JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1304-1312
https://publishoa.com
ISSN: 1309-3452

On The Stability Of Jensen Functional Equation In Neutrosophic Normed Spaces

A.N. Mangayarkkarasi[1,*] , V. Jeyanthi[2] M. Jeyaraman[3] and B. Silamparasn[4]

[1] Part Time - Research Scholar, Government Arts College for Women, Sivagangai, Affiliated to Alagappa University, Karaikudi, Tamilnadu, India.
[*]Department of Mathematics, Nachiappa Swamigal Arts & Science College, Karaikudi, Affiliated to Alagappa University, Karaikudi, Tamilnadu, India. E-mail: murugappan.mangai@gmail.com
[2]Government Arts College for Women, Sivagangai, Affiliated to Alagappa University, Karaikudi, Tamilnadu, India.
E-mail : jeykaliappa@gmail.com.
[3]P.G.and Research Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai, Affiliated to Alagappa University, Karaikudi, Tamilnadu, India. E-mail:jeya.math@gmail.com , ORCID : https://orcid.org/0000-0002-0364-1845
[4]Research Scholar, P.G. and Research Department of Mathematics, Raja Doraisingam Govt. Arts College, Sivagangai, Affiliated to Alagappa University, Karaikudi, Tamilnadu, India. E-mail: simpu2411@gmail.com

ABSTRACT

In this paper, we determine some stability results concerning the Jensen functional equation $2f((x+y)/2) = f(x) + f(y)$ in Neutrosophic Normed Spaces (NNS). We define the Neutrosophic Continuity of the Jensen mappings and prove that the existence of a solution for any approximately Jensen mapping implies the completeness of NNS.

Keywords: Jensen mapping, Completeness, Pseudo definite, Neutrosophic Normed spaces.
AMS Subject Classification (2020): 40A30 ; 40G15**.**

1. Introduction

Fuzzy set theory is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. It has also very useful applications in various fields. Stability problem of a functional equation was first posed by Ulam [20] which was answered by Hyers [6] and then generalized by Aoki [1] and Rassias [15] for additive mappings and linear mappings, respectively. Since then several stability problems for various functional equations have been investigated in [7] and [16] and various fuzzy stability results concerning Jensen functional equations were discussed. Furthermore some stability results concerning Jensen, cubic, mixed-type additive and cubic functional equations were investigated in the spirit of intuitionistic fuzzy normed spaces, while the idea of intuitionistic fuzzy normed space was introduced and further studied.

After a while, Smarandache [17] introduced the notion of Neutrosophic Sets [NS], which is the different kind of the notation of the classical set theory by adding an intermediate membership function. This set is a formal setting trying to measure the truth, indeterminacy and falsehood. Later on, the concepts of statistical convergence of double sequences have been analyzed in IFNS by Mursaleen and Mohiuddin [13]. Quite recently, Kirisci and Simsek [19] introduced the notion of Neutrosophic normed space and statistical convergence. Since Neutrosophic Normed Space [NNS] is a natural generalization of IFNS and statistical convergence.

In this paper, we determine some stability results concerning the Jensen functional equation $2f((x+y)/2) = f(x) + f(y)$ in NNS. We define the Neutrosophic Continuity of the Jensen mappings and prove that the existence of a solution for any approximately Jensen mapping implies the completeness of NNS.

2. Preliminaries

Definition 2.1:

A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is said to be continuous t-norm if it satisfies the following conditions;

i. $*$ is associative and commutative,
ii. $*$ is continuous,
iii. $a * 1 = a$, for all $a \in [0,1]$,
iv. $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$,

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1304-1312
https://publishoa.com
ISSN: 1309-3452

for each $a, b, c, d \in [0,1]$.

Definition 2.2:

A binary operation $\diamondsuit: [0,1] \times [0,1] \to [0,1]$ is said to be a continuous t-conorm if it satisfies the following conditions;

i. $\diamondsuit$ is associative and commutative,
ii. $\diamondsuit$ is continuous,
iii. $a \diamondsuit 0 = a$, for all $a \in [0,1]$,
iv. $a \diamondsuit b \leq c \diamondsuit d$ whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$.

Definition 2.3:

The seven-tuple $(X, \mu, \vartheta, \omega, *, \diamondsuit, \odot)$ is said to be Neutrosophic Normed Space (NNS) if X is a vector space, $*$ is a continuous t-norm, $\diamondsuit$ and $\odot$ are continuous t-conorm and μ, ϑ, ω are fuzzy sets on $X \times \mathbb{R}$ satisfying the following conditions; For every $x, y \in X$ and $s, t > 0$

i. $\mu(x,t) + \vartheta(x,t) + \omega(x,t) \leq 3$,
ii. $0 \leq \mu(x,t) \leq 1, 0 \leq \vartheta(x,t) \leq 1, 0 \leq \omega(x,t) \leq 1$,
iii. $\mu(x,t) > 0$,
iv. $\mu(x,t) = 1$ iff $x = 0$,
v. $\mu(\alpha x, t) = \mu\left(x, \frac{t}{|\alpha|}\right)$, for each $\alpha \neq 0$,
vi. $\mu(x,t) * \mu(y,s) \leq \mu(x+y, t+s)$,
vii. $\mu(x,\cdot): (0,\infty) \to [0,1]$ is continuous,
viii. $\lim_{t\to\infty} \mu(x,t) = 1$ and $\lim_{t\to 0} \mu(x,t) = 0$,
ix. $\vartheta(x,t) < 1$
x. $\vartheta(x,t) = 0$ iff $x = 0$,
xi. $\vartheta(\alpha x, t) = \vartheta\left(x, \frac{t}{|\alpha|}\right)$, for each $\alpha \neq 0$,
xii. $\vartheta(x,t) \diamondsuit \vartheta(y,s) \geq \vartheta(x+y, t+s)$,
xiii. $\vartheta(x,\cdot): (0,\infty) \to [0,1]$ is continuous,
xiv. $\lim_{t\to\infty} \vartheta(x,t) = 0$ and $\lim_{t\to 0} \vartheta(x,t) = 1$.
xv. $\omega(x,t) < 1$,
xvi. $\omega(x,t) = 0$ iff $x = 0$,
xvii. $\omega(\alpha x, t) = \omega\left(x, \frac{t}{|\alpha|}\right)$,for each $\alpha \neq 0$,
xviii. $\omega(x,t) \odot \omega(y,s) \geq \omega(x+y, t+s)$,
xix. $\omega(x,\cdot): (0,\infty) \to [0,1]$ is continuous,
xx. $\lim_{t\to\infty} \omega(x,t) = 0$ and $\lim_{t\to 0} \omega(x,t) = 1$.

Example 2.4:

Let $(X, \|\cdot\|)$ be a normed space, $a * b = ab$, $a \diamondsuit b = min\{a+b, 1\}$ and $a \odot b = min\{a+b, 1\}$, for all $a, b \in [0,1]$. For all $x \in X$ and every $t > 0$, Consider $\mu(x,t) = \begin{Bmatrix} \frac{t}{t+\|x\|} & \text{if } t > 0 \\ 0 & \text{if } t \leq 0 \end{Bmatrix}$, $\vartheta(x,t) = \begin{Bmatrix} \frac{\|x\|}{t+\|x\|} & \text{if } t > 0 \\ 1 & \text{if } t \leq 0 \end{Bmatrix}$ and $\omega(x,t) = \begin{Bmatrix} \frac{\|x\|}{t} & \text{if } t > 0 \\ 1 & \text{if } t \leq 0 \end{Bmatrix}$. Then $(X, \mu, \vartheta, \omega, *, \diamondsuit, \odot)$ is NNS.

