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Some New Aspects of Fibonacci Lacunary Convergence of Double Sequences in Neutrosophic Normed Spaces

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ABSTRACT

The purpose of this article is to investigate the concept of Fibonacci Lacunary ideal convergence of double sequences in Neutrosophic Normed Spaces (NNS). We examined a new concept, called Fibonacci Lacunary convergence. Fibonacci Lacunary $\mathfrak{T}_2$-limit points and Fibonacci Lacunary $\mathfrak{T}_2$- cluster points for double sequences in NNS have been defined and the significant results have been given. Additionally, Fibonacci Lacunary Cauchy and Fibonacci Lacunary $\mathfrak{T}_2$- Cauchy double sequences in NNS are explained.

Keywords: Lacunary sequence, Double sequence, Fibonacci Lacunary statistical convergence, Neutrosophic normed space.

AMS Subject Classification (2020): 40A30; 40G15.

1. Introduction

Tripathy et al. [20] gave the concept of ideal convergence of double sequences in a metric space and examined fundamental features. Using Lacunary sequence, Fridy and Orhan [5] examined Lacunary statistical convergence. Lacunary statistical convergence of double sequences was worked at initial stage by Savas and Patterson [16]. Lacunary ideal convergence of real sequences was introduced by Tripathy et al. [19]. This kind of convergence extended from single to double sequences with the study of Hazarika [6].

Fuzziness has revolutionized many areas such as mathematics, science, engineering, medicine. This concept was given by Zadeh [21]. The concept of fuzziness are using by many researchers for cybernetics, Artificial Intelligence, Expert system and Fuzzy control, pattern recognition, Operation research, Decision making, Image analysis, Projectiles, Probability theory, Agriculture, weather forecasting. Recently, the fuzzy logic became an important area of research in several branches of mathematics like metric and topological spaces, theory of function etc.

Intuitionistic fuzzy set was introduced by Atanassov [1]. The notion of intuitionistic fuzzy metric space has been established by Park [14]. The notion of neutrosophic sets was introduced by Smarandache [17,18] as an extension of the intuitionistic fuzzy set. For the situation when the aggregate of the components is 1, in the wake of satisfying the condition by applying the neutrosophic set operators, different outcomes can be acquired by applying the intuitionistic fuzzy operators, since the operators disregard the indeterminacy, while the neutrosophic operators are taken into the cognizance of the indeterminacy at a similar level as truth-membership and falsehood and non-membership. Neutrosophic set is a more adaptable and effective tool because it handles, aside from autonomous components, additionally partially independent and dependent information. In 2020 Kirisci et al. [11] defined Neutrosophic Normed space (NN Space) and statistical convergence results. Later Jeyaraman et al. [7, 8] proved several fixed point theorems and stability results in NN Space.

In this paper, we take $\mathfrak{T}_2$ as a strongly admissible ideal in $\mathbb{N} \times \mathbb{N}$. Let (X, ρ) be a metric space. A double sequence $x = (x_{mn})$ is named as $\mathfrak{T}_2$- convergent to ξ, if for any $\varepsilon > 0$ we get $P(\varepsilon) = \{(m, n) \in \mathbb{N} \times \mathbb{N}: \rho(x_{mn}, \xi) \geq \varepsilon\} \in \mathfrak{T}_2$. In this case, we write $\mathfrak{T}_2 - \lim_{m,n\to\infty} x_{mn} = \xi$.

A double sequence $\bar{\theta} = \theta_{us} = \{(k_u, l_s)\}$ is named as double Lacunary sequence if there are two increasing sequences of integers (k_u) and (l_s) such that

$$k_0 = 0, h_u = k_u - k_{u-1} \to \infty \text{ and } l_0 = 0, \bar{h}_s = l_s - l_{s-1} \to \infty,\ u, s \to \infty.$$

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We utilize the subsequent notation $k_{us} = k_u l_s, h_{us} = h_u \bar{h}_s$ and θ_{us} is determined by

$J_{us} = \{(k,l): k_{u-1} < k \leq k_u$ and $l_{s-1} < l \leq l_s\}$,

$$q_u = \frac{k_u}{k_{u-1}}, \bar{q}_s = \frac{l_s}{l_{s-1}} \text{ and } q_{us} = q_u \bar{q}_s.$$

Throughout the paper we indicate a double lacunary sequence as $\theta_2 = \theta_{us} = \{(k_u, l_s)\}$.

2. Preliminaries

Definition 2.1

A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is a continuous t-norm if $*$ satisfies the following conditions:

i. $*$ is commutative and associative,
ii. $*$ is continuous,
iii. $p * 1 = p$,
iv. If $p \leq r$ and $q \leq s$, then $p * q \leq r * s$ for all $p, q, r, s \in [0,1]$.

Definition 2.2:

A binary operation $\Diamond: [0,1] \times [0,1] \to [0,1]$ is a continuous t-conorm if $\Diamond$ satisfies the following conditions:

i. $\Diamond$ is commutative and associative,
ii. $\Diamond$ is continuous,
iii. $p \Diamond 0 = p$ for all $p \in [0,1]$,
iv. If $p \leq r$ and $q \leq s$, then $p \Diamond q \leq r \Diamond s$ for all $p, q, r, s \in [0,1]$.

Definition 2.3:

A binary operation $\odot: [0,1] \times [0,1] \to [0,1]$ is a continuous t-conorm if $\odot$ satisfies the following conditions:

i. $\odot$ is commutative and associative,
ii. $\odot$ is continuous,
iii. $p \odot 0 = p$ for all $p \in [0,1]$,
iv. If $p \leq r$ and $q \leq s$, then $p \odot q \leq r \odot s$ for all $p, q, r, s \in [0,1]$.

Definition 2.4:

The seven tuples $(\mathrm{X}, \varphi, \omega, \psi, *, \Diamond, \odot)$ is called as Neutrosophic Normed Space (NNS) if X is a vector space, $*$ is a continuous t-norm, $\Diamond$ and $\odot$ are continuous t-conorm and φ, ω and ψ are fuzzy sets on $\mathrm{X} \times (0, \infty)$ satisfying the subsequent conditions: For every $a, b \in \mathrm{X}$ and $p, q > 0$:

