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New Results on Existence of a Class of Caputo Neutral Fractional Differential Equation Dependence on Lipschitz First Derivative

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ABSTRACT

In this article, we discussed the existence and uniqueness solution for the class of Caputo fractional neutral functional differential equations with addiction on the Lipschitz first derivative conditions in Banach space. The result is established on Krasnoselskii's Fixed point theorem. As well as, an example is illustrate the results.

INTRODUCTION

Fractional differential equation is broad and comprehensive assortment of many mathematical modelling in real world application for that many researchers extent their interest in fractional differential equations. It is proven that a paramount mathematical modeling in all manner of field such as neural network system [7], dynamics [13], engineering [9], medical and health science [8], medical image enhancement [5], viscoelacity [2], modern mechanics [15,17], biological systems [3,4,11,12].

The neutral type functional differential equations depends on past and present values of the function, likewise to retarded differential equations, except it also depends on derivatives with delays. An enormous potential and applicability to solve the fractional differential equations in numerical accuracy so for that many authors gave existence results in neutral differential equations. The neutral differential equations have numerous important application in science and engineering particularly control theory is one of the interesting applications in neutral fractional system. The Initial value problem for a class of fractional neutral differential equations with infinite delay discussed by Benchohra et al [10]. For further research work yet, we have lot of proposed problems of neutral differential equations are exposed.

In this article we investigate the existence results for the IVP of fractional neutral system with boundedness is of the form

$$\begin{cases} ({}^cD^\alpha)\left(u(t) - f\left(t, u(t), u'\big(t, u(t)\big)\right)\right) = g\left(t, u(t), u'\big(t, u(t)\big)\right), t_0 \geq 0, 1 < \alpha < 2 \\ u_{t_0} = \varphi \end{cases} \quad (1)$$

Where $({}^cD^\alpha)$ is the Caputo fractional derivative, $f, g: [t_0, +\infty] \times C([-k, 0], R^n) \times C([-k, 0], R^n) \to R^n$ are given continuous functions and $\varphi \in C([-k, 0], R^n)$. In $u \in C([t_0 - k, t_0 + m], R^n)$ then for any $t \in [t_0 - k, t_0 + m]$ then for $\varpi \in [-k, 0]$ defined by $u_t(\varpi) = u(t + \varpi)$.

Consider $\mathfrak{D}\mathrm{u(t)} = \mathrm{u}'(\mathrm{t}, \mathrm{u(t)})$. Then **(1)** becomes

$$\begin{cases} ({}^cD^\alpha)\left(u(t) - f\big(t, u(t), \mathfrak{D}\mathrm{u(t)}\big)\right) = g\big(t, u(t), \mathfrak{D}\mathrm{u(t)}\big), \ t_0 \geq 0, 1 < \alpha < 2 \\ u_{t_0} = \varphi \end{cases} \quad (2)$$

This article is divided in the following sections: In section 2 we give the basic definitions, lemmas, theorems, which is we used in the upcoming section to resolve our main results. In the last section we discussed our main resuls.

2 PRELIMINARIES

In this section, we include the basic definitions, lemmas which we used throughout this paper. Let $J \subset \mathbb{R}$ and denote $C(J, \mathbb{R}^n)$ be the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm $\|u\| = \sup_{t \in J}|u(t)|$ where $|.|$ represent the complete norm on $\mathbb{R}^n$.

Definition 2.1 ([1.6]) The fractional integral of order α for a function h is defined as

$$({}_aI^\alpha h)(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-s)^{\alpha-1} h(s) ds$$

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Definition 2.2 ([1,6]) The R-L derivative of order is defined by

$$({}_aD^{\alpha}h)(t)=\frac{d}{dt}\left(\frac{1}{\Gamma(1-\alpha)}\int_a^t (t-s)^{-\alpha}h(s)ds\right)$$

