\boldsymbol{r}_{[m]\setminus T}} = (I_{\mathcal{M}(\Delta,\boldsymbol{r}_T)})_{\boldsymbol{r}_T \in \mathbb{N}^{\#T}}$, that is, there is some $\boldsymbol{d} \in \mathbb{N}^{\#T}$ such that $\mathcal{S}_{[\boldsymbol{r}_T]} \cdot I_{\mathcal{M}(\Delta,\boldsymbol{d})}$ generates in $R_{\boldsymbol{c},\boldsymbol{r}_T}$ the ideal $I_{\mathcal{M}(\Delta,\boldsymbol{r}_T)}$ whenever $\boldsymbol{r}_T \geq \boldsymbol{d}$.*

In other words, this result says that a generating set of the ideal $I_{\mathcal{M}(\Delta,\boldsymbol{r})}$ can be obtained from a fixed finite minimal generating set of $I_{\mathcal{M}(\Delta,(\boldsymbol{c},\boldsymbol{d}))}$ by applying suitable permutations whenever the number of states of every parameter in $[m] \setminus T$ is large enough.

Theorem 2.4 is not true without an assumption on the set T; see [7, Example 4.3].

Remark 2.5. An $\mathcal{S}_\infty^m$-invariant filtration can also be described using a categorical framework. Indeed, if $m = 1$ this approach has been used in [15] to study also sequences of modules by using the category FI, whose objects are finite sets and whose morphisms are injections. This approach can be extended to any $m \geq 1$ using the category FI^m (see, e.g., [12] in the case of modules over a fixed ring). For conceptional simplicity we prefer to use invariant filtrations in this paper.

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3. Equivariant Hilbert series

In order to study asymptotic properties of ideals in an $\mathcal{S}_\infty$-invariant filtration, an equivariant Hilbert series was introduced in [14]. Here we study an extension of this concept for $\mathcal{S}_\infty^m$-invariant filtrations.

We begin by recalling some basic facts. Let I be a homogeneous ideal in a polynomial ring R in finitely many variables over some field $\mathbb{K}$. We will always assume that any variable has degree 1. Thus, $R/I = \bigoplus_{j\geq 0}[R/I]_j$ is a standard graded $\mathbb{K}$-algebra. Its Hilbert series is the formal power series

$$H_{R/I}(t) = \sum_{j\geq 0} \dim_{\mathbb{K}}[R/I]_j t^j.$$

By Hilbert's theorem (see, e.g., [1, Corollary 4.1.8]), it is rational and can be uniquely written as

$$H_{R/I}(t) = \frac{g(t)}{(1-t)^{\dim R/I}},$$

with $g(t) \in \mathbb{Z}[t]$ and $g(1) > 0$, unless $I = R$. The number $g(1)$ is called the *degree* of I.

Definition 3.1. The *equivariant Hilbert series* of an $\mathcal{S}_\infty^m$-invariant filtration $\mathscr{I} = (I_{\boldsymbol{r}})_{\boldsymbol{r}\in\mathbb{N}^m}$ of ideals $I_{\boldsymbol{r}} \subset R_{\boldsymbol{r}}$ is the formal power series in variables $s_1, \ldots, s_m, t$

$$\begin{aligned}\operatorname{equivH}_{\mathscr{I}}(s_1, \ldots, s_m, t) &= \sum_{\boldsymbol{r}\in\mathbb{N}^m} H_{R_{\boldsymbol{r}}/I_{\boldsymbol{r}}}(t)\cdot s_1^{r_1}\cdots s_m^{r_m} \\ &= \sum_{\boldsymbol{r}\in\mathbb{N}^m}\sum_{j\geq 0} \dim_{\mathbb{K}}[R_{\boldsymbol{r}}/I_{\boldsymbol{r}}]_j \cdot s_1^{r_1}\cdots s_m^{r_m} t^j.\end{aligned}$$

If $m = 1$, that is, $\mathscr{I}$ is an $\mathcal{S}_\infty$-invariant filtration, the Hilbert series of $\mathscr{I}$ is always rational by [14, Theorem 7.8] or [11, Theorem 4.3]. For $m \geq 1$, one can also consider another formal power series by focusing on components whose degree is on the diagonal of $\mathbb{N}^m$. This gives

$$\sum_{r\geq 1} H_{R_{(r,\ldots,r)}/I_{(r,\ldots,r)}}(t)\cdot s^r.$$

It is open whether this formal power series is rational if $m \geq 2$, even if the ideals are trivial.

Example 3.2. Let $m = 2$ and consider the filtration $\mathscr{I} = (I_{\boldsymbol{r}})$, where every ideal $I_{\boldsymbol{r}}$ is zero. Since the ring $R_{(r_1,r_2)}$ has dimension r_1r_2, one obtains

$$\operatorname{equivH}_{\mathscr{I}}(s_1, s_2, t) = \sum_{(r_1,r_2)\in\mathbb{N}^2} H_{R_{(r_1,r_2)}}(t)\cdot s_1^{r_1}s_2^{r_2} = \sum_{(r_1,r_2)\in\mathbb{N}^2} \frac{1}{(1-t)^{r_1r_2}}\cdot s_1^{r_1}s_2^{r_2} = \sum_{r_1\geq 1}\left[-1+\frac{(1-t)^{r_1}}{(1-t)^{r_1}-s_2}s_1^{r_1}\right].$$

We do not know if this is a rational function in s_1, s_2 and t. However, if one considers the more standard Hilbert series with $r = r_1 = r_2$, one gets

$$\sum_{r\geq 0} H_{R_{(r,r)}}(t)\cdot s^r = \sum_{n\geq 1}\frac{1}{(1-t)^{r^2}}\cdot s^r.$$

This is not a rational function because the sequence $(1/(1-t)^{r^2})_{r\in\mathbb{N}}$ does not satisfy a finite linear recurrence relation with coefficients in $\mathbb{Q}(t)$.

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For the remainder of this section we restrict ourselves to considering ideals of hierarchical models $\mathcal{M}(\Delta, \boldsymbol{r})$. As pointed out in Remark 2.3, for any subset $T \neq \varnothing$ of $[m]$, these ideals give rise to $\mathcal{S}_\infty^{\#T}$-invariant filtrations. To study their equivariant Hilbert series, it is convenient to simplify notation. We may assume that $T = \{m - \#T + 1, \ldots, m\}$ and fix the entries of $\boldsymbol{r}$ in positions supported on $[m] \setminus T$; that is, we fix $\boldsymbol{c} \in \mathbb{N}^{m-\#T}$ and set $\boldsymbol{n} = (n_1, \ldots, n_{m-\#T}) = \boldsymbol{r}_T$ for $\boldsymbol{r} \in \mathbb{N}^m$ to obtain $\boldsymbol{r} = (\boldsymbol{c}, \boldsymbol{n})$. We write $\mathcal{M}(\Delta, \boldsymbol{n})$ instead of $\mathcal{M}(\Delta, (\boldsymbol{c}, \boldsymbol{n}))$ and denote the resulting $\mathcal{S}_\infty^{m-\#T}$-invariant filtration $(I_{\mathcal{M}(\Delta,\boldsymbol{n})})_{\boldsymbol{n}\in\mathbb{N}^{m-\#T}}$ by $\mathscr{I}_{\Delta,\boldsymbol{r}_{[m]\setminus T}}$, as in the independent set theorem. Its equivariant Hilbert series is

$$\operatorname{equivH}_{\mathscr{I}_{\Delta,\boldsymbol{r}_{[\mathbf{m}]\setminus\mathbf{T}}}}(s_1, s_2, \ldots, s_{\#T}, t) = \sum_{\boldsymbol{n}\in\mathbb{N}^{\#T}} H_{R_{(\boldsymbol{c},\boldsymbol{n})}/I_{\mathcal{M}(\Delta,\boldsymbol{n})}}(t) \cdot s_1^{n_1} \cdots s_{\#T}^{n_{\#T}}.$$

The independent set theorem (Theorem 2.4) guarantees stabilization of the filtration. This suggests the following problem.

Question 3.3. *If $T \subset [m]$ satisfies $\#(F \cap T) \leq 1$ for every facet F of Δ, is the equivariant Hilbert series of $\mathscr{I}_{\Delta,\boldsymbol{r}_{[m]\setminus T}}$ rational?*

The answer is affirmative if T consists of exactly one element.

Proposition 3.4. *If $\#T = 1$, then the equivariant Hilbert series of $\mathscr{I}_{\Delta,\boldsymbol{r}_{[m]\setminus T}}$ is rational.*

Proof. The assumption means $T = \{m\}$ and $\boldsymbol{r} = (\boldsymbol{c}, n)$ with $\boldsymbol{c} \in \mathbb{N}^{m-1}$ and $n \in \mathbb{N}$. Set $c = c_1 \cdots c_{m-1}$ and fix a bijection

$$\psi : [\boldsymbol{c}] = [c_1] \times \cdots \times [c_{m-1}] \to [c].$$

For every $n \in \mathbb{N}$, it induces a ring isomorphism

$$R_{(\boldsymbol{c},n)} = \mathbb{K}[x_{\boldsymbol{i},j} \mid (\boldsymbol{i}, j) \in [\boldsymbol{c}] \times [n]] \longrightarrow \mathbb{K}[x_{i,j} \mid (i, j) \in [c] \times [n]] = R'_n,$$
$$x_{\boldsymbol{i},j} \longmapsto x_{\psi(\boldsymbol{i}),j}.$$

This isomorphism maps every ideal $I_{\mathcal{M}(\Delta,n)}$ corresponding to the model $\mathcal{M}(\Delta, (\boldsymbol{c}, n))$ onto an S_n-invariant ideal I_n. In particular, the rings $R_{(\boldsymbol{c},n)}/I_{\mathcal{M}(\Delta,n)}$ and R'_n/I_n have the same Hilbert series and the family $(I_n)_{n\in\mathbb{N}}$ is an $\mathcal{S}_\infty$-invariant filtration. Thus, its equivariant Hilbert series is rational by [14, Theorem 7.8] or [11, Theorem 4.3]. □

Our main result in this section describes further cases in which the answer to Question 3.3 is affirmative.

Theorem 3.5. *The equivariant Hilbert series of $\mathscr{I}_{\Delta,\boldsymbol{r}_{[m]\setminus T}}$ is a rational function with rational coefficients if*

(1) *$F_i \cap F_j = \varnothing$ for any distinct $F_i, F_j \in \Delta$, and*

(2) *$|F \cap T| \leq 1$ for any $F \in \Delta$.*

This results applies in particular to the independence model, where it takes an attractive form.