(i) $\varphi(a,q) + \omega(a,q) + \psi(a,q) \leq 3$,
(ii) $0 \leq \varphi(a,q) \leq 1, 0 \leq \omega(a,q) \leq 1, 0 \leq \psi(a,q) \leq 1$,
(iii) $\varphi(a,q) > 0$,
(iv) $\varphi(a,q) = 1$ iff $a = 0$,
(v) $\varphi(ca,q) = \varphi\left(a, \frac{q}{|c|}\right)$ if $c \neq 0$,
(vi) $\varphi(a,q) * \varphi(b,p) \leq \varphi(a+b, q+p)$,
(vii) $\varphi(a,.): (0,\infty) \to [0,1]$ is continuous at q,
(viii) $\lim_{q\to\infty} \varphi(a,q) = 1$ and $\lim_{q\to 0} \varphi(a,q) = 0$,
(ix) $\omega(a,q) < 1$,
(x) $\omega(a,q) = 0$ iff $a = 0$,
(xi) $\omega(ca,q) = \omega(a, \frac{q}{|c|})$ if $c \neq 0$,
(xii) $\omega(a,q) \Diamond \omega(b,p) \geq \omega(a+b, q+p)$,
(xiii) $\omega(a,.): (0,\infty) \to [0,1]$ is continuous at q,
(xiv) $\lim_{q\to\infty} \omega(a,q) = 0$ and $\lim_{q\to 0} \omega(a,q) = 1$,
(xv) $\psi(a,q) < 1$,

(xvi) $\psi(a,q)=0$ iff $a=0$,
(xvii) $\psi(ca,q)=\psi(a,\frac{q}{|c|})$ if $c\neq 0$,
(xviii) $\psi(a,q)\odot\psi(b,p)\geq \psi(a+b,q+p)$,
(xix) $\psi(a,.)\colon(0,\infty)\to[0,1]$ is continuous at q,
(xx) $\lim_{q\to\infty}\psi(a,q)=0$ and $\lim_{q\to 0}\psi(a,q)=1$.

3. Main Results:

Definition 3.1:

Let $(X,\varphi,\omega,\psi,*,\Diamond,\odot)$ be a NNS. A double sequence $x=(x_{kl})$ in X is called as Fibonacci Lacunary convergent to ξ with regards to the NN(φ,ω,ψ), if for every $t>0$ and $\varepsilon\in(0,1)$, there is $r_0\in\mathbb{N}$ such that

$$\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\varphi\left(\hat{F}x_{kl}-\xi,t\right)>1-\varepsilon,\ \ \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\hat{F}x_{kl}-\xi,t\right)<\varepsilon$$

and $\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\psi\left(\hat{F}x_{kl}-\xi,t\right)<\varepsilon$, for all $u,s\geq r_0$.

In this case, we write $(\varphi,\omega,\psi)^{\theta_{us}}-lim\,Fx=\xi$.

Theorem 3.2:

If $(\varphi,\omega,\psi)^{\theta_{us}}-lim\,Fx=\xi$, then $(\varphi,\omega,\psi)^{\theta_{us}}-lim\,Fx$ is unique.

Proof:

Suppose that $(\varphi,\omega,\psi)^{\theta_{us}}-lim\,Fx=\xi_1$ and $(\varphi,\omega,\psi)^{\theta_{us}}-lim\,Fx=\xi_2$.

Given $\varepsilon>0$, choose $\gamma\in(0,1)$ such that $(1-\gamma)*(1-\gamma)>1-\varepsilon$, $\gamma\Diamond\gamma<\varepsilon$ and $\gamma\odot\gamma<\varepsilon$. Now, for all $t>0$, there is $r_1\in\mathbb{N}$ such that

$$\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\varphi\left(\hat{F}x_{kl}-\xi_1,t\right)>1-\varepsilon,\ \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\hat{F}x_{kl}-\xi_1,t\right)<\varepsilon$$

and $\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\psi\left(\hat{F}x_{kl}-\xi_1,t\right)<\varepsilon$, for all $u,s\geq r_1$. Also, there is $r_2\in\mathbb{N}$ such that

$$\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\varphi\left(\hat{F}x_{kl}-\xi_2,t\right)>1-\varepsilon,\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\hat{F}x_{kl}-\xi_2,t\right)<\varepsilon$$

and $\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\psi\left(\hat{F}x_{kl}-\xi_2,t\right)<\varepsilon$, for all $u,s\geq r_2$. Consider $r_0=max\{r_1,r_2\}$.

Then, for $u,s\geq r_0$, we take a $(m,p)\in\mathbb{N}\times\mathbb{N}$ such that

$\varphi\left(\hat{F}x_{mp}-\xi_1,\frac{t}{2}\right)>\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\varphi\left(\hat{F}x_{kl}-\xi_2,\frac{t}{2}\right)>1-\gamma$ and

$\varphi\left(\hat{F}x_{mp}-\xi_2,\frac{t}{2}\right)>\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\varphi\left(\hat{F}x_{kl}-\xi_2,\frac{t}{2}\right)>1-\gamma$. Then, we obtain

$$\varphi(\xi_1-\xi_2,t)\geq\varphi\left(\hat{F}x_{mp}-\xi_1,\frac{t}{2}\right)*\varphi\left(\hat{F}x_{mp}-\xi_2,\frac{t}{2}\right)>(1-\gamma)*(1-\gamma)>1-\varepsilon.$$

$\omega\left(\hat{F}x_{mp}-\xi_1,\frac{t}{2}\right)<\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\hat{F}x_{kl}-\xi_1,\frac{t}{2}\right)<\gamma$ and

$\omega\left(\hat{F}x_{mp}-\xi_2,\frac{t}{2}\right)<\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\hat{F}x_{kl}-\xi_2,\frac{t}{2}\right)<\gamma$.

$$\omega(\xi_1-\xi_2,t)\leq\omega\left(\hat{F}x_{mp}-\xi_1,\frac{t}{2}\right)\Diamond\omega\left(\hat{F}x_{mp}-\xi_2,\frac{t}{2}\right)<\gamma\Diamond\gamma<\varepsilon$$

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$\psi\left(\hat{F}x_{mp}-\xi_1,\frac{t}{2}\right)<\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\ \psi\left(\hat{F}x_{kl}-\xi_1,\frac{t}{2}\right)<\gamma$ and

$\psi\left(\hat{F}x_{mp}-\xi_2,\frac{t}{2}\right)<\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\ \psi\left(\hat{F}x_{kl}-\xi_2,\frac{t}{2}\right)<\gamma.$

Then, we obtain $\psi(\xi_1-\xi_2,t)\leq\psi\left(\hat{F}x_{mp}-\xi_1,\frac{t}{2}\right)\odot\psi\left(\hat{F}x_{mp}-\xi_2,\frac{t}{2}\right)<\gamma\odot\gamma<\varepsilon.$
Since $\varepsilon>0$ is arbitrary, we have $\varphi(\xi_1-\xi_2,t)=1$, $\omega(\xi_1-\xi_2,t)=0$ and $\psi(\xi_1-\xi_2,t)=0$ for every $t>0$, which gives that $\xi_1=\xi_2$.

