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Fixed Point Theorems Of Contractive Mappings In Neutrosophic B - Metric Spaces

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ABSTRACT
The main purpose of the present paper is to introduce and study the notion of neutrosophic b-metric spaces. In this way, we generalize both the notion of neutrosophic metric spaces and fuzzy b-metric spaces. Further, the formulation and proof of neutrosophic b-metric versions of some conventional theorems regarding fixed points via neutrosophic sets are presented.
Keywords: Neutrosophic *b*-metric spaces, Neutrosophic metric spaces, Fixed points.
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1. Introduction:

Amini-Harandi [1] introduced a new extension of the concept of partial metric space, called a metric-like space. The concept of *b*-metric-like space which generalizes the notions of partial metric space, metric-like space, and *b*-metric space was introduced by Alghamdi et al. in [2]. They established the existence and uniqueness of fixed points in a *b*-metric-like space as well as in a partially ordered *b*-metric-like space. In addition, as an application, they derived some new fixed point and coupled fixed point results in partial metric spaces, metric-like spaces, and *b*-metric spaces. [7,9,13,14,15]

The first successful attempt towards incorporating non-probabilistic uncertainty, i.e.uncertainty which is not caused by randomness of an event, into mathematical modelling was made in 1965 by Zadeh [24] through his remarkable theory on fuzzy sets . A fuzzy set is a set where each element of the universe belongs to it but with some 'grade' or 'degree of belongingness' which lies between 0 and 1 and such grades are called membership value of an element in that set. This gradation concept is very well suited for applications involving imprecise data such as natural language processing or in artificial intelligence, handwriting and speech recognition etc. Although Fuzzy set theory is very successful in handling uncertainties arising from vagueness or partial belongingness of an element in a set, it cannot model all sorts of uncertainties prevailing in different real physical situations specially problems involving incomplete information. Further generalization of this fuzzy set was made by Atanassov [3] in 1986, which is known as Intuitionistic fuzzy set (IFS). In IFS, instead of one 'membership grade', there is also a 'non-membership grade' attached with each element. Furthermore there is a restriction that the sum of these two grades is less or equal to unity. In IFS the 'degree of non-belongingness' is not independent but it is dependent on the 'degree of belongingness'. In 1999, a new theory has been introduced by Florentin Smarandache [17] which is known as 'Neutrosophic logic'. It is a logic in which each proposition is estimated to have a degree of truth (T), a degree of indeterminacy (I) and a degree of falsity (F). A Neutrosophic set is a set where each element of the universe has a degree of truth, indeterminacy and falsity respectively and which lies between [0- , 1+], the non-standard unit interval. Unlike in intuitionistic fuzzy sets, where the incorporated uncertainty is dependent on the degree of belongingness and degree of non-belongingness, here the uncertainty present, i.e. the indeterminacy factor, is independent of truth and falsity values. Neutrosophic sets are indeed more general in nature than IFS as there are no constraints between the 'degree of truth', 'degree of indeterminacy' and 'degree of falsity'. All these degrees can individually vary within [0- , 1+]. The main purpose of the paper is to introduce and study the notion of neutrosophic b-metric spaces. In this way, we generalize both the notion of neutrosophic metric spaces.

2. Preliminaries

Definition: 2.1

A binary operation $*: [0,1] \times [0,1] \rightarrow [0,1]$ is called continuous triangular norm (t-norm) if it satisfies the following conditions:

1. $*$ is associative and commutative;
2. $*$ is continuous;
3. $a * 1 = a$ for all $a \in [0,1]$;
4. If $a \leq c$ and $b \leq d$ with $a, b, c, d \in [0,1]$, then $a * b \leq c * d$.

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Example: 2.2

Three basic t-norms are defined as follows:

(1) The minimum t-norm, $a * b = \min(a, b)$
(2) The product t-norm, $a * b = a.b$
(3) The Lukasiewicz t-norm $a * b = max(a + b - 1,0)$.

Definition: 2.3

A binary operation $\diamond$: $[0,1]$ x $[0,1] \rightarrow [0,1]$ is called continuous triangular conorm (t-conorm) if it satisfies the following conditions:

1. $\diamond$ is associative and commutative;
2. $\diamond$ is continuous;
3. $a \diamond 0 = a$ for all $a \in [0,1]$;
4. If $a \leq c$ and $b \leq d$ with $a, b, c, d \in [0,1]$, then $a \diamond b \leq c \diamond d$.

Example: 2.4

Three basic t-conorms are given below:

(1) $a \diamond b = min(a + b, 1)$;
(2) $a \diamond b = a + b - ab$;
(3) $a \diamond b = max(a, b)$

3. Neutrosophic b-Metric Spaces

Definition: 3.1

A 7-tuple $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond, b)$ is said to be a Neutrosophic b - Metric Space (NbMS), if Σ is an arbitrary set, $b \geq 1$ is a given real number, $*$ is a continuous t-norm, $\diamond$ is a continuous t-conorm, Ξ, Θ and Υ are sets on X^2 x $[0, \infty)$ satisfying the following conditions: For all $\zeta, \eta, \lambda \in \Sigma$,

(a) $\Xi(\zeta, \eta, \lambda) + \Theta(\zeta, \eta, \lambda) + \Upsilon(\zeta, \eta, \lambda) \leq 3$;
(b) $0 \leq \Xi(\zeta, \eta, \lambda) \leq 1$; $0 \leq \Theta(\zeta, \eta, \lambda) \leq 1$; $0 \leq \Upsilon(\zeta, \eta, \lambda) \leq 1$;
(c) $\Xi(\zeta, \eta, 0) = 0$;
(d) $\Xi(\zeta, \eta, \lambda) = 1$, for all $\lambda > 0$ iff $\zeta = \eta$;
(e) $\Xi(\zeta, \eta, \lambda) = \Xi(\eta, \zeta, \lambda)$, for all $\lambda > 0$;
(f) $\Xi(\zeta, \delta, b(\lambda + \mu)) \geq \Xi(\zeta, \eta, \lambda) * \Xi(\eta, \delta, \mu)$, for all $\lambda, \mu > 0$;
(g) $\Xi(\zeta, \eta, .) : [0, \infty) \rightarrow [0,1]$ is left continuous and $\lim_{\lambda \to \infty} \Xi(\zeta, \eta, \lambda) = 1$;
(h) $\Theta(\zeta, \eta, 0) = 1$;
(i) $\Theta(\zeta, \eta, \lambda) = 0$, for all $\lambda > 0$ iff $\zeta = \eta$;
(j) $\Theta(\zeta, \eta, \lambda) = \Theta(\eta, \zeta, \lambda)$, for all $\lambda > 0$;
(k) $\Theta(\zeta, \delta, b(\lambda + \mu)) \leq \Theta(\zeta, \eta, \lambda) \diamond \Theta(\eta, \delta, \mu)$, for all $\lambda, \mu > 0$;
(l) $\Theta(\zeta, \eta, .) : [0, \infty) \rightarrow [0,1]$ is right continuous and $\lim_{\lambda \to \infty} \Theta(\zeta, \eta, \lambda) = 0$;
(m) $\Upsilon(\zeta, \eta, 0) = 1$;
(n) $\Upsilon(\zeta, \eta, \lambda) = 0$, for all $\lambda > 0$ iff $\zeta = \eta$;
(o) $\Upsilon(\zeta, \eta, \lambda) = \Upsilon(\eta, \zeta, \lambda)$, for all $\lambda > 0$;
(p) $\Upsilon(\zeta, \delta, b(\lambda + \mu)) \leq \Upsilon(\zeta, \eta, \lambda) \diamond \Upsilon(\eta, \delta, \mu)$, for all $\lambda, \mu > 0$;
(q) $\Upsilon(\zeta, \eta, .) : [0, \infty) \rightarrow [0,1]$ is right continuous and $\lim_{\lambda \to \infty} \Upsilon(\zeta, \eta, \lambda) = 0$;

Here, $\Xi(\zeta, \eta, \lambda), \Theta(\zeta, \eta, \lambda)$ and $\Upsilon(\zeta, \eta, \lambda)$ denote the degree of nearness, the degree of non-nearness and the degree of neutralness between ζ and η with respect to λ respectively.