3. Neutrosophic Stability of Jensen mapping

Let $(X, \mu_1, \vartheta_1, \omega_1, *, \diamondsuit, \odot)$ and $(Y, \mu_2, \vartheta_2, \omega_2, *, \diamondsuit, \odot)$ be two NNSs and $f: X \to Y$ be a mapping. Then f is said to be Neutrosophic Continuous at a point $x_0 \in X$ if for each sequence (x_n) converging to x_0, the sequence $(f(x_n))$ converges to $f(x_0)$. If f is Neutrosophic Continuous at each point of $x_0 \in X$ then f is said to be Neutrosophic Continuous on X. The Jensen functional equation is $2f((x+y)/2) = f(x) + f(y)$, where f is a mapping between linear spaces. It is easy to see that a mapping $f: X \to Y$ between linear spaces with $f(0) = 0$ satisfies the Jensen equation if and only if it is additive.

We begin with a generalized Hyers- Ulam- Rassias type theorem in NNS for the Jensen functional equation.

Theorem 3.1:

Let X be a linear space and f be a mapping from X to a Neutrosophic Banach Space (NBS) $(Y, \mu, \vartheta, \omega)$ such that $f(0) = 0$. Suppose that φ is a function from X to a NNS $(Z, \mu', \vartheta', \omega')$ such that

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1304-1312
https://publishoa.com
ISSN: 1309-3452

$$\begin{cases} \mu\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y),t+s\right) \geq \mu'(\varphi(x),t) * \mu'(\varphi(y),s), \\ \vartheta\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y),t+s\right) \leq \vartheta'(\varphi(x),t) \diamond \vartheta'(\varphi(y),s) \text{ and} \\ \omega\left(2f\left(\frac{x+y}{2}\right)-f(x)-f(y),t+s\right) \leq \omega'(\varphi(x),t) \odot \omega'(\varphi(y),s), \end{cases} \quad (3.1.1)$$

for all $x,y \in X \setminus \{0\}$, $t>0$ and $s>0$. If $\varphi(3x)=\alpha\varphi(x)$ for some real number α with $0<|\alpha|<3$, then there exists a unique additive mapping $T: X \to Y$ such that $T(x)=\lim_{n\to\infty} f(3^n x)/3^n$,
$\mu(T(x)-f(x),t) \geq M\left(x,\frac{(3-\alpha)t}{6}\right)$, $\vartheta(T(x)-f(x),t) \leq N\left(x,\frac{(3-\alpha)t}{6}\right)$ and $\omega(T(x)-f(x),t) \leq P\left(x,\frac{(3-\alpha)t}{6}\right)$, where
$M(x,t)=\mu'\left(\varphi(x),\frac{3}{4}t\right) * \mu'\left(\varphi(-x),\frac{3}{4}t\right) * \mu'\left(\varphi(3x),\frac{3}{4}t\right)$,
$N(x,t)=\vartheta'\left(\varphi(x),\frac{3}{4}t\right) \diamond \vartheta'\left(\varphi(-x),\frac{3}{4}t\right) \diamond \vartheta'\left(\varphi(3x),\frac{3}{4}t\right)$ and
$P(x,t)=\omega'\left(\varphi(x),\frac{3}{4}t\right) \odot \omega'\left(\varphi(-x),\frac{3}{4}t\right) \odot \omega'\left(\varphi(3x),\frac{3}{4}t\right)$.

Proof:

Without loss of generality we may assume that $0<\alpha<3$. Putting $y=-x$ and $s=t$ in (3.1.1), we get $\mu(-f(x)-f(-x),2t) \geq \mu'(\varphi(x),t) * \mu'(\varphi(-x),t)$, $\vartheta(-f(x)-f(-x),2t) \leq \vartheta'(\varphi(x),t) \diamond \vartheta'(\varphi(-x),t)$ and $\omega(-f(x)-f(-x),2t) \leq \omega'(\varphi(x),t) \odot \omega'(\varphi(-x),t)$, for all $x \in X$ and $t>0$. Replacing x by $-x$, y by $3x$ and s by t in (3.1.1), we get
$\mu(2f(x)-f(-x)-f(3x),2t) \geq \mu'(\varphi(-x),t) * \mu'(\varphi(3x),t)$,
$\vartheta(2f(x)-f(-x)-f(3x),2t) \leq \vartheta'(\varphi(-x),t) \diamond \vartheta'(\varphi(3x),t)$ and
$\omega(2f(x)-f(-x)-f(3x),2t) \leq \omega'(\varphi(-x),t) \odot \omega'(\varphi(3x),t)$.
Thus, $\mu(3f(x)-f(3x),4t) \geq \mu'(\varphi(x),t) * \mu'(\varphi(-x),t) * \mu'(\varphi(3x),t)$,
$\vartheta(3f(x)-f(3x),4t) \leq \vartheta'(\varphi(x),t) \diamond \vartheta'(\varphi(-x),t) \diamond \vartheta'(\varphi(3x),t)$ and
$\omega(3f(x)-f(3x),4t) \leq \omega'(\varphi(x),t) \odot \omega'(\varphi(-x),t) \odot \omega'(\varphi(3x),t)$. It follows that

$$\begin{cases} \mu(f(x)-3^{-1}f(3x),t) \geq \mu'\left(\varphi(x),\frac{3}{4}t\right) * \mu'\left(\varphi(-x),\frac{3}{4}t\right) * \mu'\left(\varphi(3x),\frac{3}{4}t\right), \\ \vartheta(f(x)-3^{-1}f(3x),t) \leq \vartheta'\left(\varphi(x),\frac{3}{4}t\right) \diamond \vartheta'\left(\varphi(-x),\frac{3}{4}t\right) \diamond \vartheta'\left(\varphi(3x),\frac{3}{4}t\right) \text{ and} \\ \omega(f(x)-3^{-1}f(3x),t) \geq \omega'\left(\varphi(x),\frac{3}{4}t\right) \odot \omega'\left(\varphi(-x),\frac{3}{4}t\right) \odot \omega'\left(\varphi(3x),\frac{3}{4}t\right). \end{cases} \quad (3.1.2)$$

Define, $M(x,t)=\mu'\left(\varphi(x),\frac{3}{4}t\right) * \mu'\left(\varphi(-x),\frac{3}{4}t\right) * \mu'\left(\varphi(3x),\frac{3}{4}t\right)$,
$N(x,t)=\vartheta'\left(\varphi(x),\frac{3}{4}t\right) \diamond \vartheta'\left(\varphi(-x),\frac{3}{4}t\right) \diamond \vartheta'\left(\varphi(3x),\frac{3}{4}t\right)$ and
$P(x,t)=\omega'\left(\varphi(x),\frac{3}{4}t\right) \odot \omega'\left(\varphi(-x),\frac{3}{4}t\right) \odot \omega'\left(\varphi(3x),\frac{3}{4}t\right)$.
Then by our assumption, $\{M(3x,t)=M(x,t/\alpha)$, $N(3x,t)=N(x,t/\alpha)$ and $P(3x,t)=P(x,t/\alpha).\}$ (3.1.3)
Replacing x by $3^n x$ in (3.1.2) and using (3.1.3), we get
$\mu(f(3^n x)/3^n - f(3^{n+1}x)/3^{n+1}, \alpha^n t/3^n) = \mu(f(3^n x)-3^{-1}f(3^{n+1}x),\alpha^n t) \geq M(3^n x,\alpha^n t)=M(x,t)$,
$\vartheta(f(3^n x)/3^n - f(3^{n+1}x)/3^{n+1}, \alpha^n t/3^n) = \vartheta(f(3^n x)-3^{-1}f(3^{n+1}x),\alpha^n t) \leq N(3^n x,\alpha^n t)=N(x,t)$ and
$\omega(f(3^n x)/3^n - f(3^{n+1}x)/3^{n+1}, \alpha^n t/3^n) = \omega(f(3^n x)-3^{-1}f(3^{n+1}x),\alpha^n t) \leq P(3^n x,\alpha^n t)=P(x,t)$.
Thus for each $n>m$, we have