Remark 2.3 The ultimate property of R-L fractional derivative is that for $t>t_0$ and $\alpha>0$ we have

$$D^{\alpha}\left(I^{\alpha}h(t)\right)=h(t)$$

Definition 2.4 ([1,6]) For the function h in $[a,b]$, the Caputo fractional derivative of order α is given by

$$({}_a^cD^{\alpha}h)(t)=\frac{1}{\Gamma(n-\alpha)}\int_a^t (t-s)^{n-\alpha-1}h^n(s)ds.$$

Where $n=[\alpha]+1$
Noticeably, Caputo's derivative of a constant is equal to zero.
Note 2.5 Now we need to note that ∃ hookup between R-L and Caputo's fractional derivative of order α

$$({}^cD^{\alpha}h)(t)=\frac{1}{\Gamma(n-\alpha)}\int_{t_0}^t \frac{h^n(s)}{(t-s)^{\alpha+1-n}}ds=D^{\alpha}h(t)-\sum_{i=0}^{n-1}\frac{h^k(t_0)}{\Gamma(i-\alpha+1)}(t-t_0)^{i-\alpha}$$

$$=D^{\alpha}\left[h(t)-\sum_{i=0}^{n-1}\frac{h^i(t_0)}{i\,!}(t-t_0)^i\right],\quad t>t_0,\qquad n-1<\alpha<n$$

Theorem 2.6 (Krasnoselskii Fixed Point Theorem) [16] Let B is a real Banach space in X, be a bounded, closed and convex and let $\mathbb{T}_1$ and $\mathbb{T}_2$ be operators on B satisfying the following conditions let $s\in B$.

- $\mathbb{T}_1(s)+\mathbb{T}_2(s)\subset B$
- $\mathbb{T}_1$ is a strict contraction on B, $(i.e)$., there exists a $l\in[a,b)$ such that
  $$\|\mathbb{T}_1(u)-\mathbb{T}_2(v)\|\le l\|u-v\|\quad \forall u,v\in B$$
- $\mathbb{T}_2$ is continuous on B and $\mathbb{T}_2$ is relatively compact subset of X.
  Then there exists a $y\in B$ such that $\mathbb{T}_1y+\mathbb{T}_2y=y$.

3 Existence Results

Let $I_0=[t_0,t_0+\lambda]$,

$$\mathbb{A}(\lambda,\vartheta)=\left\{u\in C([t_0+k,t_0+\lambda],\mathbb{R}^n)|u_{t_0}=\theta,\sup_{t_0\le t\le t_0+\lambda}|u(t)-\theta(0)|\le\rho\right\}$$

Where λ,ρ are positive constants.
To Prove our main result we need the following assumptions