Definition 3.3:

A double sequence $x=(x_{kl})$ in NNS is called as Fibonacci Lacunary $\mathfrak{T}_2$[FL$\mathfrak{T}_2$] Convergent to ξ with regards to the NN(φ,ω,ψ), if for every $\varepsilon>0$ and $t>0$, the set

$$\left\{\begin{array}{ll}(u,s)\in\mathbb{N}\times\mathbb{N}: & \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\varphi\left(\hat{F}x_{kl}-\xi,t\right)\leq 1-\varepsilon\\ \quad\text{or} & \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\hat{F}x_{kl}-\xi,t\right)\geq\varepsilon\\ \quad\text{and} & \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\psi\left(\hat{F}x_{kl}-\xi,t\right)\geq\varepsilon.\end{array}\right\}\in\mathfrak{T}_2$$

ξ is named the Fibonacci $\mathfrak{T}_\theta$ limit of the sequence of (x_{kl}), we note $\mathcal{F}\mathfrak{T}_{\theta_{us}}^{(\varphi,\omega,\psi)}-\lim\ x=\ \xi.$

Lemma 3.4:

For every $\varepsilon>0$ and $t>0$, the following demonstrations are equivalent.

a) $\mathcal{F}\mathfrak{T}_{\theta_{us}}^{(\varphi,\omega,\psi)}-\lim\ x=\ \xi,$

b) $\left\{(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\varphi(x_{kl}-\xi,t)\leq 1-\varepsilon\right\}\in\mathfrak{T}_2,$
$\left\{(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega(x_{kl}-\xi,t)\geq\varepsilon\right\}\in\mathfrak{T}_2$ and
$\left\{(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\psi(x_{kl}-\xi,t)\geq\varepsilon\right\}\in\mathfrak{T}_2,$

c) $\left\{\begin{array}{l}(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\varphi\left(\hat{F}x_{kl}-\xi,t\right)>1-\varepsilon,\\ \qquad\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\hat{F}x_{kl}-\xi,t\right)<\varepsilon\ \text{ and}\\ \qquad\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\ \psi\left(\hat{F}x_{kl}-\xi,t\right)<\varepsilon.\end{array}\right\}\in\mathcal{F}(\mathfrak{T}_2),$

d) $\left\{(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\varphi\left(\hat{F}x_{kl}-\xi,t\right)>1-\varepsilon\right\}\in\mathcal{F}(\mathfrak{T}_2),$
$\left\{(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\hat{F}x_{kl}-\xi,t\right)<\varepsilon\right\}\in\mathcal{F}(\mathfrak{T}_2)$ and
$\left\{(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\psi\left(\hat{F}x_{kl}-\xi,t\right)<\varepsilon\right\}\in\mathcal{F}(\mathfrak{T}_2)$ and

e) $\mathcal{F}\mathfrak{T}_{\theta_{us}}^{(\varphi,\omega,\psi)}-lim\ \varphi\left(\hat{F}x_{kl}-\xi,t\right)=\ 1,\mathcal{F}\mathfrak{T}_{\theta_{us}}^{(\varphi,\omega,\psi)}-lim\,\omega\left(\hat{F}x_{kl}-\xi,t\right)=\ 0$ and

$\mathcal{F}\mathfrak{T}_{\theta_{us}}^{(\varphi,\omega,\psi)}-lim\ \psi\left(\hat{F}x_{kl}-\xi,t\right)=\ 0.$

Theorem 3.5:

If a sequence $x=(x_{kl})$ in NNS in FL$\mathfrak{T}_2-$Convergent with regards to the NN(φ,ω,ψ), then $\mathcal{F}\mathfrak{T}_{\theta_{us}}^{(\varphi,\omega,\psi)}-\lim\ x$ is unique.

Proof:

Assume that $\mathcal{F}\mathfrak{T}_{\theta_{us}}^{(\varphi,\omega,\psi)}-\lim\ x=\xi_1$ and $\mathcal{F}\mathfrak{T}_{\theta_{us}}^{(\varphi,\omega,\psi)}-\lim\ x=\xi_2$. Given $\varepsilon\in(0,1)$, select $\gamma\in(0,1)$, such that $(1-\gamma)*(1-\gamma)>1-\varepsilon$, $\gamma\lozenge\gamma<\ \varepsilon$ and $\gamma\odot\gamma<\ \varepsilon.$

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Then, for any $t > 0$, take the following sets:

$K_{\varphi,1}(\gamma,t) = \left\{(u,s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \varphi\left(\hat{F}x_{kl} - \xi_1, \frac{t}{2}\right) \le 1-\gamma\right\}$,

$K_{\varphi,2}(\gamma,t) = \left\{(u,s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \varphi\left(\hat{F}x_{kl} - \xi_2, \frac{t}{2}\right) \le 1-\gamma\right\}$,

$K_{\omega,1}(\gamma,t) = \left\{(u,s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\hat{F}x_{kl} - \xi_1, \frac{t}{2}\right) \ge \gamma\right\}$,

$K_{\omega,2}(\gamma,t) = \left\{(u,s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\hat{F}x_{kl} - \xi_2, \frac{t}{2}\right) \ge \gamma\right\}$,

$K_{\psi,1}(\gamma,t) = \left\{(u,s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \psi\left(\hat{F}x_{kl} - \xi_1, \frac{t}{2}\right) \ge \gamma\right\}$,

$K_{\psi,2}(\gamma,t) = \left\{(u,s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \psi\left(\hat{F}x_{kl} - \xi_2, \frac{t}{2}\right) \ge \gamma\right\}$.

Since $\mathcal{F}\mathfrak{T}_{\theta_{us}}^{(\varphi,\omega,\psi)} - \lim\ x = \xi_1$, By applying Lemma (3.4), we get $K_{\varphi,1}(\gamma,t) \in \mathfrak{T}_2$, $K_{\omega,1}(\gamma,t) \in \mathfrak{T}_2$ and $K_{\psi,1}(\gamma,t) \in \mathfrak{T}_2$, for every $t > 0$. Using $\mathcal{F}\mathfrak{T}_{\theta_{us}}^{(\varphi,\omega,\psi)} - \lim\ x = \xi_2$, we have $K_{\varphi,2}(\gamma,t) \in \mathfrak{T}_2$, $K_{\omega,2}(\gamma,t) \in \mathfrak{T}_2$ and $K_{\psi,2}(\gamma,t) \in \mathfrak{T}_2$ for all $t > 0$. Now, take
$K_{\varphi,\omega,\psi}(\gamma,t) = (K_{\varphi,1}(\gamma,t) \cup K_{\varphi,2}(\gamma,t) \cap K_{\omega,1}(\gamma,t) \cup K_{\omega,2}(\gamma,t) \cap K_{\psi,1}(\gamma,t) \cup K_{\psi,2}(\gamma,t))$. Then $K_{\varphi,\omega,\psi}(\gamma,t) \in \mathfrak{T}_2$. This gives that $K^c_{\varphi,\omega,\psi}(\gamma,t) \ne \theta$ in $\mathcal{F}(\mathfrak{T}_2)$.
If $(u,s) \in K^c_{\varphi,\omega,\psi}(\gamma,t)$, contemplate the case $(u,s) \in (K^c_{\varphi,1}(\gamma,t) \cap K^c_{\varphi,2}(\gamma,t))$,

$(u,s) \in K^c_{\omega,1}(\gamma,t) \cap K^c_{\omega,2}(\gamma,t)$ and $(u,s) \in K^c_{\psi,1}(\gamma,t) \cap K^c_{\psi,2}(\gamma,t)$.