$$\begin{cases} \mu\left(f(3^m x)/3^m - f(3^n x)/3^n, \sum_{k=m}^{n-1}\frac{\alpha^k t}{3^k}\right) = \mu\left(\sum_{k=m}^{n-1} f(3^k x)\big/3^k - f(3^{k+1}x)\big/3^{k+1}, \sum_{k=m}^{n-1}\frac{\alpha^k t}{3^k}\right) \\ \qquad \geq \prod_{k=m}^{n-1}\mu\left(f(3^k x)\big/3^k - f(3^{k+1}x)\big/3^{k+1}, \sum_{k=m}^{n-1}\frac{\alpha^k t}{3^k}\right) \geq M(x,t), \\ \vartheta\left(f(3^m x)/3^m - f(3^n x)/3^n, \sum_{k=m}^{n-1}\frac{\alpha^k t}{3^k}\right) = \vartheta\left(\sum_{k=m}^{n-1} f(3^k x)\big/3^k - f(3^{k+1}x)\big/3^{k+1}, \sum_{k=m}^{n-1}\frac{\alpha^k t}{3^k}\right) \\ \qquad \leq \coprod_{k=m}^{n-1}\vartheta\left(f(3^k x)\big/3^k - f(3^{k+1}x)\big/3^{k+1}, \sum_{k=m}^{n-1}\frac{\alpha^k t}{3^k}\right) \leq N(x,t) \text{ and} \\ \omega\left(f(3^m x)/3^m - f(3^n x)/3^n, \sum_{k=m}^{n-1}\frac{\alpha^k t}{3^k}\right) = \omega\left(\sum_{k=m}^{n-1} f(3^k x)\big/3^k - f(3^{k+1}x)\big/3^{k+1}, \sum_{k=m}^{n-1}\frac{\alpha^k t}{3^k}\right) \\ \qquad \leq \coprod_{k=m}^{n-1}\omega\left(f(3^k x)\big/3^k - f(3^{k+1}x)\big/3^{k+1}, \sum_{k=m}^{n-1}\frac{\alpha^k t}{3^k}\right) \leq P(x,t), \end{cases} \quad (3.1.4)$$

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1304-1312
https://publishoa.com
ISSN: 1309-3452

where $\prod_{j=1}^{n} a_j = a_1 * a_2 * \dots * a_n$, $\coprod_{j=1}^{n} a_j = a_1 \diamond a_2 \diamond \dots \diamond a_n$ and $\coprod_{j=1}^{n} a_j = a_1 \odot a_2 \odot \dots \odot a_n$. Let $\varepsilon > 0$ and $\delta > 0$ be given. Since $\lim_{t\to\infty} M(x,t) = 1$, $\lim_{t\to\infty} N(x,t) = 0$ and $\lim_{t\to\infty} P(x,t) = 0$ there exists some $t_0 > 0$ such that $M(x,t_0) > 1-\varepsilon$, $N(x,t_0) < \varepsilon$ and $P(x,t_0) < \varepsilon$. Since $\sum_{k=0}^{\infty} \alpha^k t_0/3^k < \infty$, there exists some $n_0 \in N$ such that $\sum_{k=m}^{n-1} \alpha^k t_0/3^k < \delta$ for all $n > m \geq n_0$. It follows that

$$\mu(f(3^m x)/3^m - f(3^n x)/3^n, \delta) \geq \mu\left(f(3^m x)/3^m - f(3^n x)/3^n, \sum_{k=m}^{n-1} \frac{\alpha^k t_0}{3^k}\right) \geq M(x,t_0) \geq 1-\varepsilon,$$

$$\vartheta(f(3^m x)/3^m - f(3^n x)/3^n, \delta) \leq \vartheta\left(f(3^m x)/3^m - f(3^n x)/3^n, \sum_{k=m}^{n-1} \frac{\alpha^k t_0}{3^k}\right) \leq N(x,t_0) \leq \varepsilon \text{ and}$$

$$\omega(f(3^m x)/3^m - f(3^n x)/3^n, \delta) \leq \omega\left(f(3^m x)/3^m - f(3^n x)/3^n, \sum_{k=m}^{n-1} \frac{\alpha^k t_0}{3^k}\right) \leq P(x,t_0) \leq \varepsilon.$$

This shows that $(f(3^n x)/3^n)$ is a Cauchy Sequence in (Y,μ,ϑ,ω). Since (Y,μ,ϑ,ω) is complete, $(f(3^n x)/3^n)$ converges to some $T(x) \in Y$.

Thus, we define a mapping $T: X \to Y$ such that $T(x) = (\mu,\vartheta,\omega) - \lim_{n\to\infty} f(3^n x)/3^n$.

Moreover, if we put $m = 0$ in (3.1.4),

we get $\mu\left(f(3^n x)/3^n - f(x), \sum_{k=0}^{n-1} \frac{\alpha^k t}{3^k}\right) \geq M(x,t)$, $\vartheta\left(f(3^n x)/3^n - f(x), \sum_{k=0}^{n-1} \frac{\alpha^k t}{3^k}\right) \leq N(x,t)$ and

$\omega\left(f(3^n x)/3^n - f(x), \sum_{k=0}^{n-1} \frac{\alpha^k t}{3^k}\right) \leq P(x,t)$. Therefore

$$\left\{\begin{array}{c} \mu(f(3^n x)/3^n - f(x), t) \geq M\left(x, \frac{t}{\sum_{k=0}^{n-1}(\alpha/3)^k}\right), \\ \vartheta(f(3^n x)/3^n - f(x), t) \leq N\left(x, \frac{t}{\sum_{k=0}^{n-1}(\alpha/3)^k}\right) \text{ and} \\ \omega(f(3^n x)/3^n - f(x), t) \leq P\left(x, \frac{t}{\sum_{k=0}^{n-1}(\alpha/3)^k}\right). \end{array}\right\} \tag{3.1.5}$$