[A1] The function $g(t,\phi,\mathfrak{D}\phi)$is measurable with respect to t on I_0,
[A2] The function $g(t,\phi,\mathfrak{D}\phi)$ is continuous with respect to ϕ on $C([-k,0],\ \mathbb{R}^n)$,
[A3] Let us define a continuous function $f\in(C[a,b]\times\mathfrak{R}\times\mathfrak{R},\mathfrak{R})$ and $u\in C[a,b]$ and there exists a positive constants τ_1,τ_2 and such that
$$\|f(t,u_1,v_1)-f(t,u_2,v_2)\|\le\tau_1(\|u_1-u_2\|+\|v_1-v_2\|)$$
for each u_1,u_2,v_1,v_2in Y, $\tau_2=\max_{t\in R}\|f(t,0,0)\|$ and $\tau=\max\{\tau_1,\tau_2\}$. Let Y=C[R,X] be the set of continuous functions on R with in the Banach space X values.
[A4] Let us define a continuous function $g\in(C[a,b]\times\mathfrak{R}\times\mathfrak{R},\mathfrak{R})$ and $u\in C[a,b]$ and there exists a positive constants $\mathfrak{P}_1,\mathfrak{P}_2$ and such that
$$\|g(t,u_1,v_1)-g(t,u_2,v_2)\|\le\mathfrak{P}_1(\|u_1-u_2\|+\|v_1-v_2\|)$$
For each u_1,u_2,v_1,v_2in Y, $\mathfrak{P}_2=\max_{t\in R}\|g(t,0,0)\|$ and $\mathfrak{P}=\max\{\mathfrak{P}_1,\mathfrak{P}_2\}$. Let Y=C[R,X] be the set of continuous functions on R with in the Banach space X values.
[A5] Let u′ ∈ C[a,b] satisfy the Lipschitz condition. i.e., There exists a positive constants $\wp_1,\wp_2$ and $\wp$such that
$$\|\mathfrak{D}(t,u)-\mathfrak{D}(t,v)\|\le\wp_1(\|u-v\|),$$
for all u,v in Y. $\wp_2=\max_{t\in D}\|\mathfrak{D}(t,0)\|$ and $\wp=\max\{\wp_1,\wp_2\}$.
[A6] ∃ $\varepsilon_1\in(0,\alpha)$ and a real valued function $\mathfrak{J}(t)\in\mathfrak{L}^{\frac{1}{\varepsilon_1}}(I_0)$ such that for any $u\in\mathbb{A}(\lambda,\rho)$, $|g(t,u_t,\mathfrak{D}u_t)|\le\mathfrak{d}(t)$ $for\ t\in I_0,\ where\ \mathfrak{W}(t)=\tau+\wp t\ and\ \mathfrak{d}(t)=\mathfrak{P}+\wp t$.
[A7] For any $u\in\mathbb{A}(\lambda,\vartheta), f(t,u_t,\mathfrak{D}u_t)=f_1(t,u_t,\mathfrak{D}u_t)+f_2(t,u_t,\mathfrak{D}u_t)$,

[A8] The function f_1 is continuous and for any $u_1, u_2 \in \mathbb{A}(\lambda, \vartheta), t \in (0,1)$, by using the assumption (A4) and (A5) we have $\|f_1(t, u_1 \mathfrak{D}u_1,) - f_1(t, u_2, \mathfrak{D}u_2)\| \leq [\tau + \wp t]\|u_1 - u_2\|$. Take $\mathfrak{W}\ is\ \tau + \wp t$.

[A9] The function f_2 is bounded and it's completely continuous for any Ξ in $\mathbb{A}(\lambda, \vartheta)$, the equicontinuous set $\{t \to f_2(t, u_t, \mathfrak{D}u_t): u \in \Xi\}$ in $C(I_0, \mathbb{R}^n)$.

Lemma 3.1 If **(A3)** - **(A5)** are satisfied, then the estimate $\|\mathfrak{D}u(t))\| \leq t(\wp_1\|u\| + \wp_2)$, $\|\mathfrak{D}u(t) - \mathfrak{D}v(t)\| \leq \wp t\|u-v\|$ are satisfied for any t ∈ R and u,v ∈ Y.

Lemma 3.2 If (A1) - (A6) are satisfied and $\exists\ \lambda \in (0, a)\ and\ \vartheta \in (0, \infty)$ then the IVP of (2) is equivalent to given equation for $t \in (t_0, t_0 + \lambda]$

$$\begin{cases} u(t) = \varphi(0) - f(t_0, \varphi, \mathfrak{D}\varphi) + f(t, u_t, \mathfrak{D}u_t) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1} g(s, u_s, \mathfrak{D}u_s)ds, \quad t \in t_0 \\ u_{t_0} = \varphi \end{cases} \tag{3}$$

Proof: It is clear that by the assumption (A1) and (A2) in the interval I_0 the function $g(t, u_t, \mathfrak{D}u_t)$ is lebesque measurable and $(t-s)^{\alpha-1} \in \mathfrak{L}^{\frac{1}{1-\varepsilon_1}}([t_0, t])\ for\ t \in t_0$. By using assumption (A6) and Holders inequality we conclude that the $(t-s)^{\alpha-1} g(s, u_s, \mathfrak{D}u_s)$ is Lebesque integrable in regard to $s \in [t_0, t]\ \forall\ t \in I_0\ \&\ u \in \mathbb{A}(\lambda, \vartheta)$,

$$\int_{t_0}^{t} |(t-s)^{\alpha-1} g(s, u_s, \mathfrak{D}u_s)|\, ds \leq \|(t-s)^{\alpha-1}\|_{\mathfrak{L}^{\frac{1}{1-\varepsilon_1}}(I_0)} \|\mathfrak{d}(t)\|_{\mathfrak{L}^{\frac{1}{\varepsilon_1}}(I_0)} \tag{4}$$

