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Solving Diophantine Equations using Bronze Ratio

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ABSTRACT

The study of Diophantine equations has been of great interest among mathematicians for several centuries. In this paper, we will introduce a special type of pair of Diophantine Equations and will solve them using the continued fraction expansion of bronze ratio. In particular, the successive convergents of the continued fraction discussed in this paper provide the complete solutions to the given equations. Incidentally the solutions depend on one of the most important real numbers called Bronze Ratio. The method adopted to solve the given equations is novel and provide insights to solve many more similar equations.

Keywords: Diophantine Equations, Bronze Ratio, Continued Fraction, Convergents.

1. Introduction

Diophantine Equations were named after 3rd century CE mathematician Diophantus, who suggested solving equations whose solutions are integers. Ever since his famous book Arithmetica got released huge amount of research were carried out in number theory and several mathematicians to this day were immersed in solving different types of Diophantine Equations. In this paper, we will introduce pair of equations and provide their solutions using the successive convergents of continued fraction expansion corresponding to that of bronze ratio.

2. Describing the Diophantine Equations

The main purpose of this paper is to solve the equations $x^2 - 3xy - y^2 = \pm 1 \quad (1)$ where *x*, *y* are positive integers. In (1), we notice that we have pair of equations as when right hand side of (1) is either – 1 or 1. For solving two equations described in (1), we consider the following method.

3. Continued Fraction Expansion

In this section, we will derive the continued fraction expansion of the number $\frac{\sqrt{13}-3}{2}$ which is reciprocal of the real number called bronze ratio given by $\frac{\sqrt{13}+3}{2}$. Notice that bronze ratio is special case of class of metallic ratios (see **[1–4]**).

First we begin with noticing that $\frac{\sqrt{13}-3}{2} \times \frac{\sqrt{13}+3}{2} = 1 \quad (2)$.

From (2), we have the following computations:

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$$\frac{\sqrt{13}-3}{2}=\frac{1}{\frac{\sqrt{13}+3}{2}}=\frac{1}{3+\left(\frac{\sqrt{13}-3}{2}\right)}=\frac{1}{3+\frac{1}{3+\left(\frac{\sqrt{13}-3}{2}\right)}}$$

$$=\frac{1}{3+\frac{1}{3+\frac{1}{3+\left(\frac{\sqrt{13}-3}{2}\right)}}}=\cdots=\frac{1}{3+\frac{1}{3+\frac{1}{3+\frac{1}{3+\frac{1}{3+\cdots}}}}}$$

$$\frac{\sqrt{13}+3}{2}=3+\frac{\sqrt{13}-3}{2}=3+\frac{1}{3+\frac{1}{3+\frac{1}{3+\frac{1}{3+\frac{1}{3+\cdots}}}}}=[3;\bar{3}] \quad (3)$$

Equation (3) provides the desired continued fraction expansion of the bronze ratio $\frac{\sqrt{13}+3}{2}$.

4. Extracting the Solutions

Now using the continued fraction expansion (3), we can derive the required solutions of the given Diophantine Equations. For this, first we will compute the successive convergents of (3).

In doing so, we obtain the following sequence of rational numbers

$$\left.\begin{array}{l} c_0=\frac{3}{1}, c_1=[3;3]=3+\frac{1}{3}=\frac{10}{3}, c_2=[3;3,3]=3+\frac{3}{10}=\frac{33}{10}, c_3=[3;3,3,3]=3+\frac{10}{33}=\frac{109}{33}, \\ c_4=[3;3,3,3,3]=3+\frac{33}{109}=\frac{360}{109}, c_5=[3;3,3,3,3,3]=3+\frac{109}{360}=\frac{1189}{360}, \cdots \end{array}\right\} \quad (4)$$

Now considering the numerators and denominators of rational numbers in the successive convergents of (4) as x and y respectively, we notice that $c_0, c_2, c_4, \ldots$ form solutions to $x^2-3xy-y^2=-1$ and $c_1, c_3, c_5, \ldots$ provide solutions to $x^2-3xy-y^2=1$.

In particular, $(x,y)=(3,1);(33,10);(360,109);(3927,1189);\ \ldots$ (5) provides solutions to $x^2-3xy-y^2=-1$ and $(x,y)=(10,3);(109,33);(1189,360);(12970,3927);\ldots$ (6) provides solutions to $x^2-3xy-y^2=1$. Since, there are infinite convergents that can be generated from (3), we have infinitely many solutions to pair of equations described by (1). Moreover, for $n \geq 1$ we notice that the subsequent solutions for both (5) and (6) satisfy the recurrence relations $x_{n+2} = 11x_{n+1} - x_n, y_{n+2} = 11y_{n+1} - y_n \quad (7)$.

5. Conclusion

Considering pair of Diophantine equations as described in (1), in this paper, we had determined solutions in a novel way. In particular, noticing the fact the solutions which are directly connected to the bronze ratio $\frac{3+\sqrt{13}}{2}$, we had derived its continued fraction expansion as in (3). Using the successive convergents of the continued fraction expansion of bronze ratio, we had determined all possible solutions in positive integers for (1). Though there are several different approaches that can be adopted to solve the given pair of Diophantine Equations, in this paper, we have used a simplified idea through continued fractions. The method used in this paper can very well be extended to solve similar equations.

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