Then, we get $\frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \varphi\left(\hat{F}x_{kl} - \xi_1, \frac{t}{2}\right) > 1-\gamma$ and $\frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \varphi\left(\hat{F}x_{kl} - \xi_2, \frac{t}{2}\right) > 1-\gamma$,

$\frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\hat{F}x_{kl} - \xi_1, \frac{t}{2}\right) < \gamma$ and $\frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\hat{F}x_{kl} - \xi_2, \frac{t}{2}\right) < \gamma$ and

$\frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \psi\left(\hat{F}x_{kl} - \xi_1, \frac{t}{2}\right) < \gamma$ and $\frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \psi\left(\hat{F}x_{kl} - \xi_2, \frac{t}{2}\right) < \gamma$.

Now, obviously we will get a $p, q \in \mathbb{N} \times \mathbb{N}$ such that

$\varphi\left(\hat{F}x_{pq} - \xi_1, \frac{t}{2}\right) > \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \varphi\left(\hat{F}x_{kl} - \xi_1, \frac{t}{2}\right) > 1-\gamma$ and

$\varphi\left(\hat{F}x_{pq} - \xi_2, \frac{t}{2}\right) > \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \varphi\left(\hat{F}x_{kl} - \xi_2, \frac{t}{2}\right) > 1-\gamma$.

$\omega\left(\hat{F}x_{pq} - \xi_1, \frac{t}{2}\right) < \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\hat{F}x_{kl} - \xi_1, \frac{t}{2}\right) < \gamma$ and

$\omega\left(\hat{F}x_{pq} - \xi_2, \frac{t}{2}\right) < \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \omega\left(\hat{F}x_{kl} - \xi_2, \frac{t}{2}\right) < \gamma$.

$\psi\left(\hat{F}x_{pq} - \xi_1, \frac{t}{2}\right) < \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \psi\left(\hat{F}x_{kl} - \xi_1, \frac{t}{2}\right) < \gamma$ and

$\psi\left(\hat{F}x_{pq} - \xi_2, \frac{t}{2}\right) < \frac{1}{h_{us}} \sum_{(k,l)\in J_{us}} \psi\left(\hat{F}x_{kl} - \xi_2, \frac{t}{2}\right) < \gamma$.

Then, we obtain

$\varphi(\xi_1 - \xi_2, t) \ge \varphi\left(\hat{F}x_{pq} - \xi_1, \frac{t}{2}\right) * \varphi\left(\hat{F}x_{pq} - \xi_2, \frac{t}{2}\right) > (1-\gamma) * (1-\gamma) > 1-\varepsilon$,

$\omega(\xi_1 - \xi_2, t) \le \omega\left(\hat{F}x_{pq} - \xi_1, \frac{t}{2}\right) \Diamond \omega\left(\hat{F}x_{pq} - \xi_2, \frac{t}{2}\right) < \gamma \Diamond \gamma < \varepsilon$ and

$\psi(\xi_1 - \xi_2, t) \le \psi\left(\hat{F}x_{pq} - \xi_1, \frac{t}{2}\right) \odot \psi\left(\hat{F}x_{pq} - \xi_2, \frac{t}{2}\right) < \gamma \odot \gamma < \varepsilon$.

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Since $\varepsilon > 0$ is arbitrary, we get $\varphi(\xi_1 - \xi_2, t) = 1$, $\omega(\xi_1 - \xi_2, t) = 0$, $\psi(\xi_1 - \xi_2, t) = 0$ for each $t > 0$,which gives that $\xi_1 = \xi_2$.

Hence in all cases, we deduce $\mathcal{F}\mathfrak{I}_{\theta_{us}}^{(\varphi,\omega,\psi)} - \lim\ x$ is unique.

Theorem 3.6:

If $(\varphi, \omega, \psi)^{\theta_{us}} - lim\, Fx = \xi$, then $\mathcal{F}\mathfrak{I}_{\theta_{us}}^{(\varphi,\omega,\psi)} - \lim x = \xi$.

Proof: Let $(\varphi, \omega, \psi)^{\theta_{us}} - lim\, Fx = \xi$. Then for every $t > 0$, and $\varepsilon \in (0,1)$, there is $r_0 \in \mathbb{N}$ such that $\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}} \varphi(\hat{F}x_{kl} - \xi, t) > 1 - \varepsilon$, $\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}} \omega(\hat{F}x_{kl} - \xi, t) < \varepsilon$ and $\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}} \psi(\hat{F}x_{kl} - \xi, t) < \varepsilon$, for all $u, s \geq r_0$.

Therefore, we obtain

$$\mathcal{A} = \left\{\begin{array}{ll} (u,s) \in \mathbb{N}\times\mathbb{N}: \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}} & \varphi(\hat{F}x_{kl} - \xi, t) \leq 1 - \varepsilon \\ \quad \text{or} \qquad \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}} & \omega(\hat{F}x_{kl} - \xi, t) \geq \varepsilon \\ \quad \text{and} \qquad \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}} & \psi(\hat{F}x_{kl} - \xi, t) \geq \varepsilon \end{array}\right\}$$

$$\subseteq \{(1,1), (2,2), (3,3) \ldots (k_0 - 1, k_0 - 1\}.$$

But, with $\mathfrak{I}_2$ being admissible ideal, we get $\in \mathfrak{I}_2$.

Hence, $\mathcal{F}\mathfrak{I}_{\theta_{us}}^{(\varphi,\omega,\psi)} - \lim x = \xi$.

Theorem 3.7:

If $(\varphi, \omega, \psi)^{\theta_{us}} - lim\, Fx = \xi$, then there exists a subsequence $(x_{k'(u)l'(s)})$ of x such that $(\varphi, \omega, \psi)^{\theta_{us}} - lim\, Fx_{k'(u)l'(s)} = \xi$.