Example 3.6. A hierarchical model describing m independent parameters is called *independence model*. Its collection of facets is $\Delta = \{\{1\}, \{2\}, \ldots, \{m\}\}$. Thus, we may apply Theorem 3.5 with any subset T of $[m]$. Using $T = [m]$, we show in Example 5.5 below that

$$\operatorname{equivH}_{\mathscr{I}_{\Delta,\boldsymbol{r}_{[m]\setminus T}}}(s_1, s_2, \ldots, s_m, t) = \sum_{\boldsymbol{n}\in\mathbb{N}^m} H_{R_{\boldsymbol{n}}/I_{\mathcal{M}(\Delta,\boldsymbol{n})}}(t) \cdot s_1^{n_1} \cdots s_m^{n_m} = \frac{s_1 \cdots s_m}{(1 - s_1) \cdots (1 - s_m) - t}.$$

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The proof of Theorem 3.5 will be given in two steps. First we show that it is enough to prove the result in a special case where every facet consists of two elements. Second, we use regular languages to show the desired rationality in the following section.

In the remainder of this section we establish the reduction step.

Lemma 3.7. *Consider a collection $\Delta = \{F_1, \ldots, F_q\}$ on vertex set $[m]$ and a subset T of $[m]$ satisfying*

(1) *$F_i \cap F_j = \varnothing$ for any $F_i, F_j \in \Delta$, and*

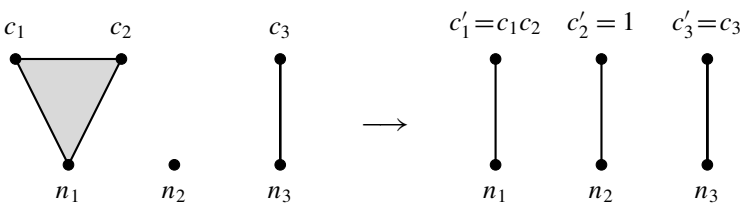
(2) *$|F \cap T| = 1$ for any $F \in \Delta$.*

Then there is a collection $\Delta' = \{F'_1, \ldots, F'_q\}$ on vertex set $[m']$ consisting of two element facets and also satisfying conditions (1) and (2) with the property that for every $\boldsymbol{c} \in \mathbb{N}^{m-\#T}$ there is some $\boldsymbol{c}' \in \mathbb{N}^{m'-\#T}$ such that the filtrations corresponding to the models $\mathcal{M}(\Delta, (\boldsymbol{c}, \boldsymbol{n}))$ and $\mathcal{M}(\Delta', (\boldsymbol{c}', \boldsymbol{n}))$ with $\boldsymbol{n} \in \mathbb{N}^{\#T}$ have the same equivariant Hilbert series.

Proof. The assumptions imply that T must have q elements. We may assume that every facet in Δ has at least two elements. Indeed, if $F \in \Delta$ has only one element then we may replace F by the union F' of F and a new vertex. Assigning to the parameter corresponding to the new vertex exactly one possible state gives a new model whose coordinate ring has the same Hilbert series as the original model.

Given such a hierarchical model $\mathcal{M}_{\boldsymbol{n}} = \mathcal{M}(\Delta, (\boldsymbol{c}, \boldsymbol{n}))$ on vertex set $[m]$, we will construct a new hierarchical model $\mathcal{M}'_{\boldsymbol{n}} = \mathcal{M}(\Delta', (\boldsymbol{c}', \boldsymbol{n}))$ on $m' = 2q$ vertices that has the same Hilbert series. The new vertex set is the disjoint union of the q vertices in $F_j \cap T$ with $j \in [q]$ and a set V of q other vertices, say $V = [q]$. For $j \in [q]$, set $F'_j = \{j\} \cup (F_j \cap T)$. Thus, the sets F'_j are pairwise disjoint because $F_1, \ldots, F_q$ have this property, and each F'_j has exactly two elements. In particular, $\Delta' = \{F'_1, \ldots, F'_q\}$ and T satisfy conditions (1) and (2).

Now let $c'_j = \prod_{e \in F_j \setminus T} c_e = \#[\boldsymbol{c}_{F_j \setminus T}]$ be the number of states of the parameter corresponding to the vertex $j \in F'_j$. Furthermore, for every $j \in [q]$, let the parameter corresponding to the vertex $F'_j \cap T$ have the same number of states as $F_j \cap T$ has in $\mathcal{M}_{\boldsymbol{n}}$. This completes the definition of a new hierarchical model $\mathcal{M}'_{\boldsymbol{n}} = \mathcal{M}(\Delta', (\boldsymbol{c}', \boldsymbol{n}))$. The passage from $\mathcal{M}_{\boldsymbol{n}}$ to $\mathcal{M}'_{\boldsymbol{n}}$ is illustrated in the example below:



$$\Delta = \{124, 5, 36\},\ r = (c_1, c_2, c_3, n_1, n_2, n_3) \quad \longrightarrow \quad \Delta' = \{14, 25, 36\},\ r' = (c'_1, 1, c'_3, n_1, n_2, n_3).$$

Varying $\boldsymbol{n} \in \mathbb{N}^q$, the ideals $I_{\mathcal{M}'_{\boldsymbol{n}}}$ form an $\mathcal{S}_\infty^q$-invariant filtration. Thus, to establish the assertion it is enough to prove that for every $\boldsymbol{n} \in \mathbb{N}^q$, the quotient rings $R_{\boldsymbol{n}}/I_{\mathcal{M}_{\boldsymbol{n}}}$ and $R'_{\boldsymbol{n}}/I_{\mathcal{M}'_{\boldsymbol{n}}}$ are isomorphic.

For every $F_j \in \Delta$, the sets $[\boldsymbol{c}_{F_j \setminus T}]$ and $[c'_j]$ have the same finite cardinality. Choose a bijection

$$\psi_j : [\boldsymbol{c}_{F_j \setminus T}] \longrightarrow [c'_j].$$

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These choices determine two further bijections

$$(\psi_1, \dots, \psi_q, \mathrm{id}_{[\boldsymbol{n}]}) : [\boldsymbol{c}_{F_1\setminus T}] \times \cdots \times [\boldsymbol{c}_{F_q\setminus T}] \times [\boldsymbol{n}] \longrightarrow [c_1'] \times \cdots \times [c_q'] \times [\boldsymbol{n}], \tag{3-1}$$

$$(\psi_j, \mathrm{id}_{[n_j]}) : [\boldsymbol{c}_{F_j\setminus T}] \times [n_j] \longrightarrow [c_j'] \times [n_j]. \tag{3-2}$$

Bijection (3-1) induces the isomorphism of polynomial rings

$$\Psi : R_{(\boldsymbol{c},\boldsymbol{n})} = \mathbb{K}[x_{\boldsymbol{i}_{F_1\setminus T},\dots,\boldsymbol{i}_{F_q\setminus T},\boldsymbol{k}} \mid \boldsymbol{i}_{F_q\setminus T} \in [\boldsymbol{c}_{F_q\setminus T}], \boldsymbol{k} \in [\boldsymbol{n}]] \longrightarrow \mathbb{K}[x_{i_1,\dots,i_q,\boldsymbol{k}} \mid i_j \in [c_j'], \boldsymbol{k} \in [\boldsymbol{n}]] = R'_{\boldsymbol{n}},$$

$$x_{\boldsymbol{i}_{F_1\setminus T},\dots,\boldsymbol{i}_{F_q\setminus T},\boldsymbol{k}} \longmapsto x_{\psi_1(\boldsymbol{i}_{F_1\setminus T}),\dots,\psi_q(\boldsymbol{i}_{F_q\setminus T}),\boldsymbol{k}}.$$

Similarly, bijection (3-2) induces the isomorphism of polynomial rings

$$\Psi' : S_{\boldsymbol{n}} = \mathbb{K}[y_{j,\boldsymbol{i}_{F_j\setminus T},k_j} \mid 1 \le j \le q, \boldsymbol{i}_{F_j\setminus T} \in [\boldsymbol{c}_{F_j\setminus T}], k_j \in [n_j]]$$
$$\longrightarrow \mathbb{K}[y_{j,i_j,k_j} \mid 1 \le j \le q, i_j \in [c_j], k_j \in [n_j]] = S'_{\boldsymbol{n}},$$

$$y_{j,\boldsymbol{i}_{F_j\setminus T},k_j} \longmapsto y_{j,\psi_j(\boldsymbol{i}_{F_j\setminus T}),k_j}.$$

We claim that the following diagram is commutative:

$$\begin{array}{ccc} R_{(\boldsymbol{c},\boldsymbol{n})} & \xrightarrow{\Phi_{\mathcal{M}}} & S_{\boldsymbol{n}} \\ \Psi \downarrow & & \downarrow \Psi' \\ R'_{\boldsymbol{n}} & \xrightarrow{\Phi_{\mathcal{M}'}} & S'_{\boldsymbol{n}} \end{array} \tag{3-3}$$

Indeed, it suffices to check this for variables. In this case commutativity is shown by the following diagram:

$$\begin{array}{ccc} x_{\boldsymbol{i}_{F_1\setminus T},\dots,\boldsymbol{i}_{F_q\setminus T},\boldsymbol{k}} & \xmapsto{\Phi_{\mathcal{M}}} & \prod_{j=1}^q y_{j,\boldsymbol{i}_{F_j\setminus T},k_j} \\ \Psi \downarrow & & \downarrow \Psi' \\ x_{\psi_1(\boldsymbol{i}_{F_1\setminus T}),\dots,\psi_q(\boldsymbol{i}_{F_q\setminus T}),\boldsymbol{k}} & \xmapsto{\Phi_{\mathcal{M}'}} & \prod_{j=1}^q y_{j,\psi_1(\boldsymbol{i}_{F_q\setminus T}),k_j} \end{array}$$

Since Ψ and Ψ' are isomorphisms, the commutativity of diagram (3-3) implies that $\mathrm{im}(\Phi) \cong \mathrm{im}(\Phi')$, which concludes the proof. □

We also need the following result.