Where $\|G\|_{\mathfrak{L}^p(J)} = \left(\int_J |G(t)|^p dt\right)^{\frac{1}{p}}$ for all $\mathfrak{L}^p$- integrable function $G: J \to \mathbb{R}$.

Sympathetic to Definition 2.1 and 2.3 obviously that the initial value problem of (2) clarification is u and it is a solution of equation (3).

Besides, if equation (3) is certain, then $\forall\ t \in (t_0, t_0 + \lambda]$,

$$\begin{aligned}({}^cD^\alpha)\big(u(t) - f(t, u_t, \mathfrak{D}u_t)\big) &= ({}^cD^\alpha)\left[\varphi(0) - f(t_0, \varphi, \mathfrak{D}\varphi) + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1} g(s, u_s, \mathfrak{D}u_s)ds\right] \\ &= ({}^cD^\alpha)\left[\frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1} g(s, u_s, \mathfrak{D}u_s)ds\right] \\ &= ({}^cD^\alpha)\big(I^\alpha g(t, u_t, \mathfrak{D}u_t)\big) \\ &= D^\alpha\big(I^\alpha g(t, u_t, \mathfrak{D}u_t)\big) - [I^\alpha g(t, u_t, \mathfrak{D}u_t)]_{t=t_0} \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)} \\ &= g(t, u_t, \mathfrak{D}u_t) - [I^\alpha g(t, u_t, \mathfrak{D}u_t)]_{t=t_0} \frac{(t-t_0)^{-\alpha}}{\Gamma(1-\alpha)}\end{aligned}$$

Sympathetic to (4) we notice that $[I^\alpha g(t, u_t, \mathfrak{D}u_t)]_{t=t_0} = 0$ which aid that $({}^cD^\alpha)\big(u(t) - f(t, u_t, \mathfrak{D}u_t)\big) = g(t, u_t, \mathfrak{D}u_t),\ \ t \in (t_0, t_0 + \boldsymbol{\lambda}]$. This completes the proof.

Theorem 3.3 If (A1)-(A9) are satisfied and assume that $\exists\ \boldsymbol{\lambda} \in (0, a)\ and\ \vartheta \in (0, \infty)$. Then the initial value problem of equation (2) has unique solution on $[t_0, t_0 + \mu]$ for some +ve number μ.

Proof: Corresponding to the assumption (A7), and system (3) is comparable to the consecutive system

$$\begin{cases} u(t) = \varphi(0) - f_1(t_0, \varphi, \mathfrak{D}\varphi) - f_2(t_0, \varphi, \mathfrak{D}\varphi) + f_1(t, u_t, \mathfrak{D}u_t) + f_2(t, u_t, \mathfrak{D}u_t) \\ \qquad + \dfrac{1}{\Gamma(\alpha)}\displaystyle\int_{t_0}^{t}(t-s)^{\alpha-1} g(s, u_s, \mathfrak{D}u_s)ds \\ u_{t_0} = \varphi \end{cases}$$

Now $\varphi^* \in \mathbb{A}(\lambda, \vartheta)$ be defined as $\varphi^*_{t_0} = \varphi,\ \ \varphi^*(t_0 + t) = \varphi(0)\ \ \ \forall\ t \in [0, \lambda]$. If the initial value problem (2) has u is a solution, let $u(t_0 + t) = \varphi^*(t_0 + t) + \mathfrak{x}(t),\ \ t \in [-k, \lambda]$, then we have $u_{t_0+t} = \varphi^*_{t_0+t} + \mathfrak{x}_t,\ \ t \in [0, \lambda]$. So $\mathfrak{x}$ satisfies the equation.

