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On Solving Euler's Quadratic Diophantine Equation

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ABSTRACT

Diophantine Equations plays a vital role not only in number theory but also in several branches of science. Leonhard Euler was considered to be the most prolific mathematician of all times. In this paper, we will solve one of the problems posed by Euler which turns out to be a Quadratic Diophantine Equation. The method adopted to solve the problem is quite novel and can be generalized for solving similar equations.

Keywords: Quadratic Diophantine Equation, Polar Form, Euler's Formula, Heegner Numbers

1. Introduction

Diophantine Equations were equations whose solutions must be in integers. Since the solutions are integers and most often positive integers, such equations have more practical applications compared to other equations in mathematics. Leonhard Euler proposed several problems in his lifetime spanning highest contributions to the knowledge of mathematics. In this paper, we will solve one of the problems that Euler posed in a novel way and present its solution in a compact form.

2. Euler's Problem

Leonhard Euler proposed to solve the quadratic Diophantine equation $7x^2 + y^2 = 2^n$ (1), where x, y are positive integers. In this paper, we will try to solve (1) and obtain a general solution in closed form. For doing this, we will make use of complex numbers and a fabulous formula proposed by Euler.

3. Solving the Problem

First we note that (1) has no solution in positive integers if $n = 1$ or 2 since in these cases, the left hand side term $7x^2 + y^2$ is always greater than the right hand side term 2^n. Hence in the following method, we consider $n \geq 3$.

Now, we will try to determine the polar form of $\left(1+i\sqrt{7}\right)^{n-2}$

$$1+i\sqrt{7} = r\left(\cos\theta + i\sin\theta\right) \Rightarrow r\cos\theta = 1, r\sin\theta = \sqrt{7}$$

From this, we obtain $r^2 = 1+7 = 8 \Rightarrow r = 2^{3/2}, \theta = \tan^{-1}(\sqrt{7})$ (2)

Hence the polar form of $\left(1+i\sqrt{7}\right)^{n-2}$ is given by

$$\left(1+i\sqrt{7}\right)^{n-2} = 2^{3(n-2)/2} e^{i(n-2)\tan^{-1}(\sqrt{7})} \quad (3)$$

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Now using Euler's Formula in (3), we obtain

$$\left(1+i\sqrt{7}\right)^{n-2} = 2^{3(n-2)/2}\left[\cos\left((n-2)\tan^{-1}(\sqrt{7})\right)+i\sin\left((n-2)\tan^{-1}(\sqrt{7})\right)\right] \quad (4)$$

If we now assume $y+i\sqrt{7}x = 2^{3-n}\times\left(1+i\sqrt{7}\right)^{n-2}$ (5) then $y-i\sqrt{7}x = 2^{3-n}\times\left(1-i\sqrt{7}\right)^{n-2}$ (6)

Now multiplying (5) and (6), we get

$$\left(y+i\sqrt{7}x\right)\times\left(y-i\sqrt{7}x\right) = \left[2^{3-n}\times\left(1+i\sqrt{7}\right)^{n-2}\right]\times\left[2^{3-n}\times\left(1-i\sqrt{7}\right)^{n-2}\right]$$

Simplifying, we obtain $7x^2+y^2 = 2^{6-2n}\times 2^{3n-6} = 2^n$ which is (1), the original problem proposed by Euler. Thus the solutions to (1) are given by equating real and imaginary parts of (5). Now using (4) in (5), and for $n \geq 3$ we get

$$\sqrt{7}x = 2^{3-n}\times 2^{3(n-2)/2}\sin\left((n-2)\tan^{-1}(\sqrt{7})\right) \Rightarrow x = \frac{2^{n/2}}{\sqrt{7}}\sin\left((n-2)\tan^{-1}(\sqrt{7})\right) \quad (7)$$

$$y = 2^{3-n}\times 2^{3(n-2)/2}\cos\left((n-2)\tan^{-1}(\sqrt{7})\right) \Rightarrow y = 2^{n/2}\cos\left((n-2)\tan^{-1}(\sqrt{7})\right) \quad (8)$$

Now from (7) and (8), if we consider $\left(|x|,|y|\right)$ then these pairs would provide all positive integer solutions to Euler's problem $7x^2+y^2 = 2^n$.

4. Conclusion

Considering a quadratic Diophantine equation $7x^2+y^2 = 2^n$ proposed by the great mathematician Leonhard Euler, we have used a novel method to solve it completely in this paper. In particular, we saw that there are no solutions in positive integers if *n* is either 1 or 2 since, in those cases is always greater than or equal to 8, which is strictly greater than $2^1 = 2$ as well as $2^2 = 4$.

Further, by considering the polar form of the complex number we have obtained nice closed expressions for the given equations. In particular, from (7) and (8), we notice that for $n \geq 3$, all positive integer solutions to $7x^2+y^2 = 2^n$ are given by

$$x = \frac{2^{n/2}}{\sqrt{7}}\left|\sin\left((n-2)\tan^{-1}(\sqrt{7})\right)\right|, y = 2^{n/2}\left|\cos\left((n-2)\tan^{-1}(\sqrt{7})\right)\right| \quad (9)$$

In fact, for *n* = 3, 4, 5, 6, 7, 8, 9, 10, . . . all positive integer solutions to $7x^2+y^2 = 2^n$ are given respectively by (1,1); (1,3); (1,5); (2,6); (1,11); (5,9); (7,13); (3,31); . . .

Since 7 is one of the nine Heegner numbers providing unique factorization in the corresponding ring of integers, the expression (9) provides all possible solutions to the Euler's Quadratic Diophantine Equation. We can adopt similar methods to solve other types of quadratic Diophantine equations using polar forms of suitable complex numbers.

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