Proposition 3.8. *Let $\mathscr{I} = \{I_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{N}^q}$ be the $\mathcal{S}_\infty^q$-invariant filtration corresponding to hierarchical models $\mathcal{M}(\Delta, (\boldsymbol{c}, \boldsymbol{n}))$ with Δ consisting of q 2-element disjoint facets $F_1, \dots, F_q$, each meeting T in exactly one vertex. Then the equivariant Hilbert series of $\mathscr{I}$ is a rational function in $s_1, \dots, s_q, t$ with rational coefficients.*

This will be shown in the following section. Assuming the result, we complete the argument for establishing Theorem 3.5.

Proof of Theorem 3.5. Let ν be the number of facets in Δ whose intersection with T is empty. We use induction on $\nu \ge 0$. If $\nu = 0$, the claimed rationality follows by combining Lemma 3.7 and Proposition 3.8.

Let $\nu \ge 1$. We may assume that $F_1 \cap T = \varnothing$ and that vertex 1 is in F_1. By assumption, it has c_1 states. Set $\tilde{\boldsymbol{n}} = (n_1, \boldsymbol{n})$, $\tilde{\boldsymbol{c}} = (c_2, \dots, c_{\#T})$ and $\widetilde{T} = T \cup \{1\}$. Then the hierarchical models $\widetilde{\mathcal{M}}(\Delta, (\tilde{\boldsymbol{c}}, \tilde{\boldsymbol{n}}))$ give rise to a

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filtration $\widetilde{\mathscr{F}} = \mathscr{I}_{\Delta, \boldsymbol{r}_{[\boldsymbol{m}]\setminus\widetilde{T}}}$. By induction on ν, it has a rational equivariant Hilbert series. By definition, it is

$$\operatorname{equivH}_{\widetilde{\mathscr{F}}}(s_1, s_2, \ldots, s_{q-\nu+1}, t) = \sum_{n_1 \geq 1} s_1^{n_1} \cdot \operatorname{equivH}_{\mathscr{I}}(s_2, \ldots, s_{q-\nu+1}, t).$$

Hence $\operatorname{equivH}_{\mathscr{I}}$ is obtained by evaluating $(1/c_1!)(\partial^{c_1} \operatorname{equivH}_{\widetilde{\mathscr{F}}}/\partial s_1^{c_1})$ at $s_1 = 0$. It follows that also $\operatorname{equivH}_{\mathscr{I}}$ is rational. □

4. Regular languages

The goal of this section is to establish Proposition 3.8. We adopt its notation.

Fix $\boldsymbol{c} \in \mathbb{N}^q$. As above, we write $x_{\boldsymbol{i},\boldsymbol{k}}$ for $x_{i_1,\ldots,i_q,k_1,\ldots,k_q}$, where $(\boldsymbol{i},\boldsymbol{k}) = (i_1, \ldots, i_q, k_1, \ldots, k_q) \in [\boldsymbol{c}] \times [\boldsymbol{n}] \subset \mathbb{N}^{2q}$. Thus, $y_{j,\boldsymbol{i}_{F_j},\boldsymbol{k}_{F_j}}$ is simply y_{j,i_j,k_j}. For any $\boldsymbol{n} \in \mathbb{N}^q$, the homomorphism associated to the model $\mathcal{M}_{\boldsymbol{n}} = \mathcal{M}(\Delta, (\boldsymbol{c}, \boldsymbol{n}))$ is

$$\Phi_{\boldsymbol{n}} : R_{\boldsymbol{n}} = \mathbb{K}[x_{\boldsymbol{i},\boldsymbol{k}} \mid (\boldsymbol{i},\boldsymbol{k}) \in [\boldsymbol{c}] \times [\boldsymbol{n}]] \longrightarrow \mathbb{K}[y_{j,i_j,k_j} \mid j \in [q],\ i_j \in [c_j],\ k_j \in [n_j]] = S_{\boldsymbol{n}},$$

$$x_{\boldsymbol{i},\boldsymbol{k}} \longmapsto \prod_{j=1}^{q} y_{j,i_j,k_j}.$$

Set

$$A_{\boldsymbol{n}} = \operatorname{im} \Phi_{\boldsymbol{n}} = \mathbb{K}\left[\prod_{j=1}^{q} y_{j,i_j,k_j} \;\middle|\; i_j \in [c_j],\ k_j \in [n_j]\right].$$

We denote the sets of monomials in $A_{\boldsymbol{n}}$ and $S_{\boldsymbol{n}}$ by $\operatorname{Mon}(A_{\boldsymbol{n}})$ and $\operatorname{Mon}(S_{\boldsymbol{n}})$, respectively. Define $\operatorname{Mon}(A)$ as the disjoint union of the sets $\operatorname{Mon}(A_{\boldsymbol{n}})$ with $\boldsymbol{n} \in \mathbb{N}^q$ and similarly define $\operatorname{Mon}(S)$, where $S = \mathbb{K}[y_{j,i_j,k} \mid j \in [q],\ i_j \in [c_j],\ k \in \mathbb{N}]$. Our next goal is to show that the elements of $\operatorname{Mon}(A)$ are in bijection to the words of a suitable formal language.

Consider a set

$$\Sigma = \{\zeta_{\boldsymbol{i}}, \tau_j \mid \boldsymbol{i} \in [\boldsymbol{c}],\ j \in [q]\}$$

with $q + \prod_{j=1}^{q} c_j$ elements. Let Σ^* be the free monoid on Σ. A *formal language* with words in the alphabet Σ is a subset of Σ^*. We refer to the elements of Σ as letters. The empty word is denoted by ε.

In order to compare subsets of Σ^* with $\operatorname{Mon}(A)$ we need suitable maps. For $j \in [q]$, define a shift operator $T_j : \operatorname{Mon}(S) \to \operatorname{Mon}(S)$ by

$$T_j(y_{l,i,k}) = \begin{cases} y_{l,i,k+1} & \text{if } l = j, \\ y_{l,i,k} & \text{if } l \neq j, \end{cases}$$

extended multiplicatively to $\operatorname{Mon}(S)$. Define a map $\boldsymbol{m} : \Sigma^\star \to \operatorname{Mon}(S)$ inductively using the three rules

(a) $\boldsymbol{m}(\varepsilon) = 1$,

(b) $\boldsymbol{m}(\zeta_{\boldsymbol{i}} w) = \prod_{j=1}^{q} y_{j,i_j,1} \boldsymbol{m}(w)$,

(c) $\boldsymbol{m}(\tau_j w) = T_j(\boldsymbol{m}(w))$,

where $w \in \Sigma^*$. In particular, this gives $\boldsymbol{m}(\zeta_{\boldsymbol{i}}) = \Phi_{\boldsymbol{n}}(x_{\boldsymbol{i},\boldsymbol{1}})$ for any $\boldsymbol{n} \in \mathbb{N}^q$, where $\boldsymbol{1}$ is the q-tuple whose entries are all equal to 1.

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Example 4.1. If $c_1 = c_2 = q = 2$, one has $\Sigma = \{\zeta_{1,1}, \zeta_{1,2}, \zeta_{2,1}, \zeta_{2,2}, \tau_1, \tau_2\}$, and, for any $\boldsymbol{n} \geq (2, 3)$,

$$\begin{aligned}\boldsymbol{m}(\tau_1\tau_2\zeta_{1,2}\tau_2\zeta_{1,1}\tau_1) &= T_1(T_2(y_{1,1,1}y_{2,2,1}T_2(y_{1,1,1}y_{2,1,1}T_1(1))))\\ &= T_1(T_2(y_{1,1,1}y_{2,2,1}y_{1,1,1}y_{2,1,2}))\\ &= y_{1,1,2}y_{2,2,2}y_{1,1,2}y_{2,1,3}\\ &= \Phi_{\boldsymbol{n}}(x_{(1,2),(2,2)})\Phi_{\boldsymbol{n}}(x_{(1,1),(2,3)}).\end{aligned}$$

The map $\boldsymbol{m}$ is certainly not injective because the variables $y_{j,i,k}$ commute. For example, if $q = 2$ one has $\boldsymbol{m}(\tau_1\tau_2) = \boldsymbol{m}(\tau_2\tau_1)$ and $\boldsymbol{m}(\zeta_{2,1}\zeta_{1,2}) = \boldsymbol{m}(\zeta_{1,2}\zeta_{2,1}) = \boldsymbol{m}(\zeta_{1,1}\zeta_{2,2})$ and $\boldsymbol{m}(\tau_1\zeta_{1,2}\tau_2\zeta_{2,1}) = \boldsymbol{m}(\tau_1\zeta_{2,2}\tau_2\zeta_{1,1})$. Thus, we introduce a suitable subset of Σ^*.

Definition 4.2. Let $\mathcal{L}$ be the set of words in Σ^* that satisfy the following conditions:

(1) Every substring $\tau_i\tau_j$ has $i \leq j$.

(2) In every substring with no τ_j, if $\zeta_{\boldsymbol{i}}$ occurs to the left of some $\zeta_{\boldsymbol{i}'}$, then the j-th entry of $\boldsymbol{i}$ is less than or equal to the j-th entry of $\boldsymbol{i}'$.

To avoid triple subscripts below, we denote the j-th entry of a q-tuple $\boldsymbol{k}_l$ by $k_{(l,j)}$; that is, we write

$$\boldsymbol{k}_l = (k_{(l,1)}, k_{(l,2)}, \ldots, k_{(l,q)}) \in \mathbb{N}^q.$$

Using multi-indices, we write τ^a for $\tau_1^{a_1}\tau_2^{a_2}\ldots\tau_q^{a_q}$ with $a = (a_1, a_2, \ldots, a_q)$. A string consisting only of τ-letters can be written as $\tau^{\boldsymbol{k}}$ if and only if it satisfies condition (1) in Definition 4.2. With this notation, one gets immediately the following explicit description of the words in $\mathcal{L}$.

Lemma 4.3. *The elements of the formal language $\mathcal{L}$ are precisely the words of the form*

$$\tau^{\boldsymbol{k}_1}\zeta_{\boldsymbol{i}_1}\tau^{\boldsymbol{k}_2}\zeta_{\boldsymbol{i}_2}\ldots\tau^{\boldsymbol{k}_d}\zeta_{\boldsymbol{i}_d}\tau^{\boldsymbol{k}_{d+1}},$$

where $\boldsymbol{i}_1, \ldots, \boldsymbol{i}_d \in [\boldsymbol{c}]$, $\boldsymbol{k}_1, \ldots, \boldsymbol{k}_{d+1} \in \mathbb{N}_0^q$, and $i_{(l-1,j)} \leq i_{(l,j)}$ whenever $k_{(l,j)} = 0$ for some (l, j) with $2 \leq l \leq d$ and $j \in [q]$.