$$\begin{aligned}\mathfrak{x}(t) = -f_1(t_0, \varphi, \mathfrak{D}\varphi) - f_2(t_0, \varphi, \mathfrak{D}\varphi) + f_1\left(t_0 + t, \mathfrak{x}_t + \varphi^*_{t_0+t}, \mathfrak{D}\big(\mathfrak{x}_t + \varphi^*_{t_0+t}\big)\right) \\ + f_2\left(t_0 + t, \mathfrak{x}_t + \varphi^*_{t_0+t}, \mathfrak{D}\big(\mathfrak{x}_t + \varphi^*_{t_0+t}\big)\right)\end{aligned}$$

$$+ \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1} g\big(t_0 + s, \mathfrak{x}_s + \varphi^*_{t_0+s}, \mathfrak{D}(\mathfrak{x}_s + \varphi^*_{t_0+s})\big)ds \tag{5}$$

Whereas f_1, f_2 are continuous and u_t is continuous in t, $\exists\ \lambda' > 0,\ when\ 0 < t < \boldsymbol{\lambda}'$

$$\left|f_1\left(t_0 + t, \mathfrak{x}_t + \varphi^*_{t_0+t}, \mathfrak{D}\big(\mathfrak{x}_t + \varphi^*_{t_0+t}\big)\right) - f_1(t_0, \varphi, \mathfrak{D}\varphi)\right| < \frac{\varsigma}{4} \tag{6}$$

and

$$\left|f_2\left(t_0 + t, \mathfrak{x}_t + \varphi^*_{t_0+t}, \mathfrak{D}\big(\mathfrak{x}_t + \varphi^*_{t_0+t}\big)\right) - f_2(t_0, \varphi, \mathfrak{D}\varphi)\right| < \frac{\varsigma}{4} \tag{7}$$

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Prefer $\mu = \left\{\lambda, \lambda', \left(\frac{\vartheta\,\Gamma(\alpha)(1+\gamma)^{1-\varepsilon_1}}{2\mathcal{G}}\right)^{\frac{1}{(1+\gamma)(1-\varepsilon_1)}}\right\}$ (8)

Where $\gamma = \frac{\alpha-1}{1-\varepsilon_1} \in (-1,0)$ & $G = \|\mathfrak{d}(t)\|_{\mathfrak{L}^{\frac{1}{\varepsilon_1}}(I_0)}$

Define $\mathbb{B}(\mu,\vartheta) = \{u \in C([-k,\mu],\mathbb{R}^n) | u(s) = 0 \; for \; s \in [-k,0] \; and \; \|u\| \le \vartheta\}$ and it is closed, convex and bounded subset of $C([-k,\mu],\mathbb{R}^n)$.

Now we define the operator $\mathcal{T}_1$ and $\mathcal{T}_2$ as

$$\mathcal{T}_1 u(t) = \begin{cases} -f_1(t_0,\varphi,\mathfrak{D}\varphi) + f_1\left(t_0+t, u_t+\varphi^*_{t_0+t}, \mathfrak{D}\left(u_t+\varphi^*_{t_0+t}\right)\right), & t\in[0,\mu] \\ 0, \quad t\in[-k,0] \end{cases}$$

$$\mathcal{T}_2 u(t) = \begin{cases} -f_2(t_0,\varphi,\mathfrak{D}\varphi) + f_2\left(t_0+t, u_t+\varphi^*_{t_0+t}, \mathfrak{D}\left(u_t+\varphi^*_{t_0+t}\right)\right) \\ \quad + \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1} g\left(t_0+s, u_s+\varphi^*_{t_0+s}, \mathfrak{D}(u_s+\varphi^*_{t_0+s})\right)ds & , t\in[0,\mu] \\ 0, \quad t\in[-k,0] \end{cases}$$

The operator equation has a solution $u \in \mathbb{B}(\mu,\vartheta) \iff$ the equation (5) has solution

$u = \mathcal{T}_1 u + \mathcal{T}_2 u.$ (9)

Hence (2) has solution $u(t_0+t) = \mathfrak{x}_t + \varphi^*_{t_0+t}$ on $[0,\mu]$.

