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Arithmetic and Polygonal Properties of Number Triangle

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ABSTRACT

Among several number triangles that exist in mathematics, Pascal's triangle is the most fascinating and prominent structure possessing numerous properties. In this paper, we will introduce a number triangle consisting of positive integers arranged in ascending to descending order so that the entries in each row would be mirror images with respect to its middle number. Using this simple number triangle, we have proved six amusing results which interestingly depend only as functions of its centred numbers.Moreover some of the concepts and results discussed in this paper have connection with Tamil literary work and triangular numbers.

Keywords: Centred Numbers, Row Sum Property, Alternating Sum Property, Polygonal Properties.

1 Introduction

Number theory is a branch of mathematics which helps to study the properties and structures of numbers. It enables us to investigate the relationships between different types of numbers. Different pattern of number triangle exhibit different mathematical and polygonal properties discussed by various authors. One such pattern is Pascal Triangle. In [3] Krcadinac proposed a generalization of the golden section based on division in mean and extreme ratio which paved way for many interesting results.In [8], Sivaraman has introduced a number triangle and had discussed polygonal properties with respect to its entries. In this paper, we have introduced a number triangle (as shown in the Figure 1) and discussed its various mathematical properties.

2 Construction of Number Triangle

Here we deal with a special pattern of number triangle namely

$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & 2 & 1 & & & \\
 & & 1 & 2 & 3 & 2 & 1 & & \\
 & 1 & 2 & 3 & 4 & 3 & 2 & 1 & \\
1 & 2 & 3 & 4 & 5 & 4 & 3 & 2 & 1 \\
 & .. & . & . & . & .. & . & . & . \\
 & .. & . & . & . & .. & . & . & .
\end{array}
$$

Figure 1

In this paper, for the above mentioned number triangle we have discussed its arithmetic and polygonal properties in detail. Each row represents a palindrome number namely 1, 121, 12321, 1234321 and so onupto$n = 9$. Also it has a special nature namely

$$1^2 = 1$$
$$11^2 = 121$$
$$111^2 = 12321$$

Figure 2

and so on which are the successive rows of the triangle in Figure 1and this property is true for $n \leq 9$. It will be interesting to study its arithmetic and polygonal properties.
In general the number triangle can be represented as

$$t_{11}$$
$$t_{21}t_{22}t_{23}$$
$$t_{31}t_{32}t_{33}t_{34}t_{35}$$
$$t_{41}t_{42}t_{43}t_{44}t_{45}t_{45}t_{46}$$

……………………..
Figure 3

here $t_{ii} = i$.

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An example of usage of this pattern of numbers can be seen in **Thiruppugazh songTiruezhukoottrirukkai which is one of the** most exquisite and poetic composition of Saint Arunagirinathar .திருவெழுகூற்றிருக்கை (Seven Tiered Pyramid – Diamante), wherein the poem follows a Seven Tiered pyramid pattern with numbers. Pattern followed in this poem is similar to that of a chariot with seven decks.

1
121
12321
1234321
123454321
12345654321
1234567654321

Deck of Chariot

1234567654321
12345654321
123454321
1234321
12321
121
1

Figure 4

Numerical values were assigned to numbers occurring in the poem in order to bring out Saint Arunagirinathar's stupendous poetic sense.

3 Arithmetic Properties

3.1 Theorem1 (Row Sum Property):

For $n \geq 1$, the sum of entries in row n is given by

$$\sum_{j=1}^{2n-1} t_{ij} = i^2 ------------- (\mathbf{3.1})$$

Proof:

$$\sum_{j=1}^{2n-1} t_{ij} = i + 2\big(1 + 2 + \cdots + (i-1)\big)$$

$$= i + 2\,\frac{(i-1)i}{2} = i^2$$

This completes the proof.

3.2 Theorem2 (Sum of squaresProperty)

For $n \geq 1$, the sum of squares of terms of each row is given by

$$\sum_{j=1}^{2n-1} \boldsymbol{t_{ij}^2} = \frac{\boldsymbol{i(2i^2+1)}}{\mathbf{3}} ---------- (\mathbf{3.2})$$

Proof:

$$\sum_{j=1}^{2n-1} t_{ij}^2 = i^2 + 2(1^2 + 2^2 + \cdots + (i-1)^2)$$

$$= i^2 + \frac{2(i-1)i(2(i-1)+1)}{6}$$

$$= \frac{i(2i^2+i)}{3}$$

This completes the proof.