Now, we will show that T is additive. Let $x, y \in X$. Then

$$\left\{\begin{array}{c} \mu\left(2T\left(\frac{x+y}{2}\right) - T(x) - T(y), t\right) \geq \mu\left(2T\left(\frac{x+y}{2}\right) - 2f\left(\frac{3^n(x+y)}{2}\right)\Big/3^n, \frac{t}{4}\right) \\ * \mu\left(f(3^n x)/3^n - T(x), \frac{t}{4}\right) * \mu\left(f(3^n y)/3^n - T(y), \frac{t}{4}\right) \\ * \mu\left(2f\left(\frac{3^n(x+y)}{2}\right)\Big/3^n - f(3^n x)/3^n - f(3^n y)/3^n, \frac{t}{4}\right), \\ \vartheta\left(2T\left(\frac{x+y}{2}\right) - T(x) - T(y), t\right) \leq \vartheta\left(2T\left(\frac{x+y}{2}\right) - 2f\left(\frac{3^n(x+y)}{2}\right)\Big/3^n, \frac{t}{4}\right) \\ \diamond \vartheta\left(f(3^n x)/3^n - T(x), \frac{t}{4}\right) \diamond \vartheta\left(f(3^n y)/3^n - T(y), \frac{t}{4}\right) \\ \diamond \vartheta\left(2f\left(\frac{3^n(x+y)}{2}\right)\Big/3^n - f(3^n x)/3^n - f(3^n y)/3^n, \frac{t}{4}\right) \text{ and} \\ \omega\left(2T\left(\frac{x+y}{2}\right) - T(x) - T(y), t\right) \leq \omega\left(2T\left(\frac{x+y}{2}\right) - 2f\left(\frac{3^n(x+y)}{2}\right)\Big/3^n, \frac{t}{4}\right) \\ \odot \omega\left(f(3^n x)/3^n - T(x), \frac{t}{4}\right) \odot \omega\left(f(3^n y)/3^n - T(y), \frac{t}{4}\right) \\ \odot \omega\left(2f\left(\frac{3^n(x+y)}{2}\right)\Big/3^n - f(3^n x)/3^n - f(3^n y)/3^n, \frac{t}{4}\right), \end{array}\right\} \tag{3.1.6}$$

and by using(3.1.1),

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1304-1312
https://publishoa.com
ISSN: 1309-3452

$$\left.\begin{cases} \mu\left(2f\left(\frac{3^n(x+y)}{2}\right)\Big/3^n - f(3^nx)/3^n - f(3^ny)/3^n, \frac{t}{4}\right) \\ \geq \mu'\left(\varphi(3^nx), \frac{3^nt}{8}\right) * \mu'\left(\varphi(3^ny), \frac{3^nt}{8}\right) = \mu'\left(\varphi(x), \left(\frac{3}{\alpha}\right)^n \frac{t}{8}\right) * \mu'\left(\varphi(y), \left(\frac{3}{\alpha}\right)^n \frac{t}{8}\right), \\ \vartheta\left(2f\left(\frac{3^n(x+y)}{2}\right)\Big/3^n - f(3^nx)/3^n - f(3^ny)/3^n, \frac{t}{4}\right) \\ \leq \vartheta'\left(\varphi(3^nx), \frac{3^nt}{8}\right) \diamond \vartheta'\left(\varphi(3^ny), \frac{3^nt}{8}\right) = \vartheta'\left(\varphi(x), \left(\frac{3}{\alpha}\right)^n \frac{t}{8}\right) \diamond \vartheta'\left(\varphi(y), \left(\frac{3}{\alpha}\right)^n \frac{t}{8}\right) \text{ and} \\ \omega\left(2f\left(\frac{3^n(x+y)}{2}\right)\Big/3^n - f(3^nx)/3^n - f(3^ny)/3^n, \frac{t}{4}\right) \\ \leq \omega'\left(\varphi(3^nx), \frac{3^nt}{8}\right) \odot \omega'\left(\varphi(3^ny), \frac{3^nt}{8}\right) = \omega'\left(\varphi(x), \left(\frac{3}{\alpha}\right)^n \frac{t}{8}\right) \odot \omega'\left(\varphi(y), \left(\frac{3}{\alpha}\right)^n \frac{t}{8}\right). \end{cases}\right\} \quad (3.1.7)$$

Letting $n \to \infty$ in (3.1.6) and (3.1.7), we get

$\mu\left(2T\left(\frac{x+y}{2}\right) - T(x) - T(y), t\right) = 1, \vartheta\left(2T\left(\frac{x+y}{2}\right) - T(x) - T(y), t\right) = 0$ and $\omega\left(2T\left(\frac{x+y}{2}\right) - T(x) - T(y), t\right) = 0$, for all $x, y \in X$ and $t > 0$. This means that T satisfies the Jensen equation and so it is additive.

Now, we approximate the difference between f and T in a Neutrosophic sense. By (3.1.5), we have

$\mu(T(x) - f(x), t) \geq \mu(T(x) - f(3^nx)/3^n, t/2) * \mu(f(3^nx)/3^n - f(x), t/2)$
$\geq M\left(x, \frac{t}{2\sum_{k=0}^{\infty}(\alpha/3)^k}\right) = M\left(x, \frac{(3-\alpha)t}{6}\right)$,
$\vartheta(T(x) - f(x), t) \leq \vartheta(T(x) - f(3^nx)/3^n, t/2) \diamond \vartheta(f(3^nx)/3^n - f(x), t/2)$
$\leq N\left(x, \frac{t}{2\sum_{k=0}^{\infty}(\alpha/3)^k}\right) = N\left(x, \frac{(3-\alpha)t}{6}\right)$ and
$\omega(T(x) - f(x), t) \leq \omega(T(x) - f(3^nx)/3^n, t/2) \odot \omega(f(3^nx)/3^n - f(x), t/2)$
$\leq P\left(x, \frac{t}{2\sum_{k=0}^{\infty}(\alpha/3)^k}\right) = P\left(x, \frac{(3-\alpha)t}{6}\right)$, for every $x \in X, t > 0$ and sufficiently large n.

To prove the uniqueness of T, assume that T' be another additive mapping from X into Y, which satisfies the required inequality. Then

$\mu(T(x) - T'(x), t) \geq \mu(T(x) - f(x), t/2) * \mu(T'(x) - f(x), t/2) \geq M\left(x, \frac{(3-\alpha)t}{12}\right)$,
$\vartheta(T(x) - T'(x), t) \leq \vartheta(T(x) - f(x), t/2) \diamond \vartheta(T'(x) - f(x), t/2) \leq N\left(x, \frac{(3-\alpha)t}{12}\right)$ and
$\omega(T(x) - T'(x), t) \leq \omega(T(x) - f(x), t/2) \odot \omega(T'(x) - f(x), t/2) \leq P\left(x, \frac{(3-\alpha)t}{12}\right)$,
for all $x \in X$ and $t > 0$ and $n \in \mathbb{N}$. Since $0 < \alpha < 3$, $\lim_{n\to\infty}(3/\alpha)^n = \infty$ and we get
$\lim_{n\to\infty} M\left(x, \frac{(3/\alpha)^n(3-\alpha)t}{12}\right) = 1, \lim_{n\to\infty} N\left(x, \frac{(3/\alpha)^n(3-\alpha)t}{12}\right) = 0$ and $\lim_{n\to\infty} P\left(x, \frac{(3/\alpha)^n(3-\alpha)t}{12}\right) = 0$.
Therefore $\mu(T(x) - T'(x), t) = 1$, $\vartheta(T(x) - T'(x), t) = 0$ and $\omega(T(x) - T'(x), t) = 0$, for all $x \in X$ and $t > 0$.
Hence $T(x) = T'(x)$ for all $x \in X$.

Remark 3.2:

We can also prove Theorem (3.1) for the case when $|\alpha| > 3$. In this case, the additive mapping T is defined by $T(x) = \lim_{n\to\infty} f(3^{-n})/3^{-n}$.