Therefore ∃ a existence solution of IVP (2) and unique fixed point in $\mathbb{B}(\mu,\vartheta)$ which is equivalent to the equation (9).

Now we prove that $\mathcal{T}_1 + \mathcal{T}_2$ has a unique fixed point in $\mathbb{B}(\mu,\vartheta)$

Step 1: $\mathcal{T}_1 u + \mathcal{T}_2 u \in \mathbb{B}(\mu,\vartheta)$ for all pair $\mathfrak{y},\mathfrak{z} \in \mathbb{B}(\mu,\vartheta)$.

For all $\mathfrak{y},\mathfrak{z} \in \mathbb{B}(\mu,\vartheta), \mathcal{T}_1\,\mathfrak{y} + \mathcal{T}_2\mathfrak{z} \in C([-k,\mu],\mathbb{R}^n)|$. As well as it is trivial that $(\mathcal{T}_1\,\mathfrak{y} + \mathcal{T}_2\mathfrak{z})(t) = 0 \;\; t \in [-k,0]$

Furthermore for $t \in [0,\mu]$, by the equation (6)-(8) and the assumption (A6) we have

$$\begin{aligned} |(\mathcal{T}_1\,\mathfrak{y} + \mathcal{T}_2\mathfrak{z})(t)| & \\ \le & \left|-f_1(t_0,\varphi,\mathfrak{D}\varphi) + f_1\left(t_0+t, \mathfrak{y}_t+\varphi^*_{t_0+t}, \mathfrak{D}\left(\mathfrak{y}_t+\varphi^*_{t_0+t}\right)\right)\right| \\ & + \left|-f_2(t_0,\varphi,\mathfrak{D}\varphi) + f_2\left(t_0+t, \mathfrak{z}_t+\varphi^*_{t_0+t}, \mathfrak{D}\left(\mathfrak{z}_t+\varphi^*_{t_0+t}\right)\right)\right| \\ & + \frac{1}{\Gamma(\alpha)}\int_0^t \left|(t-s)^{\alpha-1} g\left(t_0+s, \mathfrak{z}_s+\varphi^*_{t_0+s}, \mathfrak{D}\left(\mathfrak{z}_s+\varphi^*_{t_0+s}\right)\right)\right| ds \\ \le & \frac{2\vartheta}{4} + \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\varepsilon_1}}\,ds\right)^{1-\varepsilon}\left(\int_{t_0}^{t_0+\lambda}(\mathfrak{d}(s))^{\frac{1}{\varepsilon_1}}\right)^{\varepsilon_1} \\ \le & \frac{\vartheta}{2} + \frac{\mathcal{G}\mu^{(1+\gamma)(1-\varepsilon_1)}}{\Gamma(\alpha)(1+\gamma)^{1-\varepsilon_1}} \le \vartheta \end{aligned}$$

$$\therefore \; \|\mathcal{T}_1\,\mathfrak{y} + \mathcal{T}_2\mathfrak{z}\| = \sup_{t\in[0,\mu]} |(\mathcal{T}_1\,\mathfrak{y})(t) + (\mathcal{T}_2\,\mathfrak{z})(t)| \le \vartheta$$

$$\Longrightarrow \mathcal{T}_1\,\mathfrak{y} + \mathcal{T}_2\,\mathfrak{z} \in \mathbb{B}(\mu,\vartheta)$$

Step: 2 $\mathcal{T}_1$ is a contraction on $\mathbb{B}(\mu,\vartheta)$

For all $u', u'' \in \mathbb{B}(\mu,\vartheta), u'_t + \varphi^*_{t_0+t} \in \mathbb{A}(\lambda,\vartheta)$ by using (A8) we attain

$$\begin{aligned} |\mathcal{T}_1 u'(t) - \mathcal{T}_1 u''(t)| &= \left|f_1\left(t_0+t, u'+\varphi^*_{t_0+t}, \mathfrak{D}\left(u'+\varphi^*_{t_0+t}\right)\right) - f_1\left(t_0+t, u''+\varphi^*_{t_0+t}, \mathfrak{D}\left(u''+\varphi^*_{t_0+t}\right)\right)\right| \\ &\le \mathfrak{W}\|u'-u''\|, \end{aligned}$$