Example: 3.2

Let $(\Sigma, \mathrm{d}, \mathrm{b})$ be a b-metric space and $a * b = min\{a, b\}$, $a \diamond b = max\{a, b\}$ for all $a, b \in [0,1]$ and let Ξ_d, Θ_d and Υ_d be fuzzy sets on X^2 x $[0, \infty)$, defined as follows:

$$\Xi_d = \begin{cases} \frac{\lambda}{\lambda + d(\zeta, \eta)}, & if\ \lambda > 0 \\ 0, & if\ \lambda = 0 \end{cases}, \quad \Theta_d = \begin{cases} \frac{d(\zeta, \eta)}{\lambda + d(\zeta, \eta)} & if\ \lambda > 0 \\ 1, & if\ \lambda = 0 \end{cases} \text{ and } \Upsilon_d = \begin{cases} \frac{d(\zeta, \eta)}{\lambda} & if\ \lambda > 0 \\ 1, & if\ \lambda = 0 \end{cases}$$

We check only axioms (f), (k) and (p) of definition (3.1), because verifying the other Conditions are standard. Let $\zeta, \eta, \delta \in \Sigma$ and $\lambda, \mu > 0$. Without restraining the generality, we assume that

$\Xi_d(\zeta, \eta, \lambda) \leq \Xi_d(\eta, \theta, \mu)$, $\Theta_d(\zeta, \eta, \lambda) \geq \Theta_d(\eta, \theta, \mu)$ and $\Upsilon_d(\zeta, \eta, \lambda) \geq \Upsilon_d(\eta, \theta, \mu)$

Thus $\frac{\lambda}{\lambda + d(\zeta, \eta)} \leq \frac{\mu}{\mu + d(\eta, \theta)}$, $\frac{d(\zeta, \eta)}{\lambda + d(\zeta, \eta)} \geq \frac{d(\eta, \theta)}{\mu + d(\eta, \theta)}$ and $\frac{d(\zeta, \eta)}{\lambda} \geq \frac{d(\eta, \theta)}{\mu}$.

ie. $\lambda d(\eta, \theta) \leq \mu d(\zeta, \eta)$. On the other hand

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$\Xi_d(\zeta,\theta,\mathrm{b}(\lambda+\mu)) = \frac{\mathrm{b}(\lambda+\mu)}{\mathrm{b}(\lambda+\mu)+\mathrm{d}(\zeta,\theta)} \geq \frac{\mathrm{b}(\lambda+\mu)}{\mathrm{b}(\lambda+\mu)+\mathrm{b}[\mathrm{d}(\zeta,\eta)+\mathrm{d}(\eta,\theta)]} = \frac{\lambda+\mu}{\lambda+\mu+\mathrm{d}(\zeta,\eta)+\mathrm{d}(\eta,\theta)}$. Also,
$\Theta_d(\zeta,\theta,\mathrm{b}(\lambda+\mu)) = \frac{\mathrm{d}(\zeta,\theta)}{b(\lambda+\mu)+\mathrm{d}(\zeta,\theta)} \leq \frac{b[\mathrm{d}(\zeta,\eta)+\mathrm{d}(\eta,\theta)]}{b(\lambda+\mu)+b[\mathrm{d}(\zeta,\eta)+\mathrm{d}(\eta,\theta)]} = \frac{\mathrm{d}(\zeta,\eta)+\mathrm{d}(\eta,\theta)}{\lambda+\mu+\mathrm{d}(\zeta,\eta)+\mathrm{d}(\eta,\theta)}$ and
$\Upsilon_d(\zeta,\theta,\mathrm{b}(\lambda+\mu)) = \frac{\mathrm{d}(\zeta,\theta)}{b(\lambda+\mu)} \leq \frac{b[\mathrm{d}(\zeta,\eta)+\mathrm{d}(\eta,\theta)]}{b(\lambda+\mu)} = \frac{\mathrm{d}(\zeta,\eta)+\mathrm{d}(\eta,\theta)}{\lambda+\mu}$.
We will prove that
$\frac{\lambda+\mu}{\lambda+\mu+\mathrm{d}(\zeta,\eta)+\mathrm{d}(\eta,\theta)} \geq \frac{\lambda}{\lambda+d(\zeta,\eta)}$, $\frac{\mathrm{d}(\zeta,\eta)+\mathrm{d}(\eta,\theta)}{\lambda+\mu+\mathrm{d}(\zeta,\eta)+\mathrm{d}(\eta,\theta)} \leq \frac{\mathrm{d}(\zeta,\eta)}{\lambda+\mathrm{d}(\zeta,\eta)}$ and $\frac{\mathrm{d}(\zeta,\eta)+\mathrm{d}(\eta,\theta)}{\lambda+\mu} \leq \frac{\mathrm{d}(\zeta,\eta)}{\lambda}$.
Hence, we will obtain that
$\Xi_d(\zeta,\theta,\mathrm{b}(\lambda+\mu)) \geq \Xi_d(\zeta,\eta,\lambda) = \Xi_d(\zeta,\eta,\lambda) * \Xi_d(\eta,\theta,\mu)$
$\Theta_d(\zeta,\theta,\mathrm{b}(\lambda+\mu)) \leq \Theta_d(\zeta,\eta,\lambda) = \Theta_d(\zeta,\eta,\lambda) \diamond \Theta_d(\eta,\theta,\mu)$ and
$\Upsilon_d(\zeta,\theta,\mathrm{b}(\lambda+\mu)) \leq \Upsilon_d(\zeta,\eta,\lambda) = \Upsilon_d(\zeta,\eta,\lambda) \diamond \Upsilon_d(\eta,\theta,\mu)$.
What had to be verified . We remark that

$$\frac{\lambda+\mu}{\lambda+\mu+\mathrm{d}(\zeta,\eta)+\mathrm{d}(\eta,\theta)} \geq \frac{\lambda}{\lambda+d(\zeta,\eta)}$$

$\Leftrightarrow\ \lambda^2+\lambda\mu+\lambda d(\zeta,\eta)+\mu d(\zeta,\eta) \geq \lambda^2+\lambda\mu+\lambda d(\zeta,\eta)+\lambda\mathrm{d}(\eta,\theta)$
$\Leftrightarrow \mu d(\zeta,\eta) \geq \mathrm{d}(\eta,\theta)$, which is true. Also,

$$\frac{\mathrm{d}(\zeta,\eta)+\mathrm{d}(\eta,\theta)}{\lambda+\mu+\mathrm{d}(\zeta,\eta)+\mathrm{d}(\eta,\theta)} \leq \frac{d(\zeta,\eta)}{\lambda+d(\zeta,\eta)}$$

$\Leftrightarrow \lambda\mathrm{d}(\zeta,\eta)+\lambda\mathrm{d}(\eta,\theta)+d(\zeta,\eta)\mathrm{d}(\eta,\theta)+(d(\zeta,\eta))^2 \leq \lambda\mathrm{d}(\zeta,\eta)+\mu\mathrm{d}(\zeta,\eta)+d(\zeta,\eta)\mathrm{d}(\eta,\theta)+(d(\zeta,\eta))^2$
$\Leftrightarrow \lambda\mathrm{d}(\eta,\theta) \leq \mu\mathrm{d}(\zeta,\eta)$ and

$$\frac{\mathrm{d}(\zeta,\eta)+\mathrm{d}(\eta,\theta)}{\lambda+\mu} \leq \frac{d(\zeta,\eta)}{\lambda}$$

$\Leftrightarrow \lambda\mathrm{d}(\zeta,\eta)+\lambda\mathrm{d}(\eta,\theta) \leq \lambda\mathrm{d}(\zeta,\eta)+\mu\mathrm{d}(\zeta,\eta)$
$\Leftrightarrow \lambda\mathrm{d}(\eta,\theta) \leq \mu\mathrm{d}(\zeta,\eta)$, which is true. Hence $(\Sigma,\Xi_d,\Theta_d,\Upsilon_d,*,\diamond,b)$ is (standard) NbMS.