Theorem 3.3:

Let X be a Linear Space (LS) and $(Y, \mu', \vartheta', \omega')$ be a NBS. Let $f: X \to Y$ be a mapping with $f(0) = 0$. Suppose that $\delta > 0$ is a positive real number and z_0 is a fixed vector of a NNS $(Z, \mu'', \vartheta'', \omega'')$ such that

$$\mu'\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t + s\right) \geq \mu''(\delta z_0, t) * \mu''(\delta z_0, s),$$
$$\vartheta'\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t + s\right) \leq \vartheta''(\delta z_0, t) \diamond \vartheta''(\delta z_0, s) \text{and}$$
$$\omega'\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t + s\right) \leq \omega''(\delta z_0, t) \odot \omega''(\delta z_0, s),$$

for all $x, y \in X - \{0\}, t > 0$ and $s > 0$. Then there exists a unique additive mapping $T: X \to Y$ such that $\mu'(T(x) - f(x), t) \geq \mu''\left(z_0, \frac{t}{4\delta}\right), \vartheta'(T(x) - f(x), t) \leq \vartheta''\left(z_0, \frac{t}{4\delta}\right)$ and $\omega'(T(x) - f(x), t) \leq \omega''\left(z_0, \frac{t}{4\delta}\right)$. Moreover, T is continuous on X provided $(X, \mu, \vartheta, \omega)$ is a NNS and f is continuous at a point.

Proof :

Using Theorem (3.1) with $\varphi(x) = \delta z_0$ we deduce the existence of the required additive mapping T.

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1304-1312
https://publishoa.com
ISSN: 1309-3452

Let us put $\beta = \frac{1}{12\delta}$. Suppose that f is continuous at a point x_0. Let T be not continuous at a point.
Then there exists a sequence (x_n) such that $(\mu, \vartheta, \omega) - \lim_n x_n = 0$ and $(\mu', \vartheta', \omega') - \lim_n x_n \neq 0$. By passing to a subsequence if necessary, we may assume that $(\mu, \vartheta, \omega) - \lim_n x_n = 0$ and there exist $\varepsilon > 0$ and $t_0 > 0$ such that $\mu'(Tx_n, t_0) < 1 - \varepsilon, \vartheta'(Tx_n, t_0) > \varepsilon$ and $\omega'(Tx_n, t_0) > \varepsilon$ for all n.
Since $\lim_{t\to\infty} \mu''(z_0, \beta t) = 1$, $\lim_{t\to\infty} \vartheta''(z_0, \beta t) = 0$ and $\lim_{t\to\infty} \omega''(z_0, \beta t) = 0$, there exists t_1 such that $\mu''(z_0, \beta t_1) \geq 1 - \varepsilon, \vartheta''(z_0, \beta t_1) \leq \varepsilon$ and $\omega''(z_0, \beta t_1) \leq \varepsilon$. Take a positive integer k such that $t_1/k < t_0$.
Then, we have

$$\left\{\begin{array}{c} \mu'(T(kx_n + x_0) - Tx_0, t_1) = \mu'(Tx_n, t_1/k) \leq \mu'(Tx_0, t_0) < 1 - \varepsilon, \\ \vartheta'(T(kx_n + x_0) - Tx_0, t_1) = \vartheta'(Tx_n, t_1/k) \geq \vartheta'(Tx_0, t_0) > \varepsilon \text{ and} \\ \omega'(T(kx_n + x_0) - Tx_0, t_1) = \omega'(Tx_n, t_1/k) \geq \omega'(Tx_0, t_0) > \varepsilon. \end{array}\right\} \tag{3.3.1}$$

On the other hand,

$$\left\{\begin{array}{c} \mu'(T(kx_n + x_0) - Tx_0, t_1) \\ \geq \mu'(T(kx_n + x_0) - f(kx_n + x_0), t_1/3) * \mu'(f(kx_n + x_0) - f(x_0), t_1/3) \\ * \mu'(f(x_0) - Tx_0, t_1/3), \\ \vartheta'(T(kx_n + x_0) - Tx_0, t_1) \\ \leq \vartheta'(T(kx_n + x_0) - f(kx_n + x_0), t_1/3) \diamondsuit \vartheta'(f(kx_n + x_0) - f(x_0), t_1/3) \\ \diamondsuit \vartheta'(f(x_0) - Tx_0, t_1/3) \text{ and} \\ \omega'(T(kx_n + x_0) - Tx_0, t_1) \\ \leq \omega'(T(kx_n + x_0) - f(kx_n + x_0), t_1/3) \odot \omega'(f(kx_n + x_0) - f(x_0), t_1/3) \\ \odot \omega'(f(x_0) - Tx_0, t_1/3). \end{array}\right\} \tag{3.3.2}$$

$$\left\{\begin{array}{c} \mu'(f(x_0) - Tx_0, t_1/3) \geq \mu''(z_0, \beta t_1) \quad \text{and} \\ \mu'(T(kx_n + x_0) - f(kx_n + x_0), t_1/3) \geq \mu''(z_0, \beta t_1), \\ \vartheta'(f(x_0) - Tx_0, t_1/3) \leq \vartheta''(z_0, \beta t_1) \quad \text{and} \\ \vartheta'(T(kx_n + x_0) - f(kx_n + x_0), t_1/3) \leq \vartheta''(z_0, \beta t_1) \\ \omega'(f(x_0) - Tx_0, t_1/3) \leq \omega''(z_0, \beta t_1) \quad \text{and} \\ \omega'(T(kx_n + x_0) - f(kx_n + x_0), t_1/3) \leq \omega''(z_0, \beta t_1) \end{array}\right\} \tag{3.3.3}$$

Letting limit $n \to \infty$ in (3.3.2) and (3.3.3), we get
$\mu'(T(kx_n + x_0) - Tx_0, t_1) \geq \mu''(z_0, \beta t_1) \geq 1 - \varepsilon$,
$\vartheta'(T(kx_n + x_0) - Tx_0, t_1) \leq \vartheta''(z_0, \beta t_1) \leq \varepsilon$ and
$\omega'(T(kx_n + x_0) - Tx_0, t_1) \leq \omega''(z_0, \beta t_1) \leq \varepsilon$, which contradicts (3.3.1).

4. Neutrosophic Completeness

Definition 4.1:
Let $(X, \mu, \vartheta, \omega)$ be NNS and $\alpha \in (0,1)$. A mapping $f_\alpha: \mathbb{N} \to (X, \mu, \vartheta, \omega)$ is said to be α-approximately Jensen type if $\mu(2f_\alpha(x + y) - f_\alpha(2x) - f_\alpha(2y), \beta) \geq \alpha$, $\vartheta(2f_\alpha(x + y) - f_\alpha(2x) - f_\alpha(2y), \beta) \leq 1 - \alpha$ and
$\vartheta(2f_\alpha(x + y) - f_\alpha(2x) - f_\alpha(2y), \beta) \leq 1 - \alpha$, for some $\beta > 0$ and all $x, y \in \mathbb{N}$.
Definition 4.2:
The NNS $(X, \mu, \vartheta, \omega)$ is called definite if
$\mu(x, t) > 0, \vartheta(x, t) < 1$ and $\omega(x, t) < 1$, for all $t > 0$ implies that $x = 0$ (4.2.1)
holds. It is called pseudo-definite if for each $\alpha \in (0,1)$ the following condition holds;
$\mu(x, t) > \alpha, \vartheta(x, t) < 1 - \alpha$ and $\omega(x, t) < 1 - \alpha$ for all $t > 0$ implies that $x = 0$. (4.2.2)
Obviously, a definite NNS is pseudo-definite.
Theorem 4.3:
Let $(X, \mu, \vartheta, \omega)$be a pseudo-definite NNS. Suppose that for each $\alpha \in (0,1)$ and each α-approximately Jensen type mapping $f_\alpha: \mathbb{N} \to (X, \mu, \vartheta, \omega)$, there exists numbers $\delta_\alpha > 0, n_\alpha \in \mathbb{N}$ and an additive mapping $T_\alpha: \mathbb{N} \to X$ such that $\mu(T_\alpha(n) - f_\alpha(n), \delta_\alpha) > \alpha, \vartheta(T_\alpha(n) - f_\alpha(n), \delta_\alpha) < 1 - \alpha$ and $\omega(T_\alpha(n) - f_\alpha(n), \delta_\alpha) < 1 - \alpha$,
for all $n \geq n_\alpha$. Then $(X, \mu, \vartheta, \omega)$ is a NBS.
Proof:
Let (x_n) be a Cauchy sequence in $(X, \mu, \vartheta, \omega)$. Choose any fix value of $\alpha \in (0,1)$. There is an increasing sequence (n_k) of positive integers such that $n_k \geq k$ and $\mu(x_n - x_m, 1/4k) \geq \alpha, \vartheta(x_n - x_m, 1/4k) \leq 1 - \alpha$ and
$\omega(x_n - x_m, 1/4k) \leq 1 - \alpha$, for all $n, m \geq n_k$.
Put $y_k = x_{n_k}$ and define $f_\alpha: \mathbb{N} \to X$ by $f_\alpha(k) = ky_k, k \in \mathbb{N}$. Then
$\mu(2f_\alpha(j + k) - f_\alpha(2j) - f_\alpha(2k), 1) = \mu(2(j + k)y_{j+k} - 2jy_{2j} - 2ky_{2k}, 1)$