$\Longrightarrow |\mathcal{T}_1 u' - \mathcal{T}_1 u''| \le \mathfrak{W}\|u'-u''\|$

$\therefore \mathcal{T}_1$ is a contraction on $\mathbb{B}(\mu,\vartheta)$.

Step: 3 Here we prove that $\mathcal{T}_2$ is a completely continuous operator.

Let

$$\mathbb{T}_1 u(t) = \begin{cases} -f_2(t_0,\varphi,\mathfrak{D}\varphi) + f_2\left(t_0+t, u_t+\varphi^*_{t_0+t}, \mathfrak{D}\left(u_t+\varphi^*_{t_0+t}\right)\right) & t\in[0,\mu] \\ 0, \; t\in[-k,0] \end{cases} \quad \text{and}$$

$$\mathbb{T}_2 u(t) = \begin{cases} \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1} g\left(t_0+s, u_s+\varphi^*_{t_0+s}, \mathfrak{D}(u_s+\varphi^*_{t_0+s})\right)ds & t\in[0,\mu] \\ 0, \; t\in[-k,0] \end{cases}$$

Which means $\mathcal{T}_2 = \mathbb{T}_1 + \mathbb{T}_2$

We know that the function f_2 is completely continuous and $\mathbb{T}_1$ is continuous and $\{\mathbb{T}_1 u : u \in \mathbb{B}(\mu,\vartheta)\}$ which is uniformly bounded. By the assumption (A9) we desist that the operator $\mathbb{T}_1$ is completely continuous.

Besides for all $t \in [0,\mu]$, we have

$$|\mathcal{T}_2 u(t)| \le \frac{1}{\Gamma(\alpha)}\int_{t_0}^{t}(t-s)^{\alpha-1}\left|g\left(t_0+s, \mathfrak{x}_s+\varphi^*_{t_0+s}, \mathfrak{D}\left(\mathfrak{x}_s+\varphi^*_{t_0+s}\right)\right)\right|ds$$

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$$\leq \frac{1}{\Gamma(\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-\varepsilon_1}}\,ds\right)^{1-\varepsilon}\left(\int_{t_0}^{t_0+\lambda}\left(\mathfrak{d}(s)\right)^{\frac{1}{\varepsilon_1}}\right)^{\varepsilon_1}$$
$$\leq \frac{\mathcal{G}\mu^{(1+\gamma)(1-\varepsilon_1)}}{\Gamma(\alpha)(1+\gamma)^{1-\varepsilon_1}}$$

Thus $\{\mathcal{T}_2 u: u \in \mathbb{B}(\mu,\vartheta)\}$ is uniformly bounded.
Next, we prove that $\{\mathcal{T}_2 u: u \in \mathbb{B}(\mu,\vartheta)\}$ is equicontinuous . For all $0 \leq t_1 \leq t_2 \leq \mu$ and $u \in \mathbb{B}(\mu,\vartheta)$