Definition: 3.3

Let $b \geq 1$ be a given real number. A function $f: R \to R$ will be called b-nondecreasing if $\lambda < \mu$ implies that $f(\lambda) \leq f(b\mu)$ and f is called b-nonincreasing if $\lambda < \mu$ implies that $f(\lambda) \geq f(b\mu)$.

Proposition: 3.4

In a NbMS $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond,b)$, $\Xi(\zeta,\eta,.): [0,\infty) \to [0,1]$ is b-nondecreasing, $\Theta(\zeta,\eta,\lambda)$: $[0,\infty) \to [0,1]$ is b-nonincreasing and $\Upsilon(\zeta,\eta,\lambda)$: $[0,\infty) \to [0,1]$ is b-nonincreasing for all $\zeta,\eta \in \Sigma$.

Proof :

Fix $0 < \lambda < \mu$, we have
$\Xi(\zeta,\eta,b\mu) = \Xi(\zeta,\eta,b(\mu-\lambda+\lambda)) \geq \Xi(\zeta,\zeta,\mu-\lambda) * \Xi(\zeta,\eta,\lambda) = 1 * \Xi(\zeta,\eta,\lambda) = \Xi(\zeta,\eta,\lambda)$.
Also, $\Theta(\zeta,\eta,b\mu) = \Theta(\zeta,\eta,b(\mu-\lambda+\lambda)) \leq \Theta(\zeta,\zeta,\mu-\lambda) \diamond \Theta(\zeta,\eta,\lambda) = 0 \diamond \Theta(\zeta,\eta,\lambda) = \Theta(\zeta,\eta,\lambda)$ and
$\Upsilon(\zeta,\eta,b\mu) = \Upsilon(\zeta,\eta,b(\mu-\lambda+\lambda)) \leq \Upsilon(\zeta,\zeta,\mu-\lambda) \diamond \Upsilon(\zeta,\eta,\lambda) = 0 \diamond \Upsilon(\zeta,\eta,\lambda) = \Upsilon(\zeta,\eta,\lambda)$.

Definition: 3.5

Let $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond,b)$ be a NbMS.

(a) A sequence $\{\zeta_n\}$ in Σ is said to be convergent if there exists $\zeta \in \Sigma$ such that $\lim_{n\to\infty}\Xi(\zeta_n,\zeta,\lambda) = 1$, $\lim_{n\to\infty}\Theta(\zeta_n,\zeta,\lambda) = 0$ and $\lim_{n\to\infty}\Upsilon(\zeta_n,\zeta,\lambda) = 0$, for all $\lambda > 0$. In this case ζ is called the limit of the sequence $\{\zeta_n\}$ and we write $\lim_{n\to\infty}\zeta_n = \zeta$ or $\zeta_n \to \zeta$.

(b) A sequence $\{\zeta_n\}$ in $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond,b)$ is said to be a Cauchy sequence if for every $\epsilon \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $\Xi(\zeta_n,\zeta_m,\lambda) > 1-\epsilon$, $\Theta(\zeta_n,\zeta_m,\lambda) < \epsilon$ and $\Upsilon(\zeta_n,\zeta_m,\lambda) < \epsilon$, for all $m,n \geq n_0$ and $\lambda > 0$.

(c) The space Σ is said to be complete if and only if every Cauchy sequence is convergent and it is called compact if every sequence has a convergent subsequence.

Definition: 3.6

Let $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond,b)$ be a NbMS. We define an open ball $B(\zeta,r,\lambda)$ with centre $\zeta \in \Sigma$ and radiurs r, $0 < r < 1$, $\lambda > 0$ as $B(\zeta,r,\lambda) = \{\eta \in \Sigma : \Xi(\zeta,\eta,\lambda) > 1-r, \Theta(\zeta,\eta,\lambda) < r$ and $\Upsilon(\zeta,\eta,\lambda) < r\}$.

Definition: 3.7

Let $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond,b)$ be a NbMS and $\mathcal{A}$ be a subset of Σ. $\mathcal{A}$ is said to be open if, for each $\zeta \in \Sigma$, there is an open ball $B(\zeta,r,\lambda)$ contained in $\mathcal{A}$.

Result: 3.8

Let $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond,b)$ be a NbMS. Define $\tau_{\Xi,\Theta,\Upsilon}$ as: $\tau_{\Xi,\Theta,\Upsilon} = \{\mathcal{A} \subset \Sigma: \zeta \in \mathcal{A}$ iff there exists $\lambda > 0$ and $r \in (0,1)$: $B(\zeta,r,\lambda) \subset \mathcal{A}\}$, then $\tau_{\Xi,\Theta,\Upsilon}$ is a topology on Σ, where $\mathcal{P}(\Sigma)$ is the power set of Σ.

4. Main Results

Theorem: 4.1 (**Neutrosophic b- Metric Banach Contraction Theorem**)

Let $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond,b)$ be a complete NbMS. Let $T:\Sigma\to\Sigma$ be a mapping satisfying

$\Xi(T\zeta,T\eta,k\lambda)\geq\Xi(\zeta,\eta,\lambda)$ (4.1.1)

$\Theta(T\zeta,T\eta,k\lambda)\leq\Theta(\zeta,\eta,\lambda)$ and (4.1.2)

$\Upsilon(T\zeta,T\eta,k\lambda)\leq\Upsilon(\zeta,\eta,\lambda)$ (4.1.3)

for all $\zeta,\eta\in\Sigma$ where $0<k<1$. Then T has a unique fixed point.

Proof:

Let $\zeta_0\in\Sigma$ be an arbitrary element and let $\{\zeta_n\}$ be a sequence in Σ such that, $\zeta_n=T^n\zeta_0(n\in\mathbb{N})$. Then

$$\Xi(\zeta_n,\zeta_{n+1},k\lambda)=\Xi(T^n\zeta_0,T^{n+1}\zeta_0,k\lambda)\geq\Xi(T^{n-1}\zeta_0,T^n\zeta_0,\lambda)=\Xi(\zeta_{n-1},\zeta_n,\lambda)$$
$$\geq\Xi\left(T^{n-2}\zeta_0,T^{n-1}\zeta_0,{\lambda}/{k}\right)=\Xi\left(\zeta_{n-2},\zeta_{n-1},{\lambda}/{k}\right)\ldots\geq\Xi\left(\zeta_0,\zeta_1,{\lambda}/{k^{n-1}}\right).$$

Clearly, $1\geq\Xi(\zeta_n,\zeta_{n+1},k\lambda)\geq\Xi\left(\zeta_0,\zeta_1,{\lambda}/{k^{n-1}}\right)\to1$, when $n\to\infty$.

Thus $\lim\limits_{n\to\infty}\Xi(\zeta_n,\zeta_{n+1},k\lambda)=1$,

$$\Theta(\zeta_n,\zeta_{n+1},k\lambda)=\Theta(T^n\zeta_0,T^{n+1}\zeta_0,k\lambda)\leq\Theta(T^{n-1}\zeta_0,T^n\zeta_0,\lambda)=\Theta(\zeta_{n-1},\zeta_n,\lambda)$$
$$\leq\Theta\left(T^{n-2}\zeta_0,T^{n-1}\zeta_0,{\lambda}/{k}\right)=\Theta\left(\zeta_{n-2},\zeta_{n-1},{\lambda}/{k}\right)\ldots\leq\Theta\left(\zeta_0,\zeta_1,{\lambda}/{k^{n-1}}\right),$$

for all n and $\lambda>0$. Clearly, $0\leq\Theta(\zeta_n,\zeta_{n+1},k\lambda)\leq\Theta\left(\zeta_0,\zeta_1,{\lambda}/{k^{n-1}}\right)\to0$, when $n\to\infty$.