3.3 Theorem3 (Sum of the cubes Property)

For $n \geq 1$, the sum of cubes of terms of each row is obtained as

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$$\sum_{j=1}^{2n-1} t_{ij}^3 = \frac{i^2(i^2+1)}{2} - - - - - - - - - - - - - (3.3)$$

Proof:

$$\sum_{j=1}^{2n-1} t_{ij}^3 = i^3 + 2(1^3 + 2^3 + \cdots + (i-1)^3)$$
$$= i^3 + 2\left(\frac{(i-1)i}{2}\right)^2$$
$$= \frac{i^2(i^2+1)}{2}$$

which is the i^2th triangular number.
This completes the proof.

3.4 Theorem4 (Alternating Sum Property):
The sum of the alternating sequence formed by the terms of each row is obtained as

$$\sum_{j=1}^{2n-1} (-1)^n t_{ij} = \begin{cases} 1 & \text{when } n \text{ is odd} \\ 0 & \text{when } n \text{ is even} \end{cases} - - - - - - - - (3.4)$$

Proof:
When n is odd

$$\sum_{j=1}^{2n-1} (-1)^n t_{ij} = 2\big(1 - 2 + 3 - \cdots - (i-1)\big) + i$$

$$= 2\left\{\big(1 + 3 + 5 + \cdots + (n-2)\big) - 2\left(1 + 2 + 3 + \cdots + \frac{(i-1)}{2}\right)\right\} + i$$
$$= 2\left\{\left(\frac{i-1}{2}\right)^2 - 2\left(\frac{\left(\frac{i-1}{2}\right)\left(\frac{i-1}{2}+1\right)}{2}\right)\right\} + i$$
$$= 2\left\{\frac{i^2 - 2i + 1 - i^2 + 1}{4}\right\} + i = 1$$

When n is even

$$\sum_{j=1}^{2n-1} (-1)^{n-1} t_{ij} = 2\big(1 - 2 + 3 - \cdots - (i-1)\big) - i$$

$$= 2\left\{\big(1 + 3 + 5 + \cdots + (i-1)\big) - 2\left(1 + 2 + 3 + \cdots + \frac{(i-2)}{2}\right)\right\} - i$$
$$= 2\left\{\left(\frac{i}{2}\right)^2 - 2\left(\frac{\left(\frac{i-2}{2}\right)\left(\frac{i-2}{2}+1\right)}{2}\right)\right\} - i$$
$$= 2\left\{\frac{i^2 - i^2 + 2i}{4}\right\} - i = 0$$

This completes the proof.

4Polygonal Properties
4.1Determinant of Entries forming Square Shape
4.1.1 Definition:In a number triangle

$$\begin{array}{ccccc} & & \mathbf{a} & & \\ & \mathbf{b} & \mathbf{c} & \mathbf{d} & \\ \mathbf{e} & \mathbf{f} & \mathbf{g} & \mathbf{h} & \mathbf{i} \end{array}$$

Figure 5

The matrix obtained from the square bcgf is $\begin{pmatrix} b & c \\ f & g \end{pmatrix}$ and its determinant value is $bg - cf$

4.1.2 Theorem 5: The determinant of a 2×2 square matrix formed by entries which are in the shape of a square is -1 if $i < j$ and 1 if $i > j$
Proof:

Case 1: When $i < j$, the entries of 2×2 square matrix forming square shape would be of the form $\begin{pmatrix} t_{ij} - 1 & t_{ij} \\ t_{ij} & t_{ij} + 1 \end{pmatrix}$

The determinant of this matrix is

$$\begin{vmatrix} t_{ij} - 1 & t_{ij} \\ t_{ij} & t_{ij} + 1 \end{vmatrix} = (t_{ij} - 1)(t_{ij} + 1) - t_{ij}^2$$
$$= t_{ij}^2 - 1 - t_{ij}^2 = -1$$

Case 2: When $i > j$,the entries of 2×2 square matrix forming square shape would be of the form $\begin{pmatrix} t_{ij} & t_{ij} - 1 \\ t_{ij} + 1 & t_{ij} \end{pmatrix}$

The determinant of this matrix is

$$\begin{vmatrix} t_{ij} & t_{ij} - 1 \\ t_{ij} + 1 & t_{ij} \end{vmatrix} = t_{ij}^2 - (t_{ij} - 1)(t_{ij} + 1)$$
$$= t_{ij}^2 - t_{ij}^2 + 1 = 1$$

This completes the proof.