$$|\mathcal{T}_2 u(t_2) - \mathcal{T}_2 u(t_1)|$$
$$= \left| \frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]g\left(t_0+s,u_s+\varphi^*_{t_0+s},\mathfrak{D}(u_s+\varphi^*_{t_0+s})\right)ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}g\left(t_0+s,u_s+\varphi^*_{t_0+s},\mathfrak{D}(u_s+\varphi^*_{t_0+s})\right)ds \right|$$
$$\leq \frac{1}{\Gamma(\alpha)}\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]\left|g\left(t_0+s,u_s+\varphi^*_{t_0+s},\mathfrak{D}(u_s+\varphi^*_{t_0+s})\right)\right|ds + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2}(t_2-s)^{\alpha-1}\left|g\left(t_0+s,u_s+\varphi^*_{t_0+s},\mathfrak{D}(u_s+\varphi^*_{t_0+s})\right)\right|ds$$
$$\leq \frac{\mathcal{G}}{\Gamma(\alpha)}\left(\int_0^{t_1}[(t_2-s)^{\alpha-1}-(t_1-s)^{\alpha-1}]^{\frac{1}{1-\varepsilon_1}}\right)^{1-\varepsilon_1} + \frac{\mathcal{G}}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}[(t_2-s)^{\alpha-1}]^{\frac{1}{1-\varepsilon_1}}\right)^{1-\varepsilon_1}$$
$$\leq \frac{\mathcal{G}}{\Gamma(\alpha)}\left(\int_0^{t_1}(t_2-s)^{\gamma}-(t_1-s)^{\gamma}\right)^{1-\varepsilon_1} + \frac{\mathcal{G}}{\Gamma(\alpha)}\left(\int_{t_1}^{t_2}(t_2-s)^{\gamma}\right)^{1-\varepsilon_1}$$
$$\leq \frac{\mathcal{G}}{\Gamma(\alpha)(1+\gamma)^{(1-\varepsilon_1)}}\left(t_1^{1+\gamma}-t_2^{1+\gamma}+(t_2-t_1)^{1+\gamma}\right)^{1-\varepsilon_1} + \frac{\mathcal{G}}{\Gamma(\alpha)(1+\gamma)^{1-\varepsilon_1}}(t_2-t_1)^{(1+\gamma)(1-\varepsilon_1)}$$
$$\leq \frac{2\mathcal{G}}{\Gamma(\alpha)(1+\gamma)^{(1-\varepsilon_1)}}(t_2-t_1)^{(1+\gamma)(1-\varepsilon_1)},$$

$\therefore \mathcal{T}_2$ is equicontinuous, Furthermore , we know that $\mathcal{T}_2$ is continuous. Thus $\mathcal{T}_2$ is completely continuous operator . Also $\mathcal{T}_2 = \mathbb{T}_1 + \mathbb{T}_2$ is completely continuous operator . By using the theorem (2.6) $\mathcal{T}_1 + \mathcal{T}_2$ has unique fixed point on $\mathbb{B}(\mu,\vartheta)$.
Suppose that $f_1 = 0$, we get the following result

Theorem: 3.4 If (A1)-(A6) satisfied such that $\exists\ \lambda \in (0,a)$ *and* $\vartheta \in (0,\infty)$ and

[A10] The function f is continuous and for any $u_1, u_2 \in \mathbb{A}(\lambda,\vartheta), t \in (0,1)$, by using the assumption (A4) and (A5) we have $\|f(t,u_1\mathfrak{D}u_1,) - f(t,u_2,\mathfrak{D}u_2)\| \leq \mathfrak{W}\|u_1 - u_2\|$

Then the given system(2) has atleast one solution on the interval $[t_0, t_0+\mu]$ for some positive integer μ.
If suppose that $f_2 = 0$, we get the following result. By Theorem (2.6) $\mathcal{T}_1 + \mathcal{T}_2$ has unique fixed point on $\mathbb{B}(\mu,\vartheta)$.
Suppose that $f_1 = 0$, we get the following result

Theorem: 3.5 If (A1)-(A6) satisfied such that $\exists\ \lambda \in (0,a)$ *and* $\vartheta \in (0,\infty)$ and

[A1] The function f is continuous and for any $u_1, u_2 \in \mathbb{A}(\lambda,\vartheta), t \in (0,1)$, by using the assumption (A4) and (A5) we have $\|f(t,u_1\mathfrak{D}u_1,) - f(t,u_2,\mathfrak{D}u_2)\| \leq \mathfrak{W}\|u_1 - u_2\|$

Then the given system (2) has atleast one solution on the interval $[t_0, t_0+\mu]$ for some positive integer μ.

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