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1304-1312
https://publishoa.com
ISSN: 1309-3452

$$\geq \mu\big(2j\big(y_{j+k}-y_{2j}\big),1/2\big)*\mu\big(2k\big(y_{j+k}-y_{2k}\big),1/2\big)\geq \alpha,$$
$$\vartheta(2f_\alpha(j+k)-f_\alpha(2j)-f_\alpha(2k),1)=\vartheta(2(j+k)y_{j+k}-2jy_{2j}-2ky_{2k},1)$$
$$\leq \vartheta\big(2j\big(y_{j+k}-y_{2j}\big),1/2\big)\diamond\vartheta\big(2k\big(y_{j+k}-y_{2k}\big),1/2\big)\leq 1-\alpha \text{ and}$$
$$\omega(2f_\alpha(j+k)-f_\alpha(2j)-f_\alpha(2k),1)=\omega(2(j+k)y_{j+k}-2jy_{2j}-2ky_{2k},1)$$
$$\leq \mu\big(2j\big(y_{j+k}-y_{2j}\big),1/2\big)\odot\omega\big(2k\big(y_{j+k}-y_{2k}\big),1/2\big)\leq 1-\alpha,$$
for each $j,k\in\mathbb{N}$. Thus f_α is α-approximately Jensen type.
By our assumption, there exists $\delta_\alpha>0$, $n_\alpha\in\mathbb{N}$ and an additive mapping $T_\alpha:\mathbb{N}\to X$ such that
$\mu(T_\alpha(n)-f_\alpha(n),\delta_\alpha)>\alpha, \vartheta(T_\alpha(n)-f_\alpha(n),\delta_\alpha)<1-\alpha$ and $\omega(T_\alpha(n)-f_\alpha(n),\delta_\alpha)<1-\alpha$, for all $n\geq n_\alpha$.
Since T_α is additive, therefore $T_\alpha(n)=nT_\alpha(1)$.
Hence we have $\mu(T_\alpha(1)-y_n,\delta_\alpha/n)>\alpha, \vartheta(T_\alpha(1)-y_n,\delta_\alpha/n)<1-\alpha$ and $\omega(T_\alpha(1)-y_n,\delta_\alpha/n)<1-\alpha$,
for all $n\in\mathbb{N}$. Let $\varepsilon>0$, there exists some $n_0\geq n_\alpha$ such that $\mu(x_n-x_m,\varepsilon/2)\geq\alpha$, $\vartheta(x_n-x_m,\varepsilon/2)\leq 1-\alpha$ and $\omega(x_n-x_m,\varepsilon/2)\leq 1-\alpha$, for all $n,m\geq n_0$.
Take some $k_0\in\mathbb{N}$ such that $k_0\geq n_0$ and $\delta_\alpha/k_0<\varepsilon/2$. It follows that $n_{k_0}\geq k_0\geq n_0\geq n_\alpha$, so
$\mu(T_\alpha(1)-x_n,\varepsilon)\geq\mu\big(x_n-x_{n_k},\varepsilon/2\big)*\mu(y_{k_0}-T_\alpha(1),\varepsilon/2)\geq\alpha,$
$\vartheta(T_\alpha(1)-x_n,\varepsilon)\leq\vartheta\big(x_n-x_{n_k},\varepsilon/2\big)\diamond\vartheta(y_{k_0}-T_\alpha(1),\varepsilon/2)\leq 1-\alpha$ and
$\omega(T_\alpha(1)-x_n,\varepsilon)\leq\omega\big(x_n-x_{n_k},\varepsilon/2\big)\odot\omega(y_{k_0}-T_\alpha(1),\varepsilon/2)\leq 1-\alpha$, for all $n\geq n_{k_0}$.
This means that (x_n) is α-convergent to $T_\alpha(1)$. Let $\alpha\neq\beta$. Then, for each $\varepsilon>0$ and sufficiently large n,
$\mu\big(T_\alpha(1)-T_\beta(1),2\varepsilon\big)\geq\mu(T_\alpha(1)-x_n,\varepsilon)*\mu\big(x_n-T_\beta(1),\varepsilon\big)\geq\alpha*\beta,$
$\vartheta\big(T_\alpha(1)-T_\beta(1),2\varepsilon\big)\leq\vartheta(T_\alpha(1)-x_n,\varepsilon)\diamond\vartheta\big(x_n-T_\beta(1),\varepsilon\big)\leq(1-\alpha)\diamond(1-\beta)$ and
$\omega\big(T_\alpha(1)-T_\beta(1),2\varepsilon\big)\leq\omega(T_\alpha(1)-x_n,\varepsilon)\odot\omega\big(x_n-T_\beta(1),\varepsilon\big)\leq(1-\alpha)\odot(1-\beta).$
By (3.4.3), $T_\alpha(1)=T_\beta(1)$. Put $x=T_\alpha(1)$. Then for each $\alpha\in(0,1)$ and $\varepsilon>0$,
$\mu(x-x_n,\varepsilon)\geq\alpha, \vartheta(x-x_n,\varepsilon)\leq 1-\alpha$ and $\omega(x-x_n,\varepsilon)\leq 1-\alpha$, for sufficiently large n.
Hence $(\mu,\vartheta,\omega)-\lim_{n\to\infty}x_n=x$.