The following elementary observation is useful.

Lemma 4.4. *Every monomial in* $\mathrm{Mon}(A)$ *can be uniquely written as a string of variables such that one has the variable in any position l is of the form y_{j,i_j,k_j} with $j = l \bmod q$ and, for each $j \in [q]$, if a variable y_{j,i_j,k_j} appears to the left of y_{j,i'_j,k'_j}, then either $k_j < k'_j$ or $k_j = k'_j$ and $i_j \leq i'_j$.*

Proof. If for some j, two variables y_{j,i_j,k_j} and y_{j,i'_j,k'_j} appearing in a monomial do not satisfy the stated condition, then swap their positions. Repeating this step as long as needed results in a string meeting the requirement. It is unique, because the given condition induces an order on the variables $y_{j,i,k}$ with fixed j. In the desired string, for each fixed j, the variables $y_{j,i,k}$ occur in this order when one reads the string from left to right. □

We illustrate the above argument.

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Example 4.5. Let $q = 2$. To simplify notation write y_{jk} instead of $y_{1,j,k}$ and z_{jk} instead of $y_{2,j,k}$. Then one gets, for example,

$$y_{22}z_{21}y_{14}z_{11}y_{31}z_{21} \begin{array}{c} \nearrow \\ \searrow \end{array} \begin{array}{c} y_{22}y_{14}y_{31} \\ z_{21}z_{11}z_{21} \end{array} \begin{array}{c} \text{———} \\ \text{———} \end{array} \begin{array}{c} y_{31}y_{22}y_{14} \\ z_{11}z_{21}z_{21} \end{array} \begin{array}{c} \searrow \\ \nearrow \end{array} y_{31}z_{11}y_{22}z_{21}y_{14}z_{21}.$$

We observed above that the map $\boldsymbol{m}$ sends each letter $\zeta_{\boldsymbol{i}}$ to the monomial $\Phi_{\boldsymbol{n}}(x_{\boldsymbol{i},\mathbf{1}})$. It follows that $\boldsymbol{m}(\Sigma^*)$ is a subset of $\mathrm{Mon}(A)$. In fact, one has the following result.

Proposition 4.6. *For any $\boldsymbol{n} \in \mathbb{N}_0^q$, denote by $\mathcal{L}_{\boldsymbol{n}}$ the set of words in $\mathcal{L}$ in which, for each $j \in [q]$, the letter τ_j occurs precisely n_j times. Then $\boldsymbol{m}$ induces for every $\boldsymbol{n} \in \mathbb{N}_0^q$ a bijection*

$$\boldsymbol{m}_{\boldsymbol{n}} : \mathcal{L}_{\boldsymbol{n}} \to \mathrm{Mon}(A_{\boldsymbol{n}+\mathbf{1}}), \quad w \mapsto \boldsymbol{m}(w).$$

Proof. The definition of $\boldsymbol{m}$ readily implies $\boldsymbol{m}(w) \in \mathrm{Mon}(A_{\boldsymbol{n}+\mathbf{1}})$ if $w \in \mathcal{L}_{\boldsymbol{n}}$.

First we show that $\boldsymbol{m}_{\boldsymbol{n}}$ is surjective. Let $m \in \mathrm{Mon}(A_{\boldsymbol{n}+\mathbf{1}})$ be any monomial. Its degree is dq for some $d \in \mathbb{N}_0$. By Lemma 4.4, m can be written as

$$m = \prod_{l=1}^{d}\Big(\prod_{j=1}^{q} y_{j,i_{(l,j)},k_{(l,j)}}\Big) = \prod_{l=1}^{d} \Phi_{\boldsymbol{n}}(x_{\boldsymbol{i}_l,\boldsymbol{k}_l})$$

such that, for each $j \in [q]$, one has

$$1 \le k_{(1,j)} \le \cdots \le k_{(d,j)} \le n_j + 1,$$
$$i_{(l-1,j)} \le i_{(l,j)} \quad \text{if } k_{(l,j)} = 0 \text{ for some } l.$$

The first condition implies that all the q-tuples $\boldsymbol{k}_1 - \mathbf{1}, \boldsymbol{k}_2 - \boldsymbol{k}_1, \ldots, \boldsymbol{k}_d - \boldsymbol{k}_{d-1}$ and $\boldsymbol{n} + \mathbf{1} - \boldsymbol{k}_d$ are in $\mathbb{N}_0^q$. Hence the string

$$w = \tau^{\boldsymbol{k}_1-\mathbf{1}}\zeta_{\boldsymbol{i}_1}\tau^{\boldsymbol{k}_2-\boldsymbol{k}_1}\zeta_{\boldsymbol{i}_2}\cdots\tau^{\boldsymbol{k}_d-\boldsymbol{k}_{d-1}}\zeta_{\boldsymbol{i}_d}\tau^{\boldsymbol{n}+\mathbf{1}-\boldsymbol{k}_d}$$

is defined. The two conditions together combined with Lemma 4.3 show that in fact m is in $\mathcal{L}_{\boldsymbol{n}}$. Hence $\boldsymbol{m}(w) = m$ proves the claimed surjectivity.

Second, we establish that $\boldsymbol{m}_{\boldsymbol{n}}$ is injective. Consider any two words $w, w' \in \mathcal{L}_{\boldsymbol{n}}$ with $\boldsymbol{m}(w) = \boldsymbol{m}(w')$. We will show $w = w'$.

Write w and w' as in Lemma 4.3:

$$w = \tau^{\boldsymbol{k}_1}\zeta_{\boldsymbol{i}_1}\tau^{\boldsymbol{k}_2}\zeta_{\boldsymbol{i}_2}\cdots\tau^{\boldsymbol{k}_d}\zeta_{\boldsymbol{i}_d}\tau^{\boldsymbol{k}_{d+1}}, \quad w' = \tau^{\boldsymbol{k}'_1}\zeta_{\boldsymbol{i}'_1}\tau^{\boldsymbol{k}'_2}\zeta_{\boldsymbol{i}'_2}\cdots\tau^{\boldsymbol{k}'_{d'}}\zeta_{\boldsymbol{i}'_{d'}}\tau^{\boldsymbol{k}'_{d'+1}}.$$

Since $\boldsymbol{m}(w)$ has degree dq and $\boldsymbol{m}(w')$ has degree $d'q$, we conclude $d = d'$. Evaluating $\boldsymbol{m}$ we obtain

$$\prod_{l=1}^{d}\Big(\prod_{j=1}^{q} y_{j,i_{(l,j)},f_{(l,j)}}\Big) = \prod_{e=1}^{d}\Big(\prod_{j=1}^{q} y_{j,i'_{(l,j)},f'_{(l,j)}}\Big), \tag{4-1}$$

where $f_{(l,j)} = k_{(1,j)} + \cdots + k_{(l,j)} + 1$ and $f'_{(l,j)} = k'_{(1,j)} + \cdots + k'_{(l,j)} + 1$. Fix any $j \in [q]$. Comparing the third indices of the variables whose first index equals j and using that every index is nonnegative, we get

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for each $l \in [d]$

$$k_{(1,j)} + \cdots + k_{(l,j)} = k'_{(1,j)} + \cdots + k'_{(l,j)}.$$

It follows that $\boldsymbol{k}_l = \boldsymbol{k}'_l$ for each $l \in [d]$. Since w and w' are in $\mathcal{L}_{\boldsymbol{n}}$, we have $\boldsymbol{k}_{d+1} = \boldsymbol{n} - (\boldsymbol{k}_1 + \boldsymbol{k}_2 + \cdots + \boldsymbol{k}_d)$ and an analogous equation for $\boldsymbol{k}'_{d+1}$, which gives $\boldsymbol{k}_{d+1} = \boldsymbol{k}'_{d+1}$.

It remains to show $\boldsymbol{i}_l = \boldsymbol{i}'_l$ for every $l \in [d]$. Fix any $j \in [q]$. If for some l there is only one variable of the form $y_{j,\mu,f_{(l,j)}}$ with $\mu \in [c_j]$ that divides $\boldsymbol{m}(w)$, this implies $i_{(l,j)} = i'_{(l,j)} = \mu$, as desired. Otherwise, there is a maximal interval of consecutive indices $k_{(l,j)}$ that are equal to zero; that is, there are integers a, b such that $1 \le a \le b \le d$ and

- $k_{(l,j)} = 0$ if $a \le l \le b$,
- $k_{(a-1,j)} > 0$, unless $a = 1$, and
- $k_{(b+1,j)} > 0$, unless $b = d$.

Thus, the number of variables of the form $y_{j,\mu,f_{(l,j)}}$ that divide $\boldsymbol{m}(w)$ is $b - a + 2$ if $a \ge 2$ and $b - a + 1$ if $a = 1$. Considering these variables, Lemma 4.3 gives

$$i_{(a-1,j)} \le i_{(a,j)} \le \cdots \le i_{(b,j)} \quad \text{and} \quad i'_{(a-1,j)} \le i'_{(a,j)} \le \cdots \le i'_{(b,j)},$$

where $i_{(a-1,j)}$ and $i'_{(a-1,j)}$ are omitted if $a = 1$. Using (4-1), it now follows that $i_{(l,j)} = i'_{(l,j)}$ whenever $a - 1 \le l \le b$, unless $a = 1$. If $a = 1$, the latter equality is true whenever $a \le l \le b$.

Applying the latter argument to any interval of consecutive zero indices $k_{(l,j)}$, we conclude $i_{(l,j)} = i'_{(l,j)}$ for every $l \in [d]$. This completes the argument. □

Our next goal is to show that $\mathcal{L}$ is a regular language. By [10, Theorems 3.4 and 3.7], this is equivalent to proving that $\mathcal{L}$ is recognizable by a finite automaton. Recall that a *deterministic finite automaton* on an alphabet Σ is a 5-tuple $\mathcal{A} = (P, \Sigma, \delta, p_0, F)$ consisting of a finite set P of *states*, an *initial state* $p_0 \in P$, a set $F \subset P$ of *accepting states* and a *transition map* $\delta : D \to P$, where D is some subset of $P \times \Sigma$. We refer to $\mathcal{A}$ simply as a finite automaton because we will consider only deterministic automata. The automaton $\mathcal{A}$ *recognizes* or *accepts* a word $w = a_1 a_2 \cdots a_s \in \Sigma^*$ if there is a sequence of states $r_0, r_1, \ldots, r_s$ satisfying $r_0 = p_0$, $r_s \in F$ and

$$r_{j+1} = \delta(r_j, a_{j+1}) \quad \text{whenever } 0 \le j < s.$$

In words, the automaton starts in state p_0 and transitions from state r_j to a state r_{j+1} based on the input a_{j+1}. The word w is accepted if r_s is an accepting state. If $\delta(p, a)$ is not defined the machine halts. The automaton $\mathcal{A}$ *recognizes a formal language* $\mathcal{L} \subset \Sigma^*$ if $\mathcal{L}$ is precisely the set of words in Σ^* that are accepted by $\mathcal{A}$.