4.2 Determinant of Entries forming Rhombus Shape

4.2.1 Definition:In the number triangle
a
b c d
e f g h i
Figure 6

The matrix obtained from the rhombus adgb is $\begin{pmatrix} a & d \\ b & g \end{pmatrix}$ and its determinant value is $ag - bd$

4.2.2 Theorem 6: The determinant of a 2×2 square matrix formed by entries which are in the shape of a Rhombus is $2(i - 1)$ if $i = j$ and 0 if $i \neq j$
Case 1: Consider the rhombus(Here $i = j$)

$$\begin{array}{ccc} & i-1 & \\ i-1 & i & i-1 \\ & i+1 & \end{array}$$

Figure 7

The entries of 2×2 square matrix forming the shape of a rhombus would be of the form $\begin{pmatrix} i - 1 & i - 1 \\ i - 1 & i + 1 \end{pmatrix}$

The determinant of this matrix is

$$\begin{vmatrix} i - 1 & i - 1 \\ i - 1 & i + 1 \end{vmatrix} = (i - 1)(i + 1) - (i - 1)(i - 1)$$
$$= (i - 1)(i + 1 - i + 1) = 2(i - 1)$$

Case 2: Consider the rhombus (Here $i \neq j$)

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$$\begin{array}{ccc} & t_{ij}-1 & \\ t_{ij}-1 & t_{ij} & t_{ij}+1 \\ & t_{ij}+1 & \end{array}$$

Figure 8

The entries of 2×2 square matrix forming the shape of a rhombus would be of the form $\begin{pmatrix} t_{ij}-1 & t_{ij}+1 \\ t_{ij}-1 & t_{ij}+1 \end{pmatrix}$

The determinant of this matrix is

$$\begin{vmatrix} t_{ij}-1 & t_{ij}+1 \\ t_{ij}-1 & t_{ij}+1 \end{vmatrix} = 0$$

This completes the proof.

4.3 Determinant of Entries forming Rectangular Shape

4.3.1 Definition: In the number triangle

$$\begin{array}{ccccc} & & \mathbf{a} & & \\ & \mathbf{b} & \mathbf{c} & \mathbf{d} & \\ \mathbf{e} & \mathbf{f} & \mathbf{g} & \mathbf{h} & \mathbf{i} \end{array}$$

Figure 9

The matrix obtained from the rectangle bdhf is $\begin{pmatrix} \boldsymbol{b} & \boldsymbol{d} \\ \boldsymbol{f} & \boldsymbol{h} \end{pmatrix}$ and its determinant value is $\boldsymbol{bh - fd}$.

4.3.2 Theorem 7: The determinant of a 2×2 square matrix formed by entries which are in the shape of a Rectangle is 0 if $\boldsymbol{i = j}$, -2 if $\boldsymbol{i < j}$ and 2 if $\boldsymbol{i > j}$

Proof:

Case 1: When $i = j$ we have

$$\begin{array}{ccc} i-1 & i & i-1 \\ i & i+1 & i \end{array}$$

Figure 10

The entries of 2×2 square matrix forming the shape of a rectangle would be of the form $\begin{pmatrix} i-1 & i-1 \\ i & i \end{pmatrix}$

The determinant of this matrix is

$$\begin{vmatrix} i-1 & i-1 \\ i & i \end{vmatrix} = (i-1)i - i(i-1) = 0$$

Case 2: When $i < j$ we have

$$\begin{array}{ccc} t_{ij}-1 & t_{ij} & t_{ij}+1 \\ t_{ij} & t_{ij}+1 & t_{ij}+2 \end{array}$$