Proof:

Let $(\varphi, \omega, \psi)^{\theta_{us}} - lim\, Fx = \xi$. Then for every $t > 0$ and $\varepsilon \in (0,1)$, there exists $r_0 \in \mathbb{N}$ such that $\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}} \varphi(\hat{F}x_{kl} - \xi, t) > 1 - \varepsilon$, $\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}} \omega(\hat{F}x_{kl} - \xi, t) < \varepsilon$ and $\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}} \psi(\hat{F}x_{kl} - \xi, t) < \varepsilon$, for all $u, s \geq r_0$.

Obviously for each $u, s \geq r_0$, we can select $k'(u)l'(s) \in J_{us}$ such that

$$\varphi(\hat{F}x_{k'(u)l'(s)} - \xi, t) > \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}} \varphi(\hat{F}x_{kl} - \xi, t) > 1 - \varepsilon,$$

$\omega(\hat{F}x_{k'(u)l'(s)} - \xi, t) < \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}} \omega(\hat{F}x_{kl} - \xi, t) < \varepsilon$ and

$\psi(\hat{F}x_{k'(u)l'(s)} - \xi, t) < \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}} \psi(\hat{F}x_{kl} - \xi, t) < \varepsilon$.

It follows that$(\varphi, \omega, \psi)^{\theta_{us}} - lim\, Fx_{k'(u)l'(s)} = \xi$.

Definition 3.8:

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A double sequence $x = (x_{jk})$ in NNS is named as Fibonacci Lacunary Cauchy [FLC] with regards to the NN(φ, ω, ψ), if for every $\varepsilon > 0$ and $t > 0$, there exists $N = N(\varepsilon)$ and $M = M(\varepsilon)$ such that, for all $j, p \geq N$, $k, q \geq M$,

$\frac{1}{h_{us}} \sum_{(j,k),(p,q) \in J_{us}} \varphi(\hat{F}x_{jk} - \hat{F}x_{pq}, t) > 1 - \varepsilon$, $\frac{1}{h_{us}} \sum_{(j,k),(p,q) \in J_{us}} \omega(\hat{F}x_{jk} - \hat{F}x_{pq}, t) < \varepsilon$

and $\frac{1}{h_{us}} \sum_{(j,k),(p,q) \in J_{us}} \psi(\hat{F}x_{jk} - \hat{F}x_{pq}, t) < \varepsilon$.

Definition 3.9:

A double sequence $x = (x_{jk})$ in NNS is named as FL$\mathfrak{I}_2$- Cauchy with regards to the NN(φ, ω, ψ), if for every $\varepsilon \in (0,1)$ and $t > 0$, there exists $(p, q) \in \mathbb{N} \times \mathbb{N}$ fulfilling

$$\left\{\begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(j,k),(p,q) \in J_{us}} \varphi(\hat{F}x_{jk} - \hat{F}x_{pq}, t) > 1 - \varepsilon, \\ \frac{1}{h_{us}} \sum_{(j,k),(p,q) \in J_{us}} \omega(\hat{F}x_{jk} - \hat{F}x_{pq}, t) < \varepsilon \text{ and} \\ \frac{1}{h_{us}} \sum_{(j,k),(p,q) \in J_{us}} \psi(\hat{F}x_{jk} - \hat{F}x_{pq}, t) < \varepsilon. \end{array}\right\} \in \mathcal{F}(\mathfrak{I}_2)$$

Definition 3.10:

A double sequence $x = (x_{jk})$ in NNS is named as FL$\mathfrak{I}_2$- Cauchy with regards to the NN(φ, ω, ψ), if there is a subset $M = \{(j_m, k_m): j_1 < \ldots < k_1 < k_2 < \cdots\}$ of $\mathbb{N} \times \mathbb{N}$ such that the set $M' = \{(u,s) \in \mathbb{N} \times \mathbb{N}: (j_u, k_s) \in J_{us}\} \in \mathcal{F}(\mathfrak{I}_2)$ and the subsequence (x_{j_u,k_s}) is a FLC sequence with regards to the NN(φ, ω, ψ).

Theorem 3.11:

A double sequence $x = (x_{jk})$ is named as FL$\mathfrak{I}_2$- Convergent with regards to the NN(φ, ω, ψ), iff it is FL$\mathfrak{I}_2$-Cauchy with regards to the (φ, ω, ψ).

Proof:

Let $x = (x_{jk})$ be FL$\mathfrak{I}_2$ − Convergent to ξ with regards to the NN(φ, ω, ψ). Then

$$\left\{\begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(j,k) \in J_{us}} \varphi\left(\hat{F}x_{jk} - \xi, \frac{t}{2}\right) \leq 1 - \varepsilon \\ \text{or} \quad \frac{1}{h_{us}} \sum_{(j,k) \in J_{us}} \omega\left(\hat{F}x_{jk} - \xi, \frac{t}{2}\right) \geq \varepsilon \\ \text{and} \quad \frac{1}{h_{us}} \sum_{(j,k) \in J_{us}} \psi\left(\hat{F}x_{jk} - \xi, \frac{t}{2}\right) \geq \varepsilon \end{array}\right\} \in \mathfrak{I}_2$$

Specifically, for $j = M, k = N$

$$\left\{\begin{array}{c} (u,s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(M,N) \in J_{us}} \varphi\left(\hat{F}x_{MN} - \xi, \frac{t}{2}\right) \leq 1 - \varepsilon \\ \text{or} \quad \frac{1}{h_{us}} \sum_{(M,N) \in J_{us}} \omega\left(\hat{F}x_{MN} - \xi, \frac{t}{2}\right) \geq \varepsilon \\ \text{and} \quad \frac{1}{h_{us}} \sum_{(M,N) \in J_{us}} \psi\left(\hat{F}x_{MN} - \xi, \frac{t}{2}\right) \geq \varepsilon \end{array}\right\} \in \mathfrak{I}_2$$

Since $\varphi(\hat{F}x_{jk} - \hat{F}x_{MN}, t) = \varphi\left(\hat{F}x_{jk} - \xi - \hat{F}x_{MN} + \xi, \frac{t}{2} + \frac{t}{2}\right)$

$$\geq \varphi\left(\hat{F}x_{jk} - \xi, \frac{t}{2}\right) * \varphi\left(\hat{F}x_{MN} - \xi, \frac{t}{2}\right),$$

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$$\omega(\hat{F}x_{jk} - \hat{F}x_{MN}, t) = \omega\left(\hat{F}x_{jk} - \xi - \hat{F}x_{MN} + \xi, \tfrac{t}{2} + \tfrac{t}{2}\right)$$
$$\leq \omega\left(\hat{F}x_{jk} - \xi, \tfrac{t}{2}\right) \Diamond \omega\left(\hat{F}x_{MN} - \xi, \tfrac{t}{2}\right) \text{ and}$$

$$\psi(\hat{F}x_{jk} - \hat{F}x_{MN}, t) = \psi\left(\hat{F}x_{jk} - \xi - \hat{F}x_{MN} + \xi, \frac{t}{2} + \frac{t}{2}\right)$$
$$\leq \psi\left(\hat{F}x_{jk} - \xi, \frac{t}{2}\right) \odot \psi\left(\hat{F}x_{MN} - \xi, \frac{t}{2}\right).$$