Returning to the formal language $\mathcal{L}$ specified in Definition 4.2, we are ready to show:

Proposition 4.7. *The language $\mathcal{L}$ is recognized by a finite automaton.*

Proof. We need some further notation. We say that a sequence C of $l \ge 0$ integers $j_1, j_2, \ldots, j_l$ is an *increasing chain in* $[q]$ if $1 \le j_1 < j_2 < \cdots < j_l \le q$. Define $\max(C)$ as the largest element j_l of C. We put $\max(\varnothing) = 0$. We denote the set of increasing chains in $[q]$ by $\mathcal{C}$. Thus, the cardinality of $\mathcal{C}$ is 2^q. We

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write $j \in C$ if j occurs in the chain C. For any $\boldsymbol{k} \in \mathbb{N}_0^q$, we define the sequence of indices j with $k_j > 0$ as its support $\mathrm{Supp}(\boldsymbol{k})$. It is an element of $\mathcal{C}$. For example, one has $\mathrm{Supp}(7, 0, 1, 5, 0) = (1, 3, 4)$.

Now we define an automaton $\mathcal{A}$ as follows: Let

$$P = \{p_j, p_{\boldsymbol{i}}, p_{\boldsymbol{i},C,k} \mid 0 \le j \le q,\ \boldsymbol{i} \in [\boldsymbol{c}],\ C \in \mathcal{C},\ k \in C\}$$

be the set of states, where p_0 is the initial state of $\mathcal{A}$. Let

$$F = \{p_j, p_{\boldsymbol{i}}, p_{\boldsymbol{i},C,k} \mid 0 \le j \le q,\ \boldsymbol{i} \in [\boldsymbol{c}],\ C \in \mathcal{C},\ k = \max(C)\}$$

be the set of accepting states. Furthermore, define transitions

$$\delta(p_j, \tau_{j'}) = p_{j'} \qquad \text{if } j = 0 < j' \le q \text{ or } 1 \le j \le j' \le q, \tag{4-2}$$

$$\delta(p_j, \zeta_{\boldsymbol{i}}) = p_{\boldsymbol{i}} \qquad \text{if } 0 \le j \le q,\ \boldsymbol{i} \in [\boldsymbol{c}], \tag{4-3}$$

$$\delta(p_{\boldsymbol{i}}, \tau_j) = p_{\boldsymbol{i},C,j} \qquad \text{if } \boldsymbol{i} \in [\boldsymbol{c}],\ C \in \mathcal{C},\ j \in C, \tag{4-4}$$

$$\delta(p_{\boldsymbol{i}}, \zeta_{\boldsymbol{i}'}) = p_{\boldsymbol{i}'} \qquad \text{if } \boldsymbol{i}, \boldsymbol{i}' \in [\boldsymbol{c}],\ \boldsymbol{i} \le \boldsymbol{i}', \tag{4-5}$$

$$\delta(p_{\boldsymbol{i},C,j}, \tau_k) = p_{\boldsymbol{i},C,k} \qquad \text{if } \boldsymbol{i} \in [\boldsymbol{c}],\ C \in \mathcal{C},\ j \in C,\ k \text{ directly follows } j \text{ in } C \text{ or } k = j, \tag{4-6}$$

$$\delta(p_{\boldsymbol{i},C,j}, \zeta_{\boldsymbol{i}'}) = p_{\boldsymbol{i}'} \qquad \text{if } \boldsymbol{i}, \boldsymbol{i}' \in [\boldsymbol{c}],\ j = \max(C),\ i_k \le i'_k \text{ whenever } k \notin C. \tag{4-7}$$

If an element of $P \times \Sigma$ does not satisfy any of the above six conditions then it is not in the domain of δ.

We claim that $\mathcal{A}$ recognizes $\mathcal{L}$. Indeed, let $w \in \Sigma^*$ be a word with exactly $d \ge 0$ ζ-letters. We show by induction on d that w is recognized by $\mathcal{A}$ if $w \in \mathcal{L}$, but any word in $\Sigma^* \setminus \mathcal{L}$ is not accepted by $\mathcal{A}$. It turns out that $w \in \mathcal{L}$ is accepted

- at a state p_j for some $0 \le j \le q$ if $d = 0$,
- at a state $p_{\boldsymbol{i}}$ for some $\boldsymbol{i} \in [c]$ if $d \ge 1$ and w ends with a ζ-letter, and
- at a state $p_{\boldsymbol{i},C,j}$ for some $\boldsymbol{i} \in [\boldsymbol{c}],\ C \in \mathcal{C},\ j = \max(C)$ if $d \ge 1$ and w ends with a τ-letter.

In particular, this explains the set of accepting states.

Consider any word $w \in \Sigma^*$ with exactly $d \ge 0$ ζ-letters. Assume $d = 0$, that is, $w = \tau_{l_1}\tau_{l_2}\cdots\tau_{l_t}$. By transition rule (4-2), $\mathcal{A}$ transitions from state p_0 to any state p_j with $j \in [q]$ using input τ_j. From any p_j with $j \in [q]$ the automaton can transition to any state $p_{j'}$ with $j \le j' \le q$ by using input $\tau_{j'}$. Thus, w is accepted by $\mathcal{A}$ if and only if $l_1 \le l_2 \le \cdots \le l_t$, that is, $w \in \mathcal{L}$ (see Lemma 4.3).

Assume now that $d \ge 1$. We proceed in several steps.

(I) Assume $d = 1$ and w ends with a ζ-letter, that is,

$$w = \tau_{l_1}\tau_{l_2}\cdots\tau_{l_t}\zeta_{\boldsymbol{i}}$$

for some $t \ge 0$. The argument for $d = 0$ shows that $\tau_{l_1}\tau_{l_2}\cdots\tau_{l_t}$ is accepted if and only if it can be written as some $\tau^{\boldsymbol{k}}$. Processing input $\tau^{\boldsymbol{k}}$, the automaton arrives at state p_j with $j = \max(\mathrm{Supp}(\boldsymbol{k}))$. Using input $\zeta_{\boldsymbol{i}}$, it then transitions to $p_{\boldsymbol{i}} \in F$ by rule (4-3). Hence w is accepted if and only if $w \in \mathcal{L}$.

(II) Let $d \ge 1$ and assume w ends with a τ-letter, that is, w can be written as

$$w = w'\zeta_{\boldsymbol{i}}\tau_{l_1}\tau_{l_2}\cdots\tau_{l_t},$$

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with $t \geq 1$. Furthermore assume that $w'\zeta_{\boldsymbol{i}}$ is accepted by $\mathcal{A}$ in state $p_{\boldsymbol{i}}$. We show that w is accepted by $\mathcal{A}$ if and only if $w = w'\zeta_{\boldsymbol{i}}\tau^{\boldsymbol{k}}$ for some $\boldsymbol{k} \in \mathbb{N}_0^q$. If w is recognized, it is accepted in state $p_{\boldsymbol{i},C,\max(C)}$, where $C = \mathrm{Supp}(\boldsymbol{k})$.

Indeed, let $C \in \mathcal{C}$ be the chain corresponding to the set $\{l_1, \ldots, l_t\}$. Processing input τ_{l_1}, rule (4-3) yields that $\mathcal{A}$ transitions to state $p_{\boldsymbol{i},C,l_1}$. If $t = 1$, then $l_1 = \max(C)$ and w is accepted in $p_{\boldsymbol{i},C,l_1} \in F$, as claimed. If $t \geq 2$, rule (4-6) shows that $\mathcal{A}$ can transition from $p_{\boldsymbol{i},C,l_1}$ using input τ_{l_2} precisely if $l_2 \geq l_1$. If transition is possible, $\mathcal{A}$ gets to state $p_{\boldsymbol{i},C,l_2}$. Hence rule (4-6) guarantees that $\tau_{l_1}\tau_{l_2}\cdots\tau_{l_t}$ can be processed by $\mathcal{A}$ if and only if $\tau_{l_1}\tau_{l_2}\cdots\tau_{l_t} = \tau^{\boldsymbol{k}}$ for some nonzero $\boldsymbol{k} \in \mathbb{N}_0^q$. In this case $w = w'\zeta_{\boldsymbol{i}}\tau^{\boldsymbol{k}}$ is accepted by $\mathcal{A}$ in state $p_{\boldsymbol{i},C,\max(C)}$, where $C = \mathrm{Supp}(\boldsymbol{k})$.

(III) Assume now $w \in \Sigma^*$ ends with a ζ-letter; that is, w is of the form

$$w = w'\tau_{l_1}\tau_{l_2}\cdots\tau_{l_t}\zeta_{\boldsymbol{i}},$$

where $w' \in \mathcal{L}$ is either empty or ends with a ζ-letter and $t \geq 0$. We show by induction on $d \geq 1$ that w is recognized by $\mathcal{A}$ if and only if $w \in \mathcal{L}$. In this case, w is accepted in a state $p_{\boldsymbol{i}}$.

Indeed, if $d = 1$, i.e., w' is the empty word, this has been shown in step (I). If $d \geq 2$ write $w' = w''\zeta_{\boldsymbol{i}'}$. If w' is not accepted by $\mathcal{A}$, then neither is w. Furthermore, the induction hypothesis gives $w' \notin \mathcal{L}$, which implies $w \notin \mathcal{L}$.