Example 4.4:

Let X be athe set of all real sequences (x_n) such that x_n's are zero for all but finitely many n's.
For $x=(x_n)\in X$ and $t>0$, define $\mu(x,t)$ to be $\min\mu_{1-1/n}(x_n,t)$, $\vartheta(x,t)$ to be $\max\vartheta_{1/n}(x_n,t)$ and $\omega(x,t)$ to be $\max\omega_{1/n}(x_n,t)$, where

$$\mu_r(x,t)=\begin{Bmatrix}0 & \text{if} & t\leq 0,\\ r & \text{if} & 0<t<\|x\|,\\ 1 & \text{if} & t\geq\|x\|;\end{Bmatrix},\ \vartheta_r(x,t)=\begin{Bmatrix}1 & \text{if} & t\leq 0,\\ 1-r & \text{if} & 0<t<\|x\|,\\ 0 & \text{if} & t\geq\|x\|;\end{Bmatrix} \text{ and}$$
$$\omega_r(x,t)=\begin{Bmatrix}1 & \text{if} & t\leq 0,\\ \frac{1-r}{r} & \text{if} & 0<t<\|x\|,\\ 0 & \text{if} & t\geq\|x\|;\end{Bmatrix}.$$

Then (X,μ,ϑ,ω) is a NNS which is not pseudo-definite since, for example, $\mu\big((0,1,0,\dots,),t\big)\geq 1/2$, $\vartheta\big((0,1,0,\dots,),t\big)\leq 1/2$ and $\omega\big((0,1,0,\dots,),t\big)\leq 1/2$. This space is not complete.
To see this, we consider the sequence $\big(x^{(m)}\big)$ in X, where $x_n{}^{(m)}=\begin{Bmatrix}1 & \text{if } n\leq m,\\ 0 & \text{if } n\geq m;\end{Bmatrix}$, which is a Cauchy sequence. Given $\varepsilon>0$ and $\delta>0$, there exists $k\in\mathbb{N}$ such that $1/k<\varepsilon$. Thus for $m_1>m_2\geq k$, we have

$$\mu\big(x^{(m_1)}-x^{(m_2)},\delta\big)=\min_{m_2+1\leq n\leq m_1}\mu_{1-1/n}(1,\delta)\geq 1-\frac{1}{n}>1-\frac{1}{k}>1-\varepsilon,$$
$$\vartheta\big(x^{(m_1)}-x^{(m_2)},\delta\big)=\max_{m_2+1\leq n\leq m_1}\vartheta_{1/n}(1,\delta)\leq\frac{1}{n}<\frac{1}{k}<\varepsilon \text{ and}$$
$$\omega\big(x^{(m_1)}-x^{(m_2)},\delta\big)=\max_{m_2+1\leq n\leq m_1}\omega_{1/n}(1,\delta)\leq\frac{1}{n}<\frac{1}{k}<\varepsilon.$$

But the sequence is not convergent. On contrary, suppose that there exists some $x\in X$ such that
$\lim_{m\to\infty}\mu\big(x^{(m)}-x,t\big)=1$, $\lim_{m\to\infty}\vartheta\big(x^{(m)}-x,t\big)=0$ and $\lim_{m\to\infty}\omega\big(x^{(m)}-x,t\big)=0$.
Since $x\in X$, there exists $n_0\in\mathbb{N}$ such that $x_n=0$ for all $n\geq n_0$. Thus for $m\geq n_0$, we have
$1-1/n_0=\mu_{1-1/n_0}(1,1/2)=\mu_{1-1/n_0}\big(x_{n_0}{}^{(m)},1/2\big)\geq\mu\big(x^{(m)}-x,1/2\big),$
$1/n_0=\vartheta_{1/n_0}(1,1/2)=\vartheta_{1/n_0}\big(x_{n_0}{}^{(m)},1/2\big)\leq\vartheta\big(x^{(m)}-x,1/2\big)$ and
$1/n_0=\omega_{1/n_0}(1,1/2)=\omega_{1/n_0}\big(x_{n_0}{}^{(m)},1/2\big)\leq\omega\big(x^{(m)}-x,1/2\big).$
Taking limit $m\to\infty$, we get $1-1/n_0\geq\lim_{m\to\infty}\mu\big(x^{(m)}-x,1/2\big)=1$,
$1/n_0\leq\lim_{m\to\infty}\vartheta\big(x^{(m)}-x,1/2\big)=0$, $1/n_0\leq\lim_{m\to\infty}\omega\big(x^{(m)}-x,1/2\big)=0$, which is a contradiction.

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1304-1312
https://publishoa.com
ISSN: 1309-3452

Now, show that $(X, \mu, \vartheta, \omega)$ satisfies the hypothesis of the above theorem, of course except that being pseudo-definite. First, we find a criterion for a mapping $f: m \in \mathbb{N} \mapsto (f_n(m)) \in X$ to be α-approximately Jensen type.
Given $\alpha \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $1 - 1/n_0 \leq \alpha \leq 1 - 1/(n_0 + 1)$.
A mapping $f: m \in \mathbb{N} \mapsto (f_n(m)) \in X$ is α-approximately Jensen type if and only if there exists $\beta > 0$ such that $\mu_{1-1/n}(f_n(m+k) - f_n(2m) - f_n(2k), \beta) > \alpha$, $\vartheta_{1/n}(f_n(m+k) - f_n(2m) - f_n(2k), \beta) < 1 - \alpha$ and $\omega_{1/n}(f_n(m+k) - f_n(2m) - f_n(2k), \beta) < 1 - \alpha$, for all $n \in \mathbb{N}$.
This is satisfied if and only if $\mu_{1-1/n}(f_n(m+k) - f_n(2m) - f_n(2k), \beta) \neq 1 - \frac{1}{n}$, $\vartheta_{1/n}(f_n(m+k) - f_n(2m) - f_n(2k), \beta) \neq \frac{1}{n}$ and $\omega_{1/n}(f_n(m+k) - f_n(2m) - f_n(2k), \beta) \neq \frac{1}{n}$, for $n = 1,2, \dots, n_0$ or equivalently $|f_n(m+k) - f_n(2m) - f_n(2k)| \leq \beta$ for $n = 1, \dots, n_0$.
Thus we can say that given $\alpha \in [1 - 1/n_0, 1 - 1/(n_0 + 1)$, mapping f is α-approximately Jensen type if and only if first n_0 coordinate mappings, $f_i (1 \leq i \leq n_0)$, are Jensen bounded.
Since $\mathbb{R}$ is complete, for α-approximately Jensen type mapping f there exist additive mappings $S_n: \mathbb{N} \to \mathbb{R}$ such that $f_n(m) - S_n(m)$ is bounded for all $m \in \mathbb{N}$.
Define $T: \mathbb{N} \to X$ by $T_n(m) = \begin{Bmatrix} S_n(m) \text{ if } n \leq n_0, \\ 0 \text{ if } n > n_0 \end{Bmatrix}$.
Then T is an additive mapping. Let $\beta_n = \sup|f_n - S_n|$ for $n = 1, \dots, n_0$ and $\beta_0 = \max_{1 \leq n \leq n_0} \beta_n$.
Then $\mu(f(m) - T(m), \beta_0 > \alpha$, $\vartheta(f(m) - T(m), \beta_0) < 1 - \alpha$ and $\omega(f(m) - T(m), \beta_0) < 1 - \alpha$ for all $m \in \mathbb{N}$.
The above arguments show that for each $\alpha \in (0,1)$ and each α-approximately Jensen type mapping $f_\alpha: \mathbb{N} \to (X, \mu, \vartheta, \omega)$ there exists a number $\delta_\alpha > 0$ and an additive mapping $T_\alpha: \mathbb{N} \to X$ such that $\mu(T_\alpha(n) - f_\alpha(n), \delta_\alpha) > \alpha$, $\vartheta(T_\alpha(n) - f_\alpha(n), \delta_\alpha) < 1 - \alpha$ and $\omega(T_\alpha(n) - f_\alpha(n), \delta_\alpha) < 1 - \alpha$, for all $n \in \mathbb{N}$.

Definition 4.5:

Let $(X, \mu, \vartheta, \omega)$ be a NNS and $f: \mathbb{N} \to X$ be a mapping. Let for each $\alpha \in (0,1)$, there exists a $n_\alpha \in \mathbb{N}$ and $\delta > 0$ such that $\mu(2f(n+m) - f(2n) - f(2m), \delta) \geq \alpha$, $\vartheta(2f(n+m) - f(2n) - f(2m), \delta) \leq 1 - \alpha$ and $\omega(2f(n+m) - f(2n) - f(2m), \delta) \leq 1 - \alpha$, for each $n, m \geq n_\alpha$. Then f is said to be an approximately Jensen type mapping.