We obtain

$$\left\{\begin{array}{l}(u,s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(j,k)\in J_{us}} \varphi(\hat{F}x_{jk} - \hat{F}x_{MN}, t) \leq 1 - \varepsilon \\ \quad \text{or} \quad \frac{1}{h_{us}} \sum_{(j,k)\in J_{us}} \omega(\hat{F}x_{jk} - \hat{F}x_{MN}, t) \geq \varepsilon \\ \quad \text{and} \quad \frac{1}{h_{us}} \sum_{(j,k)\in J_{us}} \psi(\hat{F}x_{jk} - \hat{F}x_{MN}, t) \geq \varepsilon \end{array}\right\} \in \mathfrak{I}_2$$

That is, (x_{jk}) is FL$\mathfrak{I}_2$- Cauchy with regards to (φ, ω, ψ).

In contract, let $x = (x_{jk})$ be FL$\mathfrak{I}_2$- Cauchy but not FL$\mathfrak{I}_2$- Convergent with regards to the NN(φ, ω, ψ). Then, there are N and M such that set $\mathcal{A}(\varepsilon, t) \in \mathfrak{I}_2$, where

$$\mathcal{A}(\varepsilon, t) = \left\{\begin{array}{l}(u,s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(j,k)\in J_{us}} \varphi(x_{jk} - x_{MN}, t) \leq 1 - \varepsilon \text{ or} \\ \frac{1}{h_{us}} \sum_{(j,k)\in J_{us}} \omega(x_{jk} - x_{MN}, t) \geq \varepsilon \text{ and} \\ \frac{1}{h_{us}} \sum_{(j,k)\in J_{us}} \psi(x_{jk} - x_{MN}, t) \geq \varepsilon. \end{array}\right\}$$

and also $\mathcal{B}(\varepsilon, t) \in \mathfrak{I}_2$, where

$$\mathcal{B}(\varepsilon, t) = \left\{\begin{array}{l}(u,s) \in \mathbb{N} \times \mathbb{N}: \frac{1}{h_{us}} \sum_{(j,k)\in J_{us}} \varphi\left(\hat{F}x_{jk} - \xi, \frac{t}{2}\right) \leq 1 - \varepsilon \text{ or} \\ \frac{1}{h_{us}} \sum_{(j,k)\in J_{us}} \omega\left(\hat{F}x_{jk} - \xi, \frac{t}{2}\right) \geq \varepsilon \text{ and} \\ \frac{1}{h_{us}} \sum_{(j,k)\in J_{us}} \psi\left(\hat{F}x_{jk} - \xi, \frac{t}{2}\right) \geq \varepsilon. \end{array}\right\}$$

Since

$\varphi(\hat{F}x_{jk} - \hat{F}x_{MN}, t) \geq 2\varphi\left(\hat{F}x_{jk} - \xi, \frac{t}{2}\right) > 1 - \varepsilon$, $\omega(\hat{F}x_{jk} - \hat{F}x_{MN}, t) \leq 2\omega\left(\hat{F}x_{jk} - \xi, \frac{t}{2}\right) < \varepsilon$

and $\psi(\hat{F}x_{jk} - \hat{F}x_{MN}, t) \leq 2\psi\left(\hat{F}x_{jk} - \xi, \frac{t}{2}\right) < \varepsilon$,

If $\varphi\left(\hat{F}x_{jk} - \xi, \frac{t}{2}\right) > \frac{1-\varepsilon}{2}$, $\omega\left(\hat{F}x_{jk} - \xi, \frac{t}{2}\right) < \frac{\varepsilon}{2}$ and $\psi\left(\hat{F}x_{jk} - \xi, \frac{t}{2}\right) < \frac{\varepsilon}{2}$. Therefore,

$$\left\{\begin{array}{c}(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum_{(j,k)\in J_{us}}\varphi\big(\hat{F}x_{jk}-\hat{F}x_{MN},t\big)>1-\varepsilon \text{ or}\\ \frac{1}{h_{us}}\sum_{(j,k)\in J_{us}}\omega\big(\hat{F}x_{jk}-\hat{F}x_{MN},t\big)<\varepsilon \text{ and}\\ \frac{1}{h_{us}}\sum_{(j,k)\in J_{us}}\psi\big(\hat{F}x_{jk}-\hat{F}x_{MN},t\big)<\varepsilon.\end{array}\right\}\in\mathfrak{I}_2$$

that is, $\mathcal{A}^c(\varepsilon,t)\in\mathfrak{I}_2$ and hence $\mathcal{A}(\varepsilon,t)\in\mathcal{F}(\mathfrak{I}_2)$, which leads to a contradiction.

Hence x must be FL$\mathfrak{I}_2$- Convergent with regards to the NN(φ,ω,ψ).

Theorem 3.12:

If (ρ_{us}) is a double lacunary refinement of θ_{us} and $\mathcal{F}\mathfrak{I}_{\rho_{us}}^{(\varphi,\omega,\psi)}-\lim x=\xi$, then $\mathcal{F}\mathfrak{I}_{\theta_{us}}^{(\varphi,\omega,\psi)}-\lim x=\xi$.

Proof:

Suppose that each $\mathfrak{I}_{us}$ of θ_{us} involves the points $(\bar{k}_{u,i},\bar{l}_{s,j})_{i,j=1}^{v(u),w(s)}$ of (ρ_{us}) so that

$k_{u-1}<\bar{k}_{u,1}<\bar{k}_{u,2}<\cdots<\bar{k}_{u,v(u)}=k_u$, where $\bar{I}_{u,i}=\big(\bar{k}_{u,i-1},\bar{k}_{u,i}\big)$,