If $w' = w''\zeta_{\boldsymbol{i}'}$ is recognized by $\mathcal{A}$ the induction hypothesis yields $w' \in \mathcal{L}$ and w' is accepted in state $p_{\boldsymbol{i}'}$. Step (II) shows that $w''\zeta_{\boldsymbol{i}'}\tau_{l_1}\tau_{l_2}\cdots\tau_{l_t}$ is accepted by $\mathcal{A}$ if and only if it can be written as $w''\zeta_{\boldsymbol{i}'}\tau^{\boldsymbol{k}}$ for some $\boldsymbol{k} \in \mathbb{N}_0^q$, and so

$$w = w''\zeta_{\boldsymbol{i}'}\tau^{\boldsymbol{k}}\zeta_{\boldsymbol{i}}.$$

We consider two cases.

Case 1: Suppose $\boldsymbol{k}$ is zero, i.e., $\mathrm{Supp}(\boldsymbol{k}) = \varnothing$. Thus, $\mathcal{A}$ accepts $w''\zeta_{\boldsymbol{i}'} \in \mathcal{L}$ in state $p_{\boldsymbol{i}'}$. Using input $\zeta_{\boldsymbol{i}}$, rule (4-5) shows that $\mathcal{A}$ does not halt in $p_{\boldsymbol{i}'}$ if and only if $\boldsymbol{i}' \leq \boldsymbol{i}$. By Lemma 4.3, this is equivalent to $w = w''\zeta_{\boldsymbol{i}'}\zeta_{\boldsymbol{i}} \in \mathcal{L}$. Furthermore, if w is in $\mathcal{L}$ it is accepted in state $p_{\boldsymbol{i}}$, as claimed.

Case 2: Suppose $\mathrm{Supp}(\boldsymbol{k}) \neq \varnothing$. Set $C = \mathrm{Supp}(\boldsymbol{k})$. By step (II), $w''\zeta_{\boldsymbol{i}'}\tau^{\boldsymbol{k}}$ is accepted in state $p_{\boldsymbol{i}',C,j}$, where $j = \max(C)$. Hence rule (4-7) gives that input $\zeta_{\boldsymbol{i}}$ can be processed if and only if $i'_l \leq i_l$ whenever $l \notin C$. By Lemma 4.3, this is equivalent to $w = w''\zeta_{\boldsymbol{i}'}\tau^{\boldsymbol{k}}\zeta_{\boldsymbol{i}} \in \mathcal{L}$. Moreover, if w is recognized it is accepted in state $p_{\boldsymbol{i}}$, as claimed.

(IV) By steps (I) and (III) it remains to consider the case where w ends with a τ-letter, i.e., $w = w'\zeta_{\boldsymbol{i}}\tau_{l_1}\tau_{l_2}\cdots\tau_{l_t}$ with $t \geq 1$. By step (III), $w'\zeta_{\boldsymbol{i}}$ is recognized by $\mathcal{A}$ if and only if $w'\zeta_{\boldsymbol{i}} \in \mathcal{L}$. Furthermore, if $w'\zeta_{\boldsymbol{i}} \in \mathcal{L}$ then it is accepted in state $p_{\boldsymbol{i}}$. Hence, the assumption in step (II) is satisfied and we conclude that w is accepted if and only if $w = w'\zeta_{\boldsymbol{i}}\tau^{\boldsymbol{k}}$. The latter is equivalent to $w'\zeta_{\boldsymbol{i}}\tau^{\boldsymbol{k}} \in \mathcal{L}$ because $w'\zeta_{\boldsymbol{i}}$ is in $\mathcal{L}$. This completes the argument. □

Remark 4.8. Any finite automaton $\mathcal{A} = (P, \Sigma, \delta, p_0, F)$ can be represented by a labeled directed graph whose vertex set is the set of states P. Accepting states are indicated by double circles. There is an edge from vertex p to vertex p' if there is a transition $\delta(p, a) = p'$. In that case, the edge is labeled by all $a \in \Sigma$ such that $\delta(p, a) = p'$.

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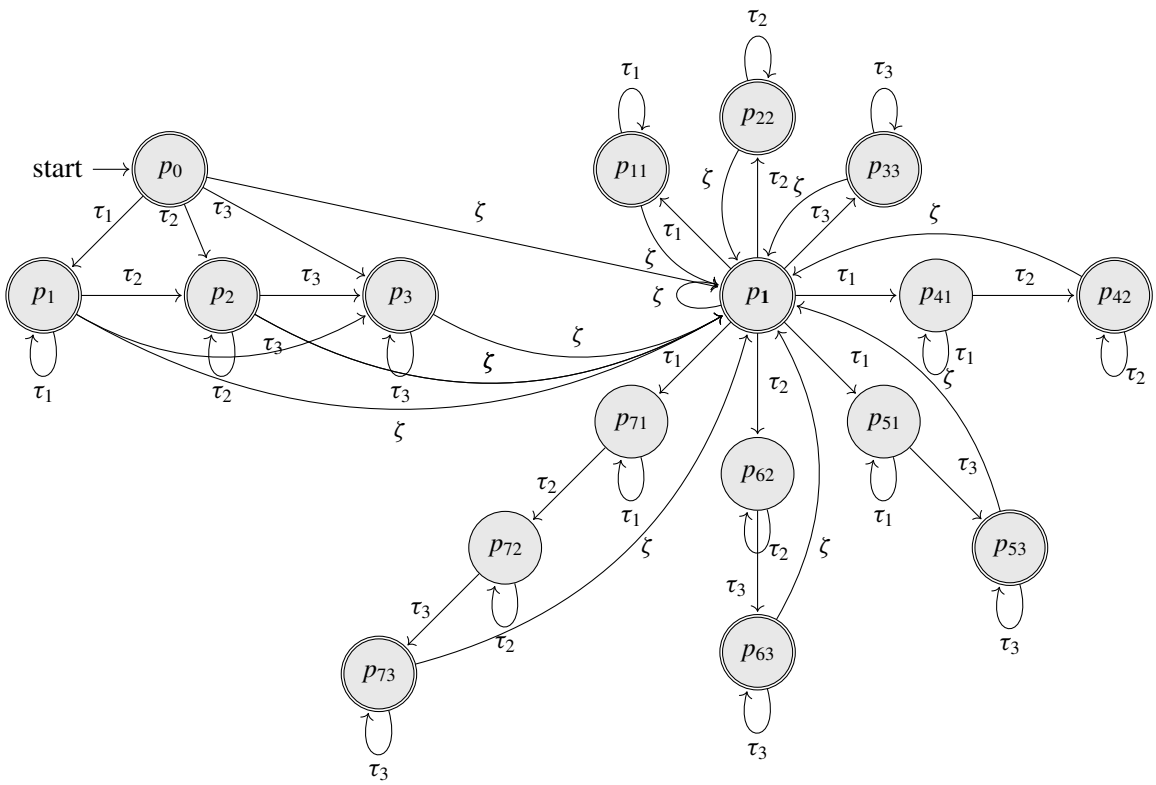

Figure 1. The automaton for $\boldsymbol{c} = (1, 1, 1)$ and $T = [3]$.

We illustrate the automata constructed in Proposition 4.7 using such a graphical representation.

Example 4.9. Let $\mathcal{A}$ be the automaton constructed in Proposition 4.7 if $q = 3$ and $\boldsymbol{c} = (1, 1, 1)$. Note the only element in $[\boldsymbol{c}]$ is $\mathbf{1} = (1, 1, 1)$. To simplify notation, we write ζ for $\zeta_{1,1,1}$ and $p_{\mathbf{1}}$ for $p_{(1,1,1)}$. We denote the nonempty increasing chains in the interval $[3]$ by $C_1 = \{1\}$, $C_2 = \{2\}$, $C_3 = \{3\}$, $C_4 = \{1, 2\}$, $C_5 = \{1, 3\}$, $C_6 = \{2, 3\}$, $C_7 = \{1, 2, 3\}$ and write $p_{i,j}$ instead of $p_{\mathbf{1},C_i,j}$. Using this notation, the constructed automaton $\mathcal{A}$ is represented by the graph in Figure 1.

Remark 4.10. The automaton constructed in Proposition 4.7 is often not the smallest automaton that recognizes the language $\mathcal{L}$. Using the minimization technique described in [10, Theorem 4.26], one can obtain an automaton with fewer states that also recognizes $\mathcal{L}$. For example, if $\boldsymbol{c} = (1, 1, 1)$, this produces an automaton with only four states, shown in Figure 2.

In order to relate a language $\mathcal{L}$ on an alphabet Σ to a Hilbert series we need a suitable weight function. Let $T = \mathbb{K}[s_1, \ldots, s_k]$ be a polynomial ring in k variables and denote by $\mathrm{Mon}(T)$ the set of monomials in T. A *weight function* is a monoid homomorphism $\rho : \Sigma^\star \to \mathrm{Mon}(T)$ such that $\rho(w) = 1$ only if w is the empty word. The corresponding generating function is a formal power series in variables $s_1, \ldots, s_k$:

$$P_{\mathcal{L},\rho}(s_1, \ldots, s_k) = \sum_{w \in \mathcal{L}} \rho(w).$$

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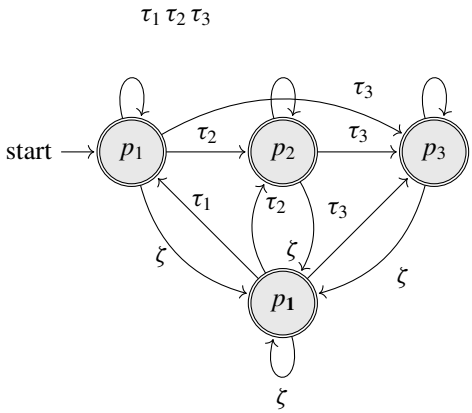

Figure 2. The reduced automaton for $\boldsymbol{c} = (1, 1, 1)$ and $T = [3]$.

We will use the following result; see, e.g., [9] or [16, Theorem 4.7.2].

Theorem 4.11. *If ρ is any weight function on a regular language $\mathcal{L}$ then $P_{\mathcal{L},\rho}$ is a rational function in $\mathbb{Q}(s_1, \ldots, s_k)$.*

We are ready to establish the ingredient of the proof of Theorem 3.5 whose proof we had postponed.