Thus $\lim\limits_{n\to\infty}\Theta(\zeta_n,\zeta_{n+1},k\lambda)=0$ and

$$\Upsilon(\zeta_n,\zeta_{n+1},k\lambda)=\Upsilon(T^n\zeta_0,T^{n+1}\zeta_0,k\lambda)\leq\Upsilon(T^{n-1}\zeta_0,T^n\zeta_0,\lambda)=\Upsilon(\zeta_{n-1},\zeta_n,\lambda)$$
$$\leq\Upsilon\left(T^{n-2}\zeta_0,T^{n-1}\zeta_0,{\lambda}/{k}\right)=\Upsilon\left(\zeta_{n-2},\zeta_{n-1},{\lambda}/{k}\right)\ldots\leq\Upsilon\left(\zeta_0,\zeta_1,{\lambda}/{k^{n-1}}\right),$$

for all n and $\lambda>0$. Clearly, $0\leq\Upsilon(\zeta_n,\zeta_{n+1},k\lambda)\leq\Upsilon\left(\zeta_0,\zeta_1,{\lambda}/{k^{n-1}}\right)\to0$ when $n\to\infty$.

Thus $\lim\limits_{n\to\infty}\Upsilon(\zeta_n,\zeta_{n+1},k\lambda)=0$.

Let $\alpha_n(\lambda)=\Xi(\zeta_n,\zeta_{n+1},\lambda)$, $\beta_n(\lambda)=\Theta(\zeta_n,\zeta_{n+1},\lambda)$ and $\gamma_n(\lambda)=\Upsilon(\zeta_n,\zeta_{n+1},\lambda)$, for all $n\in\mathbb{N}\cup\{0\}$, $\lambda>0$.

Next, we show that the sequence $\{\zeta_n\}$ is a Cauchy sequence. If it is not, then there exists $0<\epsilon<1$ and two sequences $u(n)$ and $v(n)$ such that for every $n\in\mathbb{N}\cup\{0\}$, $\lambda>0$, $u(n)>v(n)\geq n$,

$\Xi\left(\zeta_{u(n)},\zeta_{v(n)},\lambda\right)\leq1-\epsilon$, $\Theta\left(\zeta_{u(n)},\zeta_{v(n)},\lambda\right)\geq\epsilon$ and $\Upsilon\left(\zeta_{u(n)},\zeta_{v(n)},\lambda\right)\geq\epsilon$ and

$\Xi\left(\zeta_{u(n)-1},\zeta_{v(n)-1},\lambda\right)>1-\epsilon$, $\Xi\left(\zeta_{u(n)-1},\zeta_{v(n)},\lambda\right)>1-\epsilon$,

$\Theta\left(\zeta_{u(n)-1},\zeta_{v(n)-1},\lambda\right)<\epsilon$, $\Theta\left(\zeta_{u(n)-1},\zeta_{v(n)},\lambda\right)<\epsilon$ and

$\Upsilon\left(\zeta_{u(n)-1},\zeta_{v(n)-1},\lambda\right)<\epsilon$, $\Upsilon\left(\zeta_{u(n)-1},\zeta_{v(n)},\lambda\right)<\epsilon$.

Now, $1-\epsilon\geq\Xi\left(\zeta_{u(n)},\zeta_{v(n)},\lambda\right)\geq\Xi\left(\zeta_{u(n)-1},\zeta_{u(n)},{\lambda}/{2b}\right)*\Xi\left(\zeta_{u(n)-1},\zeta_{v(n)},{\lambda}/{2b}\right)$
$>\alpha_{u(n)-1}\left({\lambda}/{2b}\right)*(1-\epsilon)$,

$\epsilon\leq\Theta\left(\zeta_{u(n)},\zeta_{v(n)},\lambda\right)\leq\Theta\left(\zeta_{u(n)-1},\zeta_{u(n)},{\lambda}/{2b}\right)\diamond\Theta\left(\zeta_{u(n)-1},\zeta_{v(n)},{\lambda}/{2b}\right)$
$<\beta_{u(n)-1}\left({\lambda}/{2b}\right)\diamond\epsilon$ and

$\epsilon\leq\Upsilon\left(\zeta_{u(n)},\zeta_{v(n)},\lambda\right)\leq\Upsilon\left(\zeta_{u(n)-1},\zeta_{u(n)},{\lambda}/{2b}\right)\diamond\Upsilon\left(\zeta_{u(n)-1},\zeta_{v(n)},{\lambda}/{2b}\right)$
$<\gamma_{u(n)-1}\left({\lambda}/{2b}\right)\diamond\epsilon$.

Since $\alpha_{u(n)-1}\left({\lambda}/{2b}\right)\to1$, $\beta_{u(n)-1}\left({\lambda}/{2b}\right)\to0$ and $\gamma_{u(n)-1}\left({\lambda}/{2b}\right)\to0$ as $n\to\infty$, for every λ, therefore for $n\to\infty$, we have $1-\epsilon\geq\Xi\left(\zeta_{u(n)},\zeta_{v(n)},\lambda\right)>1-\epsilon$, $\epsilon\leq\Theta\left(\zeta_{u(n)},\zeta_{v(n)},\lambda\right)<\epsilon$ and $\epsilon\leq\Upsilon\left(\zeta_{u(n)},\zeta_{v(n)},\lambda\right)<\epsilon$.

Clearly, this leads to a contradiction. Hence ζ_n is a Cauchy sequence in Σ. Since Σ is complete so there exists a point η in Σ such that $\lim\limits_{n\to\infty}\zeta_n=\eta$.

Now, $\Xi(\eta,T\eta,\lambda)\geq\Xi\left(\eta,\zeta_{n+1},{\lambda}/{2b}\right)*\Xi\left(\zeta_{n+1},T\eta,{\lambda}/{2b}\right)$
$=\Xi\left(\eta,\zeta_{n+1},{\lambda}/{2b}\right)*\Xi\left(T\zeta_n,T\eta,{\lambda}/{2b}\right)\geq\Xi\left(\eta,\zeta_{n+1},{\lambda}/{2b}\right)*\Xi\left(\zeta_n,\eta,{\lambda}/{2bk}\right)$.

The case when $n\to\infty$, we have $\Xi(\eta,T\eta,\lambda)\geq1*1=1$.

$\Theta(\eta,T\eta,\lambda)\leq\Theta\left(\eta,\zeta_{n+1},{\lambda}/{2\mu}\right)\diamond\Theta\left(\zeta_{n+1},T\eta,{\lambda}/{2\mu}\right)$
$=\Theta\left(\eta,\zeta_{n+1},{\lambda}/{2\mu}\right)\diamond\Theta\left(T\zeta_n,T\eta,{\lambda}/{2\mu}\right)\leq\Theta\left(\eta,\zeta_{n+1},{\lambda}/{2\mu}\right)\diamond\Theta\left(\zeta_n,\eta,{\lambda}/{2\mu k}\right)$,

on $n\to\infty$, we have $\Theta(\eta,T\eta,\lambda)\leq0\diamond0=0$ and

$$\Upsilon(\eta, T\eta, \lambda) \leq \Upsilon\left(\eta, \zeta_{n+1}, {}^{\lambda}/_{2\mu}\right) \diamond \Upsilon\left(\zeta_{n+1}, T\eta, {}^{\lambda}/_{2\mu}\right)$$
$$= \Upsilon\left(\eta, \zeta_{n+1}, {}^{\lambda}/_{2\mu}\right) \diamond \Upsilon\left(T\zeta_n, T\eta, {}^{\lambda}/_{2\mu}\right) \leq \Upsilon\left(\eta, \zeta_{n+1}, {}^{\lambda}/_{2\mu}\right) \diamond \Upsilon\left(\zeta_n, \eta, {}^{\lambda}/_{2\mu k}\right),$$

on $n \to \infty$, we have $\Upsilon(\eta, T\eta, \lambda) \leq 0 \diamond 0 = 0$.
By (c) and (h) of definition (3.1), we have, $\eta = T\eta$.
For uniqueness of fixed point, let η, θ be two fixed points of the mapping T, then $\eta = T\eta$ and $\theta = T\theta$ and

$$1 \geq \Xi(\eta, \theta, \lambda) = \Xi(T\eta, T\theta, \lambda) \geq \Xi\left(\eta, \theta, {}^{\lambda}/_{k}\right) = \Xi\left(T\eta, T\theta, {}^{\lambda}/_{k}\right) \geq \Xi\left(\eta, \theta, {}^{\lambda}/_{k^2}\right)$$
$$\geq \cdots \geq \Xi\left(\eta, \theta, {}^{\lambda}/_{k^n}\right) \to 1 \text{ as } n \to \infty.$$

Also, $0 \leq \Theta(\eta, \theta, \lambda) = \Theta(T\eta, T\theta, \lambda) \leq \Theta\left(\eta, \theta, {}^{\lambda}/_{k}\right) = \Theta\left(T\eta, T\theta, {}^{\lambda}/_{k}\right) \leq \Theta\left(\eta, \theta, {}^{\lambda}/_{k^2}\right)$
$\leq \cdots \leq \Theta\left(\eta, \theta, {}^{\lambda}/_{k^n}\right) \to 1$ as $n \to \infty$ and

$$0 \leq \Upsilon(\eta, \theta, \lambda) = \Upsilon(T\eta, T\theta, \lambda) \leq \Upsilon\left(\eta, \theta, {}^{\lambda}/_{k}\right) = \Upsilon\left(T\eta, T\theta, {}^{\lambda}/_{k}\right) \leq \Upsilon\left(\eta, \theta, {}^{\lambda}/_{k^2}\right)$$
$$\leq \cdots \leq \Upsilon\left(\eta, \theta, {}^{\lambda}/_{k^n}\right) \to 1 \text{ as } n \to \infty.$$

By (d), (i) and (n) of definition (3.1), $\eta = \theta$.

Corollary: 4.2

Let (Σ, d) be a complete metric space and $T: \Sigma \to \Sigma$ be a map which satisfies the following condition for all $\zeta, \eta \in \Sigma$ and $0 < k < 1$:

$$d(T\zeta, T\eta) \leq kd(\zeta, \eta) \tag{4.2.1}$$

Then T has unique fixed point in Σ.

Proof:

We consider the corresponding NbMS $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond, b)$ where
$\Xi(\zeta, \eta, \lambda) = \frac{\lambda}{\lambda + d(\zeta, \eta)}$, $\Theta(\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda + d(\zeta, \eta)}$ and $\Upsilon(\zeta, \eta, \lambda) = \frac{d(\zeta, \eta)}{\lambda}$

(4.2.1) $\Longrightarrow$ (4.1.3)

$$d(T\zeta, T\eta) \leq kd(\zeta, \eta), \ \frac{d(T\zeta, T\eta)}{k} \leq d(\zeta, \eta) \tag{4.2.2}$$

$\frac{d(T\zeta, T\eta)}{k\lambda} \leq \frac{d(\zeta, \eta)}{\lambda}$,
$\Upsilon(T\zeta, T\eta, k\lambda) \leq \Upsilon(\zeta, \eta, \lambda)$

(4.2.1) $\Longrightarrow$ (4.1.2)

$d(T\zeta, T\eta) \leq kd(\zeta, \eta), \frac{d(T\zeta, T\eta)}{k} \leq d(\zeta, \eta)$, From inequality (4.2.2).

Note that for $a, b, c, d \geq 0$, if $\frac{a}{b} \leq \frac{c}{d}$ then $\frac{a}{a+b} < \frac{c}{d+c}$. It follows that,

$$\frac{d(T\zeta, T\eta)}{k\lambda + d(T\zeta, T\eta)} \leq \frac{d(\zeta, \eta)}{\lambda + d(\zeta, \eta)}$$

Hence $\Theta(T\zeta, T\eta, k\lambda) \leq \Theta(\zeta, \eta, \lambda)$.

(4.2.1) $\Longrightarrow$ (4.1.1) from inequality (4.2.2)

$\frac{k}{d(T\zeta, T\eta)} \geq \frac{1}{d(\zeta, \eta)} \Longrightarrow \frac{k\lambda}{k\lambda + d(T\zeta, T\eta)} \geq \frac{\lambda}{\lambda + d(\zeta, \eta)}$.

Hence $\Xi(\mathrm{T}\zeta, \mathrm{T}\eta, \lambda) \geq \Xi(\zeta, \eta, \lambda)$.

In support of above theorem we furnish the following example.

Example: 4.3

Let $\Sigma = [0,1]$ and $\Xi, \Theta, \Upsilon: \Sigma^2 \text{ x } [0, \infty) \to [0,1]$ be fuzzy sets on $\Sigma^2 \text{ x } [0, \infty)$.

For all $\zeta, \eta \in \Sigma$ and $\lambda \in [0, \infty)$, define $\Xi(\zeta, \eta, \lambda) = \begin{cases} \frac{\lambda}{\lambda + |\zeta - \eta|}, & if \ \ \lambda > 0 \\ 0, & if \ \lambda = 0 \end{cases}$,

$\Theta(\zeta, \eta, \lambda) = \begin{cases} \frac{|\zeta - \eta|}{\lambda + |\zeta - \eta|}, & if \ \ \lambda > 0 \\ 1, & if \ \lambda = 0 \end{cases}$ and $\Upsilon(\zeta, \eta, \lambda) = \begin{cases} \frac{|\zeta - \eta|}{\lambda}, & if \ \ \lambda > 0 \\ 1, & if \ \lambda = 0 \end{cases}$

Clearly, $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond, b)$ is a Complete Neutrosophic b-Metric Space, (CNbMS), where $a * b = min(a, b)$, $a \diamond b = max(a, b)$, for all $a, b \in [0,1]$.

Let $T: \Sigma \to \Sigma$ be such that $T\zeta = \frac{\zeta}{8}$. Then for $k = \frac{1}{4}$,

$$\Xi\left(\mathrm{T}\zeta, \mathrm{T}\eta, \frac{\lambda}{4}\right) = \frac{\frac{\lambda}{4}}{\frac{\lambda}{4} + |\mathrm{T}\zeta - \mathrm{T}\eta|} = \frac{\frac{\lambda}{4}}{\frac{\lambda}{4} + \frac{|\zeta - \eta|}{8}} \geq \frac{\lambda}{\lambda + |\zeta - \eta|} = \Xi(\zeta, \eta, \lambda),$$

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$\Theta\left(T\zeta, T\eta, \frac{\lambda}{4}\right) = \frac{|T\zeta - T\eta|}{\frac{\lambda}{4} + |\zeta - \eta|} = \frac{\frac{|\zeta-\eta|}{8}}{\frac{\lambda}{4} + \frac{|\zeta-\eta|}{8}} = \frac{\frac{|\zeta-\eta|}{2}}{\lambda + \frac{|\zeta-\eta|}{2}} \leq \frac{|\zeta-\eta|}{\lambda + |\zeta-\eta|} = \Theta(\zeta, \eta, \lambda)$ and

$\Upsilon\left(T\zeta, T\eta, \frac{\lambda}{4}\right) = \frac{|T\zeta - T\eta|}{\frac{\lambda}{4}} = \frac{\frac{|\zeta-\eta|}{8}}{\frac{\lambda}{4}} = \frac{\frac{|\zeta-\eta|}{2}}{\lambda} \leq \frac{|\zeta-\eta|}{\lambda} = \Upsilon(\zeta, \eta, \lambda)$.

Hence T satisfies the contractive condition of theorem (4.1.1) to obtain a fixed point.

Theorem: 4.4

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond, b)$ is a CNbMS with $*$ t-norm and $\diamond$ conorm defined as $a * b = min\{a, b\}$ and $a \diamond b = max\{a, b\}$ respectively. Also suppose that $\Xi(\zeta, \eta, .)$ is strictly increasing, $\Theta(\zeta, \eta, .)$ is strictly decreasing and $\Upsilon(\zeta, \eta, .)$ is strictly decreasing respectively. Let $\mathcal{A}: \Sigma \to \Sigma$ be a self map which satisfies the following conditions for all $\zeta, \eta \in \Sigma$

$\Xi(\mathcal{A}\zeta, \mathcal{A}\eta, k\lambda) \geq \Xi(\zeta, \mathcal{A}\zeta, \lambda) * \Xi(\eta, \mathcal{A}\eta, \lambda)$ (4.4.1)

$\Theta(\mathcal{A}\zeta, \mathcal{A}\eta, k\lambda) \leq \Theta(\zeta\ \mathcal{A}\zeta, \lambda) \diamond \Theta(\eta, \mathcal{A}\eta, \lambda)$ (4.4.2)

$\Upsilon(\mathcal{A}\zeta, \mathcal{A}\eta, k\lambda) \leq \Upsilon(\zeta, \mathcal{A}\zeta, \lambda) \diamond \Upsilon(\eta, \mathcal{A}\eta, \lambda)$ (4.4.3)

where $\lambda > 0, 0 < k < 1$. Then $\mathcal{A}$ has a unique fixed point.

Proof :

Let $\zeta_0 \in \Sigma$ be an arbitrary point. Consider a sequence $\zeta_n = \mathcal{A}\zeta_{n-1}$ of points in Σ. Then

$\Xi(\zeta_n, \zeta_{n+1}, k\lambda) = \Xi(\mathcal{A}\zeta_{n-1}, \mathcal{A}\zeta_n, k\lambda) \geq \Xi(\zeta_{n-1}, \mathcal{A}\zeta_{n-1}, \lambda) * \Xi(\zeta_n, \mathcal{A}\zeta_n, \lambda)$
$= \Xi(\zeta_{n-1}, \zeta_n, \lambda) * \Xi(\zeta_n, \zeta_{n+1}, \lambda)$,

Since $\Xi(\zeta, \eta, .)$ is strictly increasing function, $k\lambda < \lambda$ and if $min\{\Xi(\zeta_{n-1}, \zeta_n, \lambda), \Xi(\zeta_n, \zeta_{n+1}, \lambda)\} = \Xi(\zeta_n, \zeta_{n+1}, \lambda)$, then we reach to a contradiction $\Xi(\zeta_n, \zeta_{n+1}, k\lambda) \geq \Xi(\zeta_n, \zeta_{n+1}, \lambda)$. Therefore,

$\Xi(\zeta_n, \zeta_{n+1}, k\lambda) \geq \Xi(\zeta_{n-1}, \zeta_n, \lambda) = \Xi(\mathcal{A}\zeta_{n-2}, \mathcal{A}\zeta_{n-1}, \lambda)$
$\geq \Xi\left(\zeta_{n-1}, \mathcal{A}\zeta_{n-1}, \frac{\lambda}{k}\right) * \Xi\left(\zeta_{n-2}, \mathcal{A}\zeta_{n-2}, \frac{\lambda}{k}\right) = \Xi\left(\zeta_{n-1}, \zeta_n, \frac{\lambda}{k}\right) * \Xi\left(\zeta_{n-2}, \zeta_{n-1}, \frac{\lambda}{k}\right)$
$= \Xi\left(\zeta_{n-2}, \zeta_{n-1}, \frac{\lambda}{k}\right) \ldots \Xi\left(\zeta_0, \zeta_1, \frac{\lambda}{k^{n-1}}\right)$.

Clearly, $1 \geq \Xi(\zeta_n, \zeta_{n+1}, k\lambda) \geq \Xi\left(\zeta_0, \zeta_1, \frac{\lambda}{k^{n-1}}\right) \to 1$, when $n \to \infty$.

Thus $\underset{n \to \infty}{Lim} \Xi(\zeta_n, \zeta_{n+1}, k\lambda) = 1$.

Now, $\Theta(\zeta_n, \zeta_{n+1}, k\lambda) = \Theta(\mathcal{A}\zeta_{n-1}, \mathcal{A}\zeta_n, k\lambda) \leq \Theta(\zeta_{n-1}, \mathcal{A}\zeta_{n-1}, \lambda) \diamond \Theta(\zeta_n, \mathcal{A}\zeta_n, \lambda)$
$= \Theta(\zeta_{n-1}, \zeta_n, \lambda) \diamond \Theta(\zeta_n, \zeta_{n+1}, \lambda)$,

since $\Theta(\zeta, \eta, .)$ is strictly decreasing function, $k\lambda < \lambda$, by the same argument $\Theta(\zeta_n, \zeta_{n+1}, k\lambda) \leq \Theta(\zeta_n, \zeta_{n+1}, \lambda)$ is not possible. Therefore,

$\Theta(\zeta_n, \zeta_{n+1}, k\lambda) \leq \Theta(\zeta_{n-1}, \zeta_n, \lambda) = \Theta(\mathcal{A}\zeta_{n-2}, \mathcal{A}\zeta_{n-1}, \lambda)$
$\leq \Theta\left(\zeta_{n-1}, \mathcal{A}\zeta_{n-1}, \frac{\lambda}{k}\right) \diamond \Theta\left(\zeta_{n-2}, \mathcal{A}\zeta_{n-2}, \frac{\lambda}{k}\right) = \Theta\left(\zeta_{n-1}, \zeta_n, \frac{\lambda}{k}\right) \diamond \Theta\left(\zeta_{n-2}, \zeta_{n-1}, \frac{\lambda}{k}\right)$
$= \Theta\left(\zeta_{n-2}, \zeta_{n-1}, \frac{\lambda}{k}\right) \ldots \leq \Theta\left(\zeta_0, \zeta_1, \frac{\lambda}{k^{n-1}}\right)$.

Clearly, $0 \leq \Theta(\zeta_n, \zeta_{n+1}, k\lambda) \leq \Theta\left(\zeta_0, \zeta_1, \frac{\lambda}{k^{n-1}}\right) \to 0$, when $n \to \infty$.

Hence $\underset{n \to \infty}{\text{Lim}} \Theta(\zeta_n, \zeta_{n+1}, k\lambda) = 0$.

Also, $\Upsilon(\zeta_n, \zeta_{n+1}, k\lambda) = \Upsilon(\mathcal{A}\zeta_{n-1}, \mathcal{A}\zeta_n, k\lambda) \leq \Upsilon(\zeta_{n-1}, \mathcal{A}\zeta_{n-1}, \lambda) \diamond \Upsilon(\zeta_n, \mathcal{A}\zeta_n, \lambda)$
$= \Upsilon(\zeta_{n-1}, \zeta_n, \lambda) \diamond \Upsilon(\zeta_n, \zeta_{n+1}, \lambda)$,

Since $\Upsilon(\zeta, \eta, .)$ is strictly decreasing function, $k\lambda < \lambda$, by the same argument $\Upsilon(\zeta_n, \zeta_{n+1}, k\lambda) \leq \Upsilon(\zeta_n, \zeta_{n+1}, \lambda)$ is not possible. Therefore,

$\Upsilon(\zeta_n, \zeta_{n+1}, k\lambda) \leq \Upsilon(\zeta_{n-1}, \zeta_n, \lambda) = \Upsilon(\mathcal{A}\zeta_{n-2}, \mathcal{A}\zeta_{n-1}, \lambda)$
$\leq \Upsilon\left(\zeta_{n-1}, \mathcal{A}\zeta_{n-1}, \frac{\lambda}{k}\right) \diamond \Upsilon\left(\zeta_{n-2}, \mathcal{A}\zeta_{n-2}, \frac{\lambda}{k}\right) = \Upsilon\left(\zeta_{n-1}, \zeta_n, \frac{\lambda}{k}\right) \diamond \Upsilon\left(\zeta_{n-2}, \zeta_{n-1}, \frac{\lambda}{k}\right)$
$= \Upsilon\left(\zeta_{n-2}, \zeta_{n-1}, \frac{\lambda}{k}\right) \ldots \leq \Upsilon\left(\zeta_0, \zeta_1, \frac{\lambda}{k^{n-1}}\right)$.

Clearly, $0 \leq \Upsilon(\zeta_n, \zeta_{n+1}, k\lambda) \leq \Upsilon\left(\zeta_0, \zeta_1, \frac{\lambda}{k^{n-1}}\right) \to 0$, when $n \to \infty$.

Thus $\underset{n \to \infty}{\text{Lim}} \Upsilon(\zeta_n, \zeta_{n+1}, k\lambda) = 0$.

Let $\alpha_n(\lambda) = \Xi(\zeta_n, \zeta_{n+1}, \lambda)$, $\beta_n(\lambda) = \Theta(\zeta_n, \zeta_{n+1}, \lambda)$ and $\gamma_n(\lambda) = \Upsilon(\zeta_n, \zeta_{n+1}, \lambda)$, for all $n \in \mathbb{N} \cup \{0\}, \lambda > 0$. Clearly, $\underset{\lambda \to \infty}{\lim} \alpha_n(\lambda) = 1$, $\underset{\lambda \to \infty}{\lim} \beta_n(\lambda) = 0$ and $\underset{\lambda \to \infty}{\lim} \gamma_n(\lambda) = 0$. Next, we show that the sequence $\{\zeta_n\}$ is a Cauchy sequence. If it is not, then there exists $0 < \epsilon < 1$ and the sequences $\{u(n)\}, \{v(n)\}$ such that for every $n \in \mathbb{N} \cup \{0\}$, $\lambda > 0$, $u(n) > v(n) \geq n$, $\Xi(\zeta_{u(n)}, \zeta_{v(n)}, \lambda) \leq 1 - \epsilon$, $\Theta(\zeta_{u(n)}, \zeta_{v(n)}, \lambda) \geq \epsilon$ and $\Upsilon(\zeta_{u(n)}, \zeta_{v(n)}, \lambda) \geq \epsilon$ and $\Xi(\zeta_{u(n)-1}, \zeta_{v(n)-1}, \lambda) > 1 - \epsilon$, $\Xi(\zeta_{u(n)-1}, \zeta_{v(n)}, \lambda) > 1 - \epsilon$,

$\Theta\left(\zeta_{u(n)-1},\zeta_{v(n)-1},\lambda\right) < \epsilon,\ \Theta\left(\zeta_{u(n)-1},\zeta_{v(n)},\lambda\right) < \epsilon$ and $\Upsilon\left(\zeta_{u(n)-1},\zeta_{v(n)-1},\lambda\right) < \epsilon,\ \Upsilon\left(\zeta_{u(n)-1},\zeta_{v(n)},\lambda\right) < \epsilon$.

Now, $1-\epsilon \geq \Xi\left(\zeta_{u(n)},\zeta_{v(n)},\lambda\right) \geq \Xi\left(\zeta_{u(n)-1},\zeta_{u(n)},{}^{\lambda}/_{2b}\right) * \Xi\left(\zeta_{u(n)-1},\zeta_{v(n)},{}^{\lambda}/_{2b}\right)$

$> \alpha_{u(n)-1}\left({}^{\lambda}/_{2b}\right) * (1-\epsilon)$,

$\epsilon \leq \Theta\left(\zeta_{u(n)},\zeta_{v(n)},\lambda\right) \leq \Theta\left(\zeta_{u(n)-1},\zeta_{u(n)},{}^{\lambda}/_{2b}\right) \diamond \Theta\left(\zeta_{u(n)-1},\zeta_{v(n)},{}^{\lambda}/_{2b}\right)$

$< \beta_{u(n)-1}\left({}^{\lambda}/_{2b}\right) \diamond \epsilon$ and

$\epsilon \leq \Upsilon\left(\zeta_{u(n)},\zeta_{v(n)},\lambda\right) \leq \Upsilon\left(\zeta_{u(n)-1},\zeta_{u(n)},{}^{\lambda}/_{2b}\right) \diamond \Upsilon\left(\zeta_{u(n)-1},\zeta_{v(n)},{}^{\lambda}/_{2b}\right)$

$< \gamma_{u(n)-1}\left({}^{\lambda}/_{2b}\right) \diamond \epsilon$.

Since $\alpha_{u(n)-1}\left({}^{\lambda}/_{2b}\right) \to 1,\ \beta_{u(n)-1}\left({}^{\lambda}/_{2b}\right) \to 0$ and $\gamma_{u(n)-1}\left({}^{\lambda}/_{2b}\right) \to 0$ as $n \to \infty$ for every λ, it follows that $1-\epsilon \geq \Xi\left(\zeta_{u(n)},\zeta_{v(n)},\lambda\right) > 1-\epsilon,\ \epsilon \leq \Theta\left(\zeta_{u(n)},\zeta_{v(n)},\lambda\right) < \epsilon$ and $\epsilon \leq \Upsilon\left(\zeta_{u(n)},\zeta_{v(n)},\lambda\right) < \epsilon$.

Clearly, this leads to a contradiction. Hence ζ_n is a Cauchy sequence in Σ.

Since Σ is complete so there exist $\eta \in \Sigma$ such that $\lim_{n\to\infty} \zeta_n = \eta$.

Assume that $\eta \neq \mathcal{A}\eta$, then there exists $\lambda > 0$ such that $\Xi(\eta,\mathcal{A}\eta,\lambda) \neq 1$ or $\Theta(\eta,\mathcal{A}\eta,\lambda) \neq 0$ and $\Upsilon(\eta,\mathcal{A}\eta,\lambda) \neq 0$. For this $\lambda > 0$.

$\Xi(\mathcal{A}\zeta_n,\mathcal{A}\eta,k\lambda) \geq \Xi(\zeta_n,\mathcal{A}\zeta_n,\lambda) * \Xi(\eta,\mathcal{A}\eta,\lambda)$ by contractive condition (4.4.1).

That is $\Xi(\zeta_{n+1},\mathcal{A}\eta,k\lambda) \geq \Xi(\zeta_n,\zeta_{n+1},\lambda) * \Xi(\eta,\mathcal{A}\eta,\lambda)$.

In limiting case as $n \to \infty$, $\Xi(\eta,\mathcal{A}\eta,k\lambda) \geq \Xi(\eta,\mathcal{A}\eta,\lambda)$.

As $\Xi(\eta,\mathcal{A}\eta,\lambda) \neq 1$, the above inequality yields a contradiction to the fact that $\Xi(\zeta,\eta,.)$ is strictly increasing. Moreover,

$\Theta(\mathcal{A}\zeta_n,\mathcal{A}\eta,k\lambda) \leq \Theta(\zeta_n,\mathcal{A}\zeta_n,\lambda) \diamond \Theta(\eta,\mathcal{A}\eta,\lambda)$, by contractive condition (4.4.2).

That is $\Theta(\zeta_{n+1},\mathcal{A}\eta,k\lambda) \leq \Theta(\zeta_n,\zeta_{n+1},\lambda) \diamond \Theta(\eta,\mathcal{A}\eta,\lambda)$.

In limiting case as $n \to \infty$, $\Theta(\eta,\mathcal{A}\eta,k\lambda) \leq \Theta(\eta,\mathcal{A}\eta,\lambda)$.

As $\Theta(\eta,\mathcal{A}\eta,\lambda) \neq 0$, the above inequality yields a contradiction to the fact that $\Theta(\zeta,\eta,.)$ is strictly decreasing. Also, $\Upsilon(\mathcal{A}\zeta_n,\mathcal{A}\eta,k\lambda) \leq \Upsilon(\zeta_n,\mathcal{A}\zeta_n,\lambda) \diamond \Upsilon(\eta,\mathcal{A}\eta,\lambda)$, by contractive condition (4.4.3).

That is $\Upsilon(\zeta_{n+1},\mathcal{A}\eta,k\lambda) \leq \Upsilon(\zeta_n,\zeta_{n+1},\lambda) \diamond \Upsilon(\eta,\mathcal{A}\eta,\lambda)$.

In limiting case as $n \to \infty$ $\Upsilon(\eta,\mathcal{A}\eta,k\lambda) \leq \Upsilon(\eta,\mathcal{A}\eta,\lambda)$.

As $\Upsilon(\eta,\mathcal{A}\eta,\lambda) \neq 0$, the above inequality yields a contradiction to the fact that $\Upsilon(\zeta,\eta,.)$ is strictly decreasing. Hence, $\eta = \mathcal{A}\eta$.

For uniqueness, let η and θ be two fixed points of $\mathcal{A}$. So, $\eta = \mathcal{A}\eta$ and $\theta = \mathcal{A}\theta$.

Then $\Xi(\eta,\mathcal{A}\eta,\lambda) = 1, \Xi(\theta,\mathcal{A}\theta,\lambda) = 1$,

$\Theta(\eta,\mathcal{A}\eta,\lambda) = 0, \Theta(\theta,\mathcal{A}\theta,\lambda) = 0$ and $\Upsilon(\eta,\mathcal{A}\eta,\lambda) = 0, \Upsilon(\theta,\mathcal{A}\theta,\lambda) = 0$, for all $\lambda > 0$.

Now, $1 \geq \Xi(\eta,\theta,\lambda) = \Xi(\mathcal{A}\eta,\mathcal{A}\theta,\lambda) \geq \Xi\left(\eta,\mathcal{A}\eta,{}^{\lambda}/_{k}\right) * \Xi\left(\theta,\mathcal{A}\theta,{}^{\lambda}/_{k}\right) = 1 * 1 = 1$,

$0 \leq \Theta(\eta,\theta,\lambda) = \Theta(\mathcal{A}\eta,\mathcal{A}\theta,\lambda) \leq \Theta\left(\eta,\mathcal{A}\eta,{}^{\lambda}/_{k}\right) \diamond \Theta\left(\theta,\mathcal{A}\theta,{}^{\lambda}/_{k}\right) = 0 \diamond 0 = 0$ and

$0 \leq \Upsilon(\eta,\theta,\lambda) = \Upsilon(\mathcal{A}\eta,\mathcal{A}\theta,\lambda) \leq \Upsilon\left(\eta,\mathcal{A}\eta,{}^{\lambda}/_{k}\right) \diamond \Upsilon\left(\theta,\mathcal{A}\theta,{}^{\lambda}/_{k}\right) = 0 \diamond 0 = 0$.

From (d), (i) and (n) of definition 3.1, we have $\eta = \theta$.

Corollary: 4.5

Let (Σ, d) be a complete metric space and $T:\Sigma \to \Sigma$ be a map which satisfies the following condition for all $\zeta,\eta \in \Sigma$ and $0 < k < 1$:

$$d(T\zeta,T\eta) \leq \frac{k}{2}[d(\zeta,T\zeta) + d(\eta,T\eta)]. \qquad (4.5.1)$$

Then T has a unique fixed point in Σ.

Proof:

We consider the corresponding NbMS $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond,b)$ where

$\Xi(\zeta,\eta,\lambda) = \begin{cases} \frac{\lambda}{\lambda+d(\zeta,\eta)} & if\ \lambda > 0 \\ 0\ if\ \lambda = 0 \end{cases}$, $\Theta(\zeta,\eta,\lambda) = \begin{cases} \frac{d(\zeta,\eta)}{\lambda+d(\zeta,\eta)} & if\ \lambda > 0 \\ 1\ if\ \lambda = 0 \end{cases}$ and

$\Upsilon(\zeta,\eta,\lambda) = \begin{cases} \frac{d(\zeta,\eta)}{\lambda} & if\ \lambda > 0 \\ 1\ if\ \lambda = 0 \end{cases}$

Replacing $\mathcal{A}$ with T in inequalities (4.4.1), (4.4.2) and (4.4.3)

Now, (4.5.1) $\Longrightarrow$ (4.4.1). If otherwise, then from (4.4.1), for some $\lambda > 0$

$\Xi(T\zeta,T\eta,k\lambda) < min\{\Xi(\zeta,T\zeta,\lambda),\Xi(\eta,T\eta,\lambda)\}$,

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ie., $\frac{\lambda}{\lambda+\frac{1}{k}d(T\zeta,T\eta)} < min\left\{\frac{\lambda}{\lambda+d(\zeta,T\zeta)},\frac{\lambda}{\lambda+d(\eta,T\eta)}\right\}$.
This implies that $\lambda+\frac{1}{k}d(T\zeta,T\eta) > \lambda+d(\zeta,T\zeta)$ and $\lambda+\frac{1}{k}d(T\zeta,T\eta) > \lambda+d(\eta,T\eta)$
$\Rightarrow \frac{2}{k}d(T\zeta,T\eta) > [d(\zeta,T\zeta)+d(\eta,T\eta)]$ or $d(T\zeta,T\eta) > \frac{k}{2}\,[d(\zeta,T\zeta)+d(\eta,T\eta)]$,
which is contradiction to (4.5.1).
Now (4.5.1) $\Rightarrow$ (4.4.2)

$$d(T\zeta,T\eta) \le \frac{k}{2}\,[d(\zeta,T\zeta)+d(\eta,T\eta)] \le \frac{2k}{2}max\,[d(\zeta,T\zeta),d(\eta,T\eta)]$$
$$\le k\,max[d(\zeta,T\zeta),d(\eta,T\eta)]$$
$$\frac{d(T\zeta,T\eta)}{k} \le max[d(\zeta,T\zeta),d(\eta,T\eta)].$$

Without loss of generality, we assume that $max[d(\zeta,T\zeta),d(\eta,T\eta)] = d(\zeta,T\zeta)$.
This implies that $\frac{d(T\zeta,T\eta)}{k} \le d(\zeta,T\zeta)$,
So, $\frac{d(T\zeta,T\eta)}{k\lambda} \le \frac{d(\zeta,T\zeta)}{\lambda}$ and $\frac{d(T\zeta,T\eta)}{k\lambda+d(T\zeta,T\eta)} \le \frac{d(\zeta,T\zeta)}{\lambda+d(\zeta,T\zeta)}$.
Then, $\frac{d(T\zeta,T\eta)}{k\lambda+d(T\zeta,T\eta)} \le max\left\{\frac{d(\zeta,T\zeta)}{\lambda+d(\zeta,T\zeta)},\frac{d(\eta,T\eta)}{\lambda+d(\eta,T\eta)}\right\}$.
Hence, $\Theta(T\zeta,T\eta,k\lambda) \le max\{\Theta(\zeta,T\zeta,\lambda),\Theta(\eta,T\eta,\lambda)\}$
Now (4.5.1) $\Rightarrow$ (4.4.3).

$$d(T\zeta,T\eta) \le \frac{k}{2}\,[d(\zeta,T\zeta)+d(\eta,T\eta)] \le \frac{2k}{2}max\,[d(\zeta,T\zeta),d(\eta,T\eta)]$$
$$\le k\,max[d(\zeta,T\zeta),d(\eta,T\eta)]$$
$$\frac{d(T\zeta,T\eta)}{k} \le max[d(\zeta,T\zeta),d(\eta,T\eta)].$$

Without loss of generality, we assume that $max[d(\zeta,T\zeta),d(\eta,T\eta)] = d(\zeta,T\zeta)$.
This implies that $\frac{d(T\zeta,T\eta)}{k} \le d(\zeta,T\zeta)$,
So, $\frac{d(T\zeta,T\eta)}{k\lambda} \le \frac{d(\zeta,T\zeta)}{\lambda}$.
Then, $\frac{d(T\zeta,T\eta)}{k\lambda} \le max\left\{\frac{d(\zeta,T\zeta)}{\lambda},\frac{d(\eta,T\eta)}{\lambda}\right\}$.
Hence, $\Upsilon(T\zeta,T\eta,k\lambda) \le max\{\Upsilon(\zeta,T\zeta,\lambda),\Upsilon(\eta,T\eta,\lambda)\}$.
In support of theorem (4.1.2) we establish an example.

Example: 4.6

Let $\Sigma=[0,1]$ and $\Xi,\Theta,\Upsilon:\Sigma^2$ x $[0,\infty)\to[0,1]$ by fuzzy sets on Σ^2 x $[0,\infty)$. For all $\zeta,\eta\in\Sigma$ and $\lambda\in[0,\infty)$, define

$$\Xi(\zeta,\eta,\lambda)=\begin{cases}\frac{\lambda}{\lambda+|\zeta-\eta|} & if\ \lambda>0\\ 0 & if\ \lambda=0\end{cases},\quad \Theta(\zeta,\eta,\lambda)=\begin{cases}\frac{|\zeta-\eta|}{\lambda+|\zeta-\eta|} & if\ \lambda>0\\ 1 & if\ \lambda=0\end{cases}\text{ and}$$
$$\Upsilon(\zeta,\eta,\lambda)=\begin{cases}\frac{|\zeta-\eta|}{\lambda} & if\ \lambda>0\\ 1 & if\ \lambda=0.\end{cases}$$

Clearly, $(\