$l_{s-1}<\bar{l}_{s,1}<\bar{l}_{s,2}<\cdots<\bar{l}_{s,w(s)}=l_s$, where $\bar{J}_{s,j}=(\bar{l}_{s,j-1},\bar{l}_{s,j})$

and $\bar{J}_{u,s,i,j}=\{(k,l):\bar{k}_{u,i-1}<k\le\bar{k}_u;\bar{l}_{s,j-1}<l\le\bar{l}_s\}$, for all u,s and $v(u)\ge 1, w(s)\ge 1$ this gives that $(k_u,l_s)\subseteq(\bar{k}_u,\bar{l}_s)$. Let $\big(\bar{J}_{ij}\big)_{i,j=1}^{\infty,\infty}$ be the sequence of abutting blocks of $(\bar{J}_{u,s,i,j})$ ordered by increasing a lower right index point. Since $\mathcal{F}\mathfrak{I}_{\rho_{us}}^{(\varphi,\omega,\psi)}-\lim x=\xi$, we obtain the following for each $t>0$ and $\varepsilon\in(0,1)$

$$\left\{\begin{array}{c}(i,j)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\bar{h}_{ij}}\sum_{\bar{J}_{ij}\subset J_{us}}\varphi\big(\hat{F}x_{kl}-\xi,t\big)\le 1-\varepsilon\\ \text{or}\quad \frac{1}{\bar{h}_{ij}}\sum_{\bar{J}_{ij}\subset J_{us}}\omega\big(\hat{F}x_{kl}-\xi,t\big)\ge\varepsilon\\ \text{and}\quad \frac{1}{\bar{h}_{ij}}\sum_{\bar{J}_{ij}\subset J_{us}}\psi\big(\hat{F}x_{kl}-\xi,t\big)\ge\varepsilon.\end{array}\right\}\in\mathfrak{I}_2 \qquad (3.12.1)$$

As before, we take $h_{us}=h_u\bar{h}_s:\bar{h}_{ui}=\bar{k}_{ui}-\bar{k}_{u,i-1},\bar{h}_{sj}=\bar{l}_{s,j}-\bar{l}_{s,j-1}$.

for each $t>0$ and $\varepsilon\in(0,1)$ we get

$$\left\{\begin{array}{c}(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\varphi\big(\hat{F}x_{kl}-\xi,t\big)\le 1-\varepsilon \text{ or}\\ \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\big(\hat{F}x_{kl}-\xi,t\big)\ge\varepsilon \text{ and}\\ \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\psi\big(\hat{F}x_{kl}-\xi,t\big)\ge\varepsilon.\end{array}\right\}$$

$$\subseteq\left\{(u,s)\in\mathbb{N}\times\mathbb{N}:\tfrac{1}{h_{us}}\textstyle\sum_{(k,l),\in J_{us}}\left\{\begin{array}{c}(i,j)\in\mathbb{N}\times\mathbb{N}:\frac{1}{\bar{h}_{ij}}\sum_{\bar{J}_{ij}\subset J_{us};(k,l)\in\bar{J}_{ij}}\varphi\big(\hat{F}x_{kl}-\xi,t\big)\le 1-\varepsilon \text{ or}\\ \frac{1}{\bar{h}_{ij}}\sum_{\bar{J}_{ij}\subset J_{us};(k,l)\in\bar{J}_{ij}}\omega\big(\hat{F}x_{kl}-\xi,t\big)\ge\varepsilon \text{ and}\\ \frac{1}{\bar{h}_{ij}}\sum_{\bar{J}_{ij}\subset J_{us};(k,l)\in\bar{J}_{ij}}\psi\big(\hat{F}x_{kl}-\xi,t\big)\ge\varepsilon\end{array}\right\}\right\}.$$

By (3.12.1), for each $t>0$ and $\varepsilon\in(0,1)$ if we define

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$$t_{ij}=\begin{bmatrix} \frac{1}{\bar{h}_{ij}}\sum_{(k,l)\in\bar{J}_{ij}}\varphi\big(\hat{F}x_{kl}-\xi,t\big)\le 1-\varepsilon \text{ or} \\ \frac{1}{\bar{h}_{ij}}\sum_{\bar{J}_{ij}\subset J_{us},(k,l)\in\bar{J}_{ij}}\omega\big(\hat{F}x_{kl}-\xi,t\big)\ge\varepsilon \text{ and} \\ \frac{1}{\bar{h}_{ij}}\sum_{\bar{J}_{ij}\subset J_{us},(k,l)\in\bar{J}_{ij}}\psi\big(\hat{F}x_{kl}-\xi,t\big)\ge\varepsilon \end{bmatrix}_{i,j=1}^{\infty,\infty}$$

then $(t_{(i,j)})$ is a pringsheim null sequence. The transformation

$$(\mathcal{A}t)_{us}=\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\begin{bmatrix} \bar{h}_{ij}\frac{1}{\bar{h}_{ij}}\sum_{(k,l)\in\bar{J}_{ij}}\varphi\big(\hat{F}x_{kl}-\xi,t\big)\le 1-\varepsilon \text{ or} \\ \bar{h}_{ij}\frac{1}{\bar{h}_{ij}}\sum_{\bar{J}_{ij}\subset J_{us};(k,l)\in\bar{J}_{ij}}\omega\big(\hat{F}x_{kl}-\xi,t\big)\ge\varepsilon \text{ and} \\ \bar{h}_{ij}\frac{1}{\bar{h}_{ij}}\sum_{\bar{J}_{ij}\subset J_{us};(k,l)\in\bar{J}_{ij}}\psi\big(\hat{F}x_{kl}-\xi,t\big)\ge\varepsilon \end{bmatrix}$$

fulfills all situations for a matrix transformation to map a pringsheim null sequence.

Hence, $\mathcal{F}\mathfrak{I}_{\theta_{us}}^{(\varphi,\omega,\psi)}-\lim x=\xi$.

Definition 3.13:

Let $(X,\varphi,\omega,\psi,*,\Diamond,\odot)$ be an NNS.

a) An element $\xi\in X$ is named as FL$\mathfrak{I}_2$- limit point of $x=(x_{kl})$ if there is set $M=\{(k_1,l_1)<(k_2,l_2)<\cdots<(k_u,l_s)<\cdots\}\subset\mathbb{N}\times\mathbb{N}$ such that the set $M'=\{(u,s)\in\mathbb{N}\times\mathbb{N}:(k_u,l_s)\in J_{us}\}\notin\mathfrak{I}_2$ and $(\varphi,\omega,\psi)^{\theta_{us}}-lim\,Fx_{k_u,l_s}=\xi$.

b) $\xi\in X$ is named as FL$\mathfrak{I}_2$- cluster point of $x=(x_{kl})$ if, for every $t>0$ and $\varepsilon\in(0,1)$, we get

$$\left\{\begin{matrix} (u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\varphi\big(\hat{F}x_{kl}-\xi,t\big)>1-\varepsilon, \\ \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\big(\hat{F}x_{kl}-\xi,t\big)<\varepsilon \text{ and} \\ \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\psi\big(\hat{F}x_{kl}-\xi,t\big)<\varepsilon. \end{matrix}\right\}\notin\mathfrak{I}_2$$

$\Lambda_{(\varphi,\omega,\psi)^{\theta_{us}}}^{\mathcal{F}\mathfrak{I}_2}(x)$ and $\Gamma_{(\varphi,\omega,\psi)^{\theta_{us}}}^{\mathcal{F}\mathfrak{I}_2}(x)$ indicate the set of all FL$\mathfrak{I}_2$- limit points and the set of all FL$\mathfrak{I}_2$- cluster points in NNS, respectively.