Proof of Proposition 3.8. Since $I_{\boldsymbol{n}} = \ker \Phi_n$ and Φ_n is a homomorphism of degree q, we get $R_{\boldsymbol{n}}/I_n \cong A_{\boldsymbol{n}}$ and, for each $d \in \mathbb{Z}$,

$$\dim_{\mathbb{K}}[R_{\boldsymbol{n}}/I_{\boldsymbol{n}}]_d = \dim_{\mathbb{K}}[A_{\boldsymbol{n}}]_{dq}.$$

Recall that the algebra $A_{\boldsymbol{n}}$ is generated by monomials. Hence, every graded component has a $\mathbb{K}$-basis consisting of monomials. It follows that $\dim_{\mathbb{K}}[A_{\boldsymbol{n}}]_{dq} = \#\mathrm{Mon}([A_{\boldsymbol{n}}]_{dq})$. Therefore we get for the equivariant Hilbert series of the filtration $\mathscr{I}$

$$\mathrm{equivH}_{\mathscr{I}}(s_1, \ldots, s_q, t) = \sum_{\boldsymbol{n}\in\mathbb{N}^q} \sum_{d\geq 0} \#\mathrm{Mon}([A_{\boldsymbol{n}}]_{dq}) \cdot s^{\boldsymbol{n}} t^d,$$

where $s^{\boldsymbol{n}} = s_1^{n_1} \cdots s_q^{n_q}$ if $\boldsymbol{n} = (n_1, \ldots, n_q)$.

Consider now the language $\mathcal{L}$ described in Definition 4.2. Define a weight function $\rho : \Sigma^* \to \mathrm{Mon}(\mathbb{K}[s_1, \ldots, s_q, t])$ by $\rho(\tau_j) = s_j$ and $\rho(\zeta_{\boldsymbol{i}}) = t$ for $\boldsymbol{i} \in [\boldsymbol{c}]$. Thus, for $w \in \mathcal{L}$, one obtains $\rho(w) = s^{\boldsymbol{n}} t^d$ if d is the number of ζ-letters occurring in w and n_j is the number of appearances of τ_j in w. Hence Proposition 4.6 gives that the number of words $w \in \mathcal{L}_{\boldsymbol{n}}$ with $\rho(w) = s^{\boldsymbol{n}} t^d$ is precisely $\#\mathrm{Mon}([A_{\boldsymbol{n}+\mathbf{1}}]_{dq})$. Since $\mathcal{L}$ is the disjoint union of all $\mathcal{L}_{\boldsymbol{n}}$, it follows

$$s_1 \cdots s_q \cdot \mathrm{equivH}_{\mathscr{I}}(s_1, \ldots, s_q, t) = \sum_{\boldsymbol{n}\in\mathbb{N}_0^q} \sum_{w\in\mathcal{L}_n} \rho(w) = P_{\mathcal{L},\rho}(s_1, \ldots, s_q, t). \tag{4-8}$$

As the right-hand side is rational by Theorem 4.11, the claim follows. □

Remark 4.12. The method of proof for Theorem 3.5 is rather general and can also be used in other situations. An easy generalization is obtained as follows. Fix $(a_1, \ldots, a_q) \in \mathbb{N}^q$. For $\boldsymbol{n} \in \mathbb{N}^q$, consider the

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homomorphism

$$\widetilde{\Phi}_{\boldsymbol{n}} : R_{\boldsymbol{n}} = \mathbb{K}[x_{\boldsymbol{i},\boldsymbol{k}} \mid (\boldsymbol{i},\boldsymbol{k}) \in [\boldsymbol{c}] \times [\boldsymbol{n}]] \longrightarrow \mathbb{K}[y_{j,i_j,k_j} \mid j \in [q], i_j \in [c_j], k_j \in [n_j]] = S_{\boldsymbol{n}},$$
$$x_{\boldsymbol{i},\boldsymbol{k}} \longmapsto \prod_{j=1}^{q} y_{j,i_j,k_j}^{a_j},$$

and set

$$\tilde{A}_{\boldsymbol{n}} = \operatorname{im} \Phi_{\boldsymbol{n}} = \mathbb{K}\left[\prod_{j=1}^{q} y_{j,i_j,k_j}^{a_j} \;\middle|\; i_j \in [c_j], k_j \in [n_j]\right],$$
$$\tilde{I}_{\boldsymbol{n}} = \ker \widetilde{\Phi}_{\boldsymbol{n}}.$$

Then $\widetilde{\mathscr{I}} = \{\tilde{I}_{\boldsymbol{n}}\}_{\boldsymbol{n}\in\mathbb{N}^q}$ also is an $\mathcal{S}_\infty^q$-invariant filtration whose equivariant Hilbert series is rational. Indeed, this follows using the language $\mathcal{L}$ as above with the following modifications. In the definition of the map $\boldsymbol{m}$ change rule (b) to $\tilde{\boldsymbol{m}}(\zeta_{\boldsymbol{i}} w) = \prod_{j=1}^{q} y_{j,i_j,1}^{a_j} \tilde{\boldsymbol{m}}(w)$, but keep rules (a) and (c) to obtain a map $\tilde{\boldsymbol{m}} : \Sigma^\star \to \operatorname{Mon}(S)$. It induces bijections $\mathcal{L}_n \to \operatorname{Mon}(\tilde{A}_{\boldsymbol{n}+\boldsymbol{1}})$ as in Proposition 4.6. Observe that $[R_{\boldsymbol{n}}/\tilde{I}_{\boldsymbol{n}}]_d \cong [\tilde{A}_{\boldsymbol{n}}]_{da}$, where $a = a_1 + \cdots + a_q$. Thus, using the same weight function ρ as above, we obtain $s_1 \cdots s_q \cdot \operatorname{equivH}_{\mathscr{I}}(s_1, \ldots, s_q, t) = P_{\mathcal{L},\rho}(s_1, \ldots, s_q, t)$.

A systematic study of substantial generalizations will be presented in [13].

5. Explicit formulas

We provide explicit formulas for the Hilbert series of hierarchical models considered in Theorem 3.5.

It is useful to begin by discussing Segre products more generally. To this end we temporarily use some new notation.

Lemma 5.1. *Let $A = \mathbb{K}[a_1, \ldots, a_s] \subset R$ and $B = \mathbb{K}[b_1, \ldots, b_t] \subset S$ be subalgebras of polynomial rings $R = \mathbb{K}[x_1, \ldots, x_m]$ and $S = \mathbb{K}[y_1, \ldots, y_n]$ that are generated by monomials $a_1, \ldots, a_s$ of degree d_1 and monomials $b_1, \ldots, b_t$ of degree d_2, respectively. Let C be the subalgebra of $\mathbb{K}[x_1, \ldots, x_m, y_1, \ldots, y_n]$ that is generated by all monomials $a_i b_j$ with $i \in [s]$ and $j \in [t]$. Using the gradings induced from the corresponding polynomials rings one has, for all $k \in \mathbb{Z}$,*

$$\dim_{\mathbb{K}}[C]_{k(d_1+d_2)} = \dim_{\mathbb{K}}[A]_{kd_1} \cdot \dim_{\mathbb{K}}[B]_{kd_2}.$$

Proof. This follows from the fact that the nontrivial degree components of the algebras A, B, C have $\mathbb{K}$-bases generated by monomials in the respective algebra generators of suitable degrees. □

It is customary to consider the algebras occurring in Lemma 5.1 as standard graded algebras that are generated in degree 1 by redefining their grading. In the new gradings, the degree k elements of A are elements that have degree kd_1, considered as polynomials in R, and similarly the degree-k elements of C have degree $k(d_1 + d_2)$ when considered as elements of $\mathbb{K}[x_1, \ldots, x_m, y_1, \ldots, y_n]$. Using this new grading, the statement in the above lemma reads

$$\dim_{\mathbb{K}}[C]_d = \dim_{\mathbb{K}}[A]_d \cdot \dim_{\mathbb{K}}[B]_d. \tag{5-1}$$

This justifies calling C the *Segre product* of the algebras A and B. We denote it by $A \boxtimes B$.

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Iterating the above construction we get the following consequence.

Corollary 5.2. *Let $A_1, \ldots, A_k$ be subalgebras of polynomial rings and assume every A_i is generated by finitely many monomials of degree d_i. Regrade such that every A_i is an algebra that is generated in degree* 1. *Then one has*

$$\dim_{\mathbb{K}}[A_1 \boxtimes \cdots \boxtimes A_k]_d = \prod_{i=1}^{k} \dim_{\mathbb{K}}[A_i]_d.$$

We need an elementary observation.

Lemma 5.3. *Let $\omega \in \mathbb{C}$ be a primitive k-th root of unity. If*

$$f(t) = \sum_{n=0}^{\infty} c_n t^n$$

is a formal power series in t with complex coefficients, then

$$\sum_{n=0}^{\infty} c_{kn} x^{kn} = \frac{1}{k}[f(t) + f(\omega t) + \cdots + f(\omega^{k-1} t)].$$

Proof. Using geometric sums one gets, for every $n \in \mathbb{N}_0$,

$$\sum_{j=0}^{k-1} (\omega^j)^n = \begin{cases} k & \text{if } k \text{ divides } n, \\ 0 & \text{otherwise.} \end{cases}$$

The claim follows. □

Proposition 5.4. *Fix any $q \in \mathbb{N}$ and let $\mathscr{I}$ be the $\mathcal{S}_\infty^q$-invariant filtration considered in Proposition 3.8. For $j \in [q]$, let ω_j be a c_j-th primitive root of unity. Then the equivariant Hilbert series of $\mathscr{I}$ is*

$$\operatorname{equivH}_{\mathscr{I}}(s_1, \ldots, s_q, t) = \frac{1}{c_1 \cdots c_q} \sum_{m_1 \in [c_1], \ldots, m_q \in [c_q]} \frac{\omega_1^{m_1} s_1^{1/c_1} \cdots \omega_q^{m_q} s_q^{1/c_q}}{(1 - \omega_1^{m_1} s_1^{1/c_1}) \cdots (1 - \omega_q^{m_q} s_q^{1/c_q}) - t}.$$

Proof. By definition of the map $\Phi_{\mathcal{M}_n}$, its image is isomorphic to the Segre product of polynomial rings of dimension $c_j n_j$ with $j = 1, \ldots, q$. Hence Corollary 5.2 gives for the equivariant Hilbert series

$$\begin{aligned} \operatorname{equivH}_{\mathscr{I}}(s_1, \ldots, s_q, t) &= \sum_{d \geq 0, \boldsymbol{n} \in \mathbb{N}^q} \binom{c_1 n_1 + d - 1}{d} \cdots \binom{c_q n_q + d - 1}{d} s_1^{n_1} \ldots s_q^{n_q} t^d \\ &= \sum_{d \geq 0} \left\{ \prod_{j=1}^{q} \left[\sum_{n_j \in \mathbb{N}} \binom{c_j n_j + d - 1}{d} s_j^{n_j} \right] \right\} t^d. \end{aligned} \tag{5-2}$$