Figure 11

The entries of 2×2 square matrix forming the shape of a rectangle would be of the form $\begin{pmatrix} t_{ij}-1 & t_{ij}+1 \\ t_{ij} & t_{ij}+2 \end{pmatrix}$

The determinant of this matrix is

$$\begin{vmatrix} t_{ij}-1 & t_{ij}+1 \\ t_{ij} & t_{ij}+2 \end{vmatrix} = (t_{ij}-1)(t_{ij}+2) - t_{ij}(t_{ij}+1)$$
$$= t_{ij}^2 + 2t_{ij} - t_{ij} - 2 - t_{ij}^2 - t_{ij} = -2$$

Case 3: When $i > j$ we have

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$$\begin{array}{ccc} t_{ij}+1 & t_{ij} & t_{ij}-1 \\ t_{ij}+2 & t_{ij}+1 & t_{ij} \end{array}$$

Figure 12

The entries of 2×2 square matrix forming the shape of a rectangle would be of the form $\begin{pmatrix} t_{ij}+1 & t_{ij}-1 \\ t_{ij}+2 & t_{ij} \end{pmatrix}$

The determinant of this matrix is

$$\begin{vmatrix} t_{ij}+1 & t_{ij}-1 \\ t_{ij}+2 & t_{ij} \end{vmatrix} = (t_{ij}+1)t_{ij} - (t_{ij}+2)(t_{ij}-1)$$
$$= t_{ij}^2 + t_{ij} - t_{ij}^2 + t_{ij} - 2t_{ij} + 2 = 2$$

This completes the proof.

4.4 Determinant of entries forming Parallelogram Shape
4.4.1 Definition:In the Number triangle

$$\begin{array}{ccccc} & & \mathbf{a} & & \\ & \mathbf{b} & \mathbf{c} & \mathbf{d} & \\ \mathbf{e} & \mathbf{f} & \mathbf{g} & \mathbf{h} & \mathbf{i} \end{array}$$

Figure 13

The matrix obtained from the parallelogram bcfe is $\begin{pmatrix} \boldsymbol{b} & \boldsymbol{c} \\ \boldsymbol{e} & \boldsymbol{f} \end{pmatrix}$ and its determinant value is $\boldsymbol{bf - ce}$

4.4.2 Theorem 8: The determinant of a 2×2 square matrix formed by entries which are in the shape of a Parallelogram is $2\boldsymbol{i}$ if $\boldsymbol{i} = \boldsymbol{j}$, 0 if $\boldsymbol{i} < j$ and 2 if $\boldsymbol{i} > j$

Proof:
Case 1: When $i = j$ we have

$$\begin{array}{ccc} i-1 & i & i-1 \\ i & i+1 & i \end{array}$$

Figure 14

The entries of 2×2 square matrix forming the shape of a parallelogram would be of the form $\begin{pmatrix} i & i-1 \\ i & i+1 \end{pmatrix}$
The determinant of this matrix is

$$\begin{vmatrix} i & i-1 \\ i & i+1 \end{vmatrix} = i(i+1) - i(i-1) = 2i$$

Case 2: When $i < j$ we have

$$\begin{array}{ccc} t_{ij}-1 & t_{ij} & t_{ij}+1 \\ t_{ij} & t_{ij}+1 & t_{ij}+2 \end{array}$$

Figure 15

The entries of 2×2 square matrix forming the shape of a parallelogram would be of the form $\begin{pmatrix} t_{ij} & t_{ij}+1 \\ t_{ij} & t_{ij}+1 \end{pmatrix}$

The determinant of this matrix is

$$\begin{vmatrix} t_{ij} & t_{ij}+1 \\ t_{ij} & t_{ij}+1 \end{vmatrix} = t_{ij}(t_{ij}+1) - t_{ij}(t_{ij}+1)$$
$$= t_{ij}^2 + t_{ij} - t_{ij}^2 - t_{ij} = 0$$

Case 3: When $i > j$ we have

$$\begin{array}{ccc} t_{ij}+1 & t_{ij} & t_{ij}-1 \\ t_{ij}+2 & t_{ij}+1 & t_{ij} \end{array}$$

Figure 16

The entries of 2×2 square matrix forming the shape of a parallelogram would be of the form$\begin{pmatrix} t_{ij} & t_{ij}-1 \\ t_{ij}+2 & t_{ij}+1 \end{pmatrix}$

The determinant of this matrix is

$$\begin{vmatrix} t_{ij} & t_{ij}-1 \\ t_{ij}+2 & t_{ij}+1 \end{vmatrix} = (t_{ij}+1)t_{ij} - (t_{ij}+2)(t_{ij}-1)$$
$$= t_{ij}^2 + t_{ij} - t_{ij}^2 + t_{ij} - 2t_{ij} + 2 = 2$$

This completes the proof.