Theorem 3.14:

For every sequence $x=(x_{kl})$ in NNS, we have $\Lambda_{(\varphi,\omega,\psi)^{\theta_{us}}}^{\mathcal{F}\mathfrak{I}_2}(x)\subseteq\Gamma_{(\varphi,\omega,\psi)^{\theta_{us}}}^{\mathcal{F}\mathfrak{I}_2}(x)$.

Proof:

Let $\xi\in\Lambda_{(\varphi,\omega,\psi)^{\theta_{us}}}^{\mathcal{F}\mathfrak{I}_2}(x)$. Then, there is a set $M\subset\mathbb{N}\times\mathbb{N}$ such that the set $M'\notin\mathfrak{I}_2$, where M and M' are as in definition (3.13), fulfills $(\varphi,\omega,\psi)^{\theta_{us}}-lim\,Fx_{k_u,l_s}=\xi$. Hence, for every $t>0$ and $\varepsilon\in(0,1)$, there are $u_0,s_0\in\mathbb{N}$ such that

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$$\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\varphi\left(\hat{F}x_{k_u,l_s}-\xi,t\right)>1-\varepsilon,\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\hat{F}x_{k_u,l_s}-\xi,t\right)<\varepsilon \text{ and}$$
$$\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\psi\left(\hat{F}x_{k_u,l_s}-\xi,t\right)<\varepsilon \text{ , for all } u\geq u_0, s\geq s_0.$$

Therefore,

$$\mathcal{B}=\left\{\begin{array}{c}(u,s)\in\mathbb{N}\times\mathbb{N}:\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\varphi\left(\hat{F}x_{kl}-\xi,t\right)>1-\varepsilon,\\ \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\hat{F}x_{kl}-\xi,t\right)<\varepsilon \text{ and}\\ \frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\psi\left(\hat{F}x_{kl}-\xi,t\right)<\varepsilon.\end{array}\right\}$$
$$\supseteq M'\backslash\left\{(k_1,l_1)<(k_2,l_2)<\cdots<\left(k_{u_0},l_{s_0}\right)\right\}.$$

Now, with $\mathfrak{I}_2$ being admissible, we must have $M'\backslash\left\{(k_1,l_1)<(k_2,l_2)<\cdots<\left(k_{u_0},l_{s_0}\right)\right\}\notin\mathfrak{I}_2$ and as such $B\in\mathfrak{I}_2$. Hence, $\xi\in\Gamma^{\mathcal{F}\mathfrak{I}_2}_{(\varphi,\omega,\psi)^{\theta_{us}}}(x)$.

Theorem 3.15:

The following observations are equivalent

a) ξ is FL$\mathfrak{I}_2$ is limit point of x.
b) There are two sequences $y=(y_{kl})$ and $z=(z_{kl})$ in NNS such that $x=y+z$ and $(\varphi,\omega,\psi)^{\theta_{us}}-lim\,Fy=\xi$ and $\{(u,s)\in\mathbb{N}\times\mathbb{N}:(k,l)\in J_{us},z_{kl}\neq\bar{0}\}\in\mathfrak{I}_2$.

Proof:

Suppose that (a) holds. Then there are M and M' are as in Definition (3.13) such that $M'\notin\mathfrak{I}_2$ and $(\varphi,\omega,\psi)^{\theta_{us}}-lim\,Fx=\xi$. Take the sequences y and z as follows:

$$y_{kl}=\begin{cases}x_{kl}, & \text{if } (k,l)\in J_{us},(u,s)\in M'\\ \xi, & \text{otherwise.}\end{cases} \quad\text{and}\quad z_{kl}=\begin{cases}\bar{0}, & \text{if } (k,l)\in J_{us},(u,s)\in M'\\ x_{kl}-\xi, & \text{otherwise.}\end{cases}$$

It is adequate to think the case $(k,l)\in J_{us}$ such that $(u,s)\in\mathbb{N}\times\mathbb{N}/M'$.

Then for each $t>0$ and $\varepsilon\in(0,1)$. Then, we have

$\varphi\left(\hat{F}y_{kl}-\xi,t\right)=1>1-\varepsilon$, $\omega\left(\hat{F}y_{kl}-\xi,t\right)=0<\varepsilon$ and $\psi\left(\hat{F}y_{kl}-\xi,t\right)=0<\varepsilon$.

Thus, we write

$\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\varphi\left(\hat{F}y_{kl}-\xi,t\right)=1>1-\varepsilon$, $\frac{1}{h_{us}}\sum_{(k,l)\in J_{us}}\omega\left(\hat{F}y_{kl}-\xi,t\right)=0<\varepsilon$,

and $\frac{1}{h_{us}}\sum_{(k,l),\in J_{us}}\psi\left(\hat{F}y_{kl}-\xi,t\right)=0<\varepsilon$. Hence, $(\varphi,\omega,\psi)^{\theta_{us}}-lim\,y=\xi$.

Now, we have $\{(u,s)\in\mathbb{N}\times\mathbb{N}:(k,l)\in J_{us},z_{kl}\neq\bar{0}\}\subset\mathbb{N}\times\mathbb{N}\backslash M'$.

But $\mathbb{N}\times\mathbb{N}\backslash M'\in\mathfrak{I}_2$ and so $\{(u,s)\in\mathbb{N}\times\mathbb{N}:(k,l)\in J_{us},z_{kl}\neq\bar{0}\}\in\mathfrak{I}_2$.

Now, presume that (b) holds. Let $\{(u,s)\in\mathbb{N}\times\mathbb{N}:(k,l)\in J_{us},z_{kl}=\bar{0}\}$.

Then, obviously $M'\in\mathcal{F}(\mathfrak{I}_2)$ and so it is an infinite set. Construct the set

$M=\{(k_1,l_1)<(k_2,l_2)<\cdots<(k_u,l_s)<\cdots\}\subset\mathbb{N}\times\mathbb{N}$ such that $(k_u,l_s)\in J_{us}$ and $z_{k_u,l_s}=\bar{0}$. Since $x_{k_u,l_s}=y_{k_u,l_s}$ and $(\varphi,\omega,\psi)^{\theta_{us}}-lim\,y=\xi$, we obtain $(\varphi,\omega,\psi)^{\theta_{us}}-lim\,Fx_{k_u,l_s}=\xi$.

4. Conclusion

In this paper, we have introduced the concept of $FL\mathfrak{T}_2$- Convergent of double sequence in NNS. Also, we proved some basic results in this space. The definition of $FL\mathfrak{T}_2$- Convergent and $FL\mathfrak{T}_2$- Cauchy with respect to NNS are discussed. After that, the definitions of $FL\mathfrak{T}_2$- limit point and $FL\mathfrak{T}_2$- cluster point are discussed.

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