Theorem 4.6:

Let $(X, \mu, \vartheta, \omega)$ be a NNS such that for every approximately Jensen type mapping $f: \mathbb{N} \to X$, there exists an additive mapping $T: \mathbb{N} \to X$ such that $\mu(x_m - x_n, 1/4k) \geq \alpha_k$, $\vartheta(x_m - x_n, 1/4k) \leq 1 - \alpha_k$ and $\omega(x_m - x_n, 1/4k) \leq 1 - \alpha_k$, for each $n, m \geq n_k$.
Let $y_k = x_{n_k}$ for each $k \geq 1$. Define $f: \mathbb{N} \to X$ by $f(k) = k y_k, k \in \mathbb{N}$. If $\alpha \in (0,1)$, take some $m_0 \in \mathbb{N}$ such that $\alpha_{m_0} > \alpha$ and let $n_\alpha = m_0$. Then for each $n \geq m \geq n_\alpha$, we obtain

$$\begin{aligned}
\mu(2f(n+m) - f(2n) - f(2m), 1) &= \mu(2(n+m)y_{n+m} - 2ny_{2n} - 2my_{2m}, 1) \\
&\geq \mu(2n(y_{n+m} - y_{2n}), 1/2) * \mu(2m(y_{n+m} - y_{2m}), 1/2) \\
&= \mu(y_{n+m} - y_{2n}, 1/4n) * \mu(y_{n+m} - y_{2m}, 1/4m) \geq \alpha_n * \alpha_m \geq \alpha, \\
\vartheta(2f(n+m) - f(2n) - f(2m), 1) &= \vartheta(2(n+m)y_{n+m} - 2ny_{2n} - 2my_{2m}, 1) \\
&\leq \vartheta(2n(y_{n+m} - y_{2n}), 1/2) \diamond \vartheta(2m(y_{n+m} - y_{2m}), 1/2) \\
&= \vartheta(y_{n+m} - y_{2n}, 1/4n) \diamond \vartheta(y_{n+m} - y_{2m}, 1/4m) \\
&\leq (1 - \alpha_n) \diamond (1 - \alpha_m) \leq 1 - \alpha \text{ and} \\
\omega(2f(n+m) - f(2n) - f(2m), 1) &= \omega(2(n+m)y_{n+m} - 2ny_{2n} - 2my_{2m}, 1) \\
&\leq \omega(2n(y_{n+m} - y_{2n}), 1/2) \odot \omega(2m(y_{n+m} - y_{2m}), 1/2) \\
&= \omega(y_{n+m} - y_{2n}, 1/4n) \odot \omega(y_{n+m} - y_{2m}, 1/4m) \\
&\leq (1 - \alpha_n) \odot (1 - \alpha_m) \leq 1 - \alpha.
\end{aligned}$$

Therefore f is an approximately Jensen type mapping. By our assumption, there is an additive mapping $T: \mathbb{N} \to X$ such that $\lim_{n\to\infty} \mu(T(n) - f(n), t) = 1$, $\lim_{n\to\infty} \vartheta(T(n) - f(n), t) = 0$ and $\lim_{n\to\infty} \omega(T(n) - f(n), t) = 0$.
This means that $\lim_{n\to\infty} \mu(T(1) - y_n, t/n) = 1$, $\lim_{n\to\infty} \vartheta(T(1) - y_n, t/n) = 0$ and $\lim_{n\to\infty} \omega(T(1) - y_n, t/n) = 0$. That is, the subsequence (y_n) of the Cauchy sequence (x_n) converges to $x = T(1)$ and hence (x_n) is also convergent to x.

Conclusion

We linked here two different disciplines, namely, the fuzzy spaces and functional equations. We established Hyers–Ulam–Rassias stability of a Jensen functional equation $2f((x+y)/2) = f(x) + f(y)$ in NNS.
We also studied the Neutrosophic Continuity and Completeness through the existence of a certain solution of a fuzzy stability problem for approximately Jensen functional equation.

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1304-1312
https://publishoa.com
ISSN: 1309-3452

References

[1] Aoki T. On the stability of the linear transformation in Banach spaces. J Math Soc Jpn, 2:64–6, (1950).
[2] Barros LC, Bassanezi RC, Tonelli PA.. Fuzzy modelling in population dynamics, Ecol Model, 128:27–33, (2000).
[3] Fradkov AL, Evans RJ. Control of chaos: methods and applications in engineering, Chaos, Solitons and Fractals, 29:33–56, (2005).
[4] Giles R. A computer program for fuzzy reasoning, Fuzzy Sets Syst,4:221–34, (1980).
[5] Hong L, Sun JQ. Bifurcations of fuzzy nonlinear dynamical systems, Commun Nonlinear Sci Numer Simul, 1:1–12, (2006).
[6] Hyers DH. On the stability of the linear functional equation, Proc Natl Acad Sci USA, 27:222–4, (1941).
[7] Hyers DH, Isac G, Rassias TM. Stability of functional equations in several variables, Basel: Birkhäuser, (1998).
[8] Jeyaraman M, Mangayarkkarasi AN, Jeyanthi V, Pandiselvi R. Hyers-Ulam-Rassias stability for functional equation in Neutrosophic Normed Spaces, International Journal of Neutrosophic Science, Vol.18, No.1,127-143, (2022).
[9] Jeyaraman M, Ramachandran A and Shakila VB, Approximate fixed point Theorems for weak contractions on Neutrosophic Normed space, Journal of computational Mathematics, 6(1), 134-158 (2022).
[10] Mohiuddine SA, Danish Lohani QM. On generalized statistical convergence in intuitionistic fuzzy normed space, Chaos, Solitons and Fractals, 42:1731–7, (2009).
[11] Mursaleen M, Mohiuddine SA. Statistical convergence of double sequences in intuitionistic fuzzy normed spaces, Chaos, Solitons and Fractals, 41:2414–21, (2009).
[12] Mursaleen M, Mohiuddine SA. Nonlinear operators between intuitionistic fuzzy normed spaces and Fréchet differentiation, Chaos, Solitons and Fractals, 42:1010–5, (2009).
[13] Mursaleen M, Mohiuddine SA. On stability of a cubic functional equation in intuitionistic fuzzy normed spaces, Chaos, Solitons and Fractals, 42:2997–3005, (2009).
[14] Parnami JC, Vasudeva HL. On Jensen’s functional equation, Aequationes Math, 43:211–8, (1992).
[15] Rassias TM. On the stability of the linear mapping in Banach spaces, Proc Am Math Soc, 72:297–300, (1978).
[16] Rassias TM. On the stability of functional equations and a problem of Ulam. Acta Appl Math, 62:23–130, (2000).
[17] Smarandache F. Neutrosophic set, a generalisation of the intuitionistic fuzzy sets, Int. J. Pure Appl. Math., 24, 287-297, (2005).
[18] Smarandache F. Neutrosophy, Neutrosophic Probability, Set, and Logic, Pro Quest Information & Learning, Ann Arbor, Michigan, USA (1998).
[19] Simsek N, Kirisci M. Fixed point theorems in Neutrosophic Metric Spaces, Sigma J. Eng. Nat. Sci., 10(2), 221-230, (2019).
[20] Ulam SM. Problems in modern mathematics, Science ed. New York: John Wiley & Sons, [Chapter VI, Some Questions in Analysis: Section 1,Stability], (1940).
[21] Zadeh. L. A, Fuzzy Sets, Information and Control, 8, 338-353, (1965).