4.4.3 Theorem 9:The determinant of a 2×2 square matrix formed by entries which are in the shape of a Parallelogram is $-2i$ if $i = j$, -2 if $i < j$ and 0 if $i > j$

Proof:
Case 1: When $i = j$ we have

$$\begin{array}{ccc} i-1 & i & i-1 \\ i & i+1 & i \end{array}$$

Figure 14

The entries of 2×2 square matrix forming the shape of a parallelogram would be of the form$\begin{pmatrix} i-1 & i \\ i+1 & i \end{pmatrix}$

The determinant of this matrix is

$$\begin{vmatrix} i-1 & i \\ i+1 & i \end{vmatrix} = i(i-1) - i(i+1) = -2i$$

Case 2: When $i < j$ we have

$$\begin{array}{ccc} t_{ij}-1 & t_{ij} & t_{ij}+1 \\ t_{ij} & t_{ij}+1 & t_{ij}+2 \end{array}$$

Figure 15

The entries of 2×2 square matrix forming the shape of a parallelogram would be of the form$\begin{pmatrix} t_{ij}-1 & t_{ij} \\ t_{ij}+1 & t_{ij}+2 \end{pmatrix}$

The determinant of this matrix is

$$\begin{vmatrix} t_{ij}-1 & t_{ij} \\ t_{ij}+1 & t_{ij}+2 \end{vmatrix} = (t_{ij}-1)(t_{ij}+2) - t_{ij}(t_{ij}+1)$$
$$= t_{ij}^2 - t_{ij} + 2t_{ij} - 2 - t_{ij}^2 - t_{ij} = -2$$

Case 3: When $i > j$ we have

$$\begin{array}{ccc} t_{ij}+1 & t_{ij} & t_{ij}-1 \\ t_{ij}+2 & t_{ij}+1 & t_{ij} \end{array}$$

Figure 16

The entries of 2×2 square matrix forming the shape of a parallelogram would be of the form$\begin{pmatrix} t_{ij}+1 & t_{ij} \\ t_{ij}+1 & t_{ij} \end{pmatrix}$

The determinant of this matrix is

$$\begin{vmatrix} t_{ij}+1 & t_{ij} \\ t_{ij}+1 & t_{ij} \end{vmatrix} = (t_{ij}+1)t_{ij} - t_{ij}(t_{ij}+1)$$
$$= t_{ij}^2 + t_{ij} - t_{ij}^2 - t_{ij} = 0$$

This completes the proof.

Conclusion:

In this paper we have proved nine interesting mathematical properties regarding the entries of Figure 1. Investigation on other polygons can also be done which might give rise to more mathematical properties. In Theorem1, for $n \geq 1$, the sum of entries in row i($i = t_{ii}$)is obtained as i^2 which is the sum of $i(= t_{ii})$ **odd numbers. In Theorem 2, the sum of squares of terms of each row of the number triangle is obtained as $\frac{2i^3+i}{3}$. In Theorem 3, the sum of cubes of terms of each row is obtained as $\frac{i^2(i^2+1)}{2}$ which is the sum of first i^2 natural numbers. The result obtained in Theorem 3.4 is that the sum of the alternating sequence formed by the terms of each row is 1, when n is odd and 0 if n is even. The determinants of a 2×2 square matrix formed by entries which are in the shape of a square, Rhombus, rectangle and parallelogram are obtained in theorem 5, 6, 7, 8 and 9 respectively. Thus a simple number triangle has generated several interesting properties, a few of which were discussed in sections 3 and 4. There is always scope for further investigation of similar number triangle.**

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