For any integer $d \geq 0$, one computes

$$\sum_{n \in \mathbb{N}} \binom{n + d - 1}{d} s^n = s \sum_{n \in \mathbb{N}_0} \binom{d + n}{n} s^n = \frac{s}{(1 - s)^{d+1}}.$$

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Combined with Lemma 5.3 and using a c-th primitive root of unity $\omega \in \mathbb{C}$, we obtain, for any integer $c > 0$,

$$\sum_{n\in\mathbb{N}} \binom{cn+d-1}{d} s^n = \frac{1}{c} \sum_{m\in[c]} \frac{\omega^m s^{1/c}}{(1-\omega^m s^{1/c})^{d+1}}.$$

Applying the last formula to the inner sums in (5-2) we get

$$\begin{aligned}
\mathrm{equivH}_{\mathscr{J}}(s_1,\dots,s_q,t) &= \sum_{d\geq 0} \left\{ \prod_{j=1}^{q} \left[\frac{1}{c_j} \frac{\omega_j^m s_j^{1/c_j}}{(1-\omega_j^m s_j^{1/c_j})^{d+1}} \right] \right\} t^d \\
&= \sum_{d\geq 0} \frac{1}{c_1\cdots c_q} \left\{ \sum_{m_1\in[c_1],\dots,m_q\in[c_q]} \frac{\omega_1^{m_1} s_1^{1/c_1}}{(1-\omega_1^{m_1} s_1^{1/c_1})^{d+1}} \cdots \frac{\omega_q^{m_q} s_q^{1/c_q}}{(1-\omega_q^{m_q} s_q^{1/c_q})^{d+1}} \right\} t^d \\
&= \frac{1}{c_1\cdots c_q} \sum_{m_1\in[c_1],\dots,m_q\in[c_q]} \frac{\omega_1^{m_1} s_1^{1/c_1} \cdots \omega_q^{m_q} s_q^{1/c_q}}{(1-\omega_1^{m_1} s_1^{1/c_1})\cdots(1-\omega_q^{m_q} s_q^{1/c_q}) - t},
\end{aligned}$$

as claimed. □

By Theorem 3.5, the above formula for the equivariant Hilbert series can be rewritten as a rational function with rational coefficients.

Example 5.5. (i) Let $c_1 = \cdots = c_q = 1$. Then Proposition 5.4 gives

$$\mathrm{equivH}_{\mathscr{J}}(s_1, s_2, \dots, s_q, t) = \frac{s_1 \dots s_q}{(1-s_1)\cdots(1-s_q) - t}.$$

By the argument at the beginning of the proof of Lemma 3.7, this model has the same equivariant Hilbert series as the corresponding independence model (see Example 3.6).

(ii) Let $q = c_1 = c_2 = 2$. Then Proposition 5.4 yields

$$\begin{aligned}
4 \cdot \mathrm{equivH}_{\mathscr{J}}(s_1, s_2, t) = {} & \frac{\sqrt{s_1 s_2}}{(1-\sqrt{s_1})(1-\sqrt{s_2}) - t} - \frac{\sqrt{s_1 s_2}}{(1-\sqrt{s_1})(1+\sqrt{s_2}) - t} \\
& - \frac{\sqrt{s_1 s_2}}{(1+\sqrt{s_1})(1-\sqrt{s_2}) - t} + \frac{\sqrt{s_1 s_2}}{(1+\sqrt{s_1})(1+\sqrt{s_2}) - t}.
\end{aligned}$$

Now a straightforward computation gives

$$\mathrm{equivH}_{\mathscr{J}}(s_1, s_2, t) = \frac{s_1 s_2 (s_1 s_2 - s_1 - s_2 - t^2)}{f},$$

where

$$f = s_1 s_2 (s_1 - 2)(s_2 - 2) + s_1(s_1 - 2) + s_2(s_2 - 2) - 2t^2(s_1 s_2 + s_1 + s_2) - 4t(s_1 s_2 - s_1 - s_2) + (1-t)^4.$$

There is an alternative method to determine the equivariant Hilbert series whose rationality is guaranteed by Proposition 3.8. It directly produces a rational function with rational coefficients. This approach applies to any equivariant Hilbert series that is equal to the generating function $P_{\mathcal{L},\rho}$ determined by a weight function ρ on a regular language $\mathcal{L}$. Indeed, let $\mathcal{A} = (P, \Sigma, \delta, p_0, F)$ be a finite automaton that recognizes $\mathcal{L}$. Suppose P has N elements $p_0, \dots, p_{N-1}$. For every letter $a \in \Sigma$ define a 0-1 matrix $M_{\mathcal{A},a}$

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of size $N \times N$. Its entry at position (i, j) is 1 precisely if there is a transition $\delta(p_j, a) = p_i$. Let $\boldsymbol{e}_i \in \mathbb{K}^N$ be the canonical basis vector corresponding to state p_{i-1}. Let $\boldsymbol{u} = \sum_{p_{i-1} \in F} \boldsymbol{e}_i \in \mathbb{K}^N$ be the sum of the basis vectors corresponding to the accepting states. Then, for any word $w = w_1 \cdots w_d$ with $w_i \in \Sigma$, one has

$$\boldsymbol{u}^T M_{\mathcal{A},w_d} \cdots A_{\mathcal{A},w_1} \boldsymbol{e}_1 = \begin{cases} 1 & \text{if } \mathcal{A} \text{ accepts } w, \\ 0 & \text{if } \mathcal{A} \text{ rejects } w. \end{cases}$$

Let $\rho : \Sigma^* \to \mathrm{Mon}(\mathbb{K}[s_1, \ldots, s_k])$ be a weight function. Thus, $\rho(w_1 w_2) = \rho(w_1) \cdot \rho(w_2)$ for any $w_1, w_2 \in \Sigma^*$. It follows (see, e.g, [16, Section 4.7])

$$\begin{aligned} P_{\mathcal{L},\rho}(s_1, \ldots, s_k) &= \sum_{w \in \mathcal{L}} \rho(w) \\ &= \sum_{d \geq 0} \sum_{w_1, \ldots, w_d \in \Sigma} \boldsymbol{u}^T (\rho(w_1 \cdots w_d) M_{\mathcal{A},w_d} \cdots A_{\mathcal{A},w_1}) \boldsymbol{e}_1 \\ &= \sum_{d \geq 0} \boldsymbol{u}^T \left(\sum_{a \in \Sigma} \rho(a) M_{\mathcal{A},a} \right)^d \boldsymbol{e}_1 = \boldsymbol{u}^T \left(\mathrm{id}_N - \sum_{a \in \Sigma} \rho(a) M_{\mathcal{A},a} \right)^{-1} \boldsymbol{e}_1. \end{aligned}$$

Thus, the generating function $P_{\mathcal{L},\rho}(s_1, \ldots, s_k)$ is rational with rational coefficients and can be explicitly computed from the automaton $\mathcal{A}$ using linear algebra.

In the proof of Proposition 3.8, we showed (see (4-8)) that the equivariant Hilbert series of a considered filtration is, up to a degree shift, equal to a generating function. Hence, the above approach can be used to compute directly this Hilbert series as a rational function with rational coefficients. We implemented the resulting algorithm in Macaulay2 [6]. It is available at http://www.sites.google.com/view/aidamaraj/research.

Example 5.6. In Proposition 3.8, consider the case where $\boldsymbol{c} = (1, 1, \ldots, 1) \in \mathbb{N}^q$. The automaton constructed in Proposition 4.7 can be reduced to one with only $q + 1$ states (see Remark 4.10 if $q = 3$), shown in Figure 3.

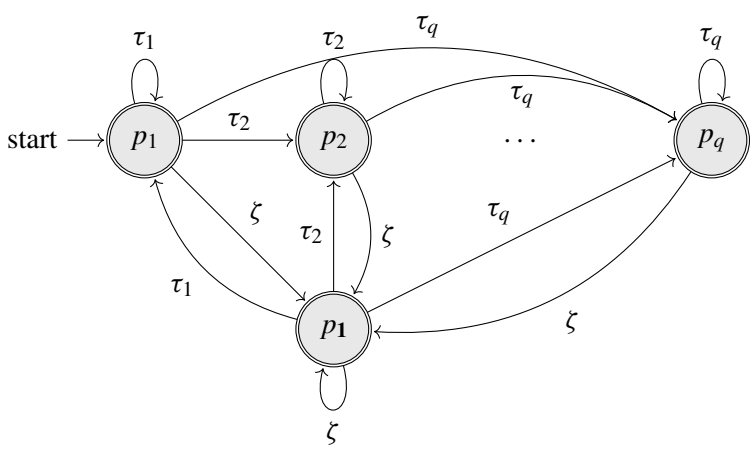

Figure 3. The reduced automaton for $\boldsymbol{c} = (1, \ldots, 1)$ and $T = [q]$.

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Hence, listing $p_{\mathbf{1}}$ as the last state, we obtain for the equivariant Hilbert series of the filtration $\mathscr{J}$

$$\begin{aligned}\operatorname{equivH}_{\mathscr{J}}(s_1,\dots,s_q,t) &= s_1s_2\cdots s_q\cdot \boldsymbol{u}^T\left(\mathrm{id}_{q+1}-\sum_{a\in\Sigma}\rho(a)M_{\mathcal{A},w}\right)^{-1}\boldsymbol{e}_1\\ &= s_1s_2\cdots s_q \begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}^T \begin{bmatrix} 1-s_1 & 0 & 0 & \dots & 0 & 0 & -s_1\\ -s_2 & 1-s_2 & 0 & \dots & 0 & 0 & -s_2\\ -s_3 & -s_3 & 1-s_3 & \dots & 0 & 0 & -s_3\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots\\ -s_{q-1} & -s_{q-1} & -s_{q-1} & \dots & 1-s_{q-1} & 0 & -s_{q-1}\\ -s_q & -s_q & -s_q & \dots & -s_q & 1-s_q & -s_q\\ -t & -t & -t & \dots & -t & -t & 1-t \end{bmatrix}^{-1} \begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix}\\ &= \frac{s_1\cdots s_q}{(1-s_1)\cdots(1-s_q)-t},\end{aligned}$$

where the first column of the inverse matrix can be determined using suitable minors. Of course, the result is the same as in Example 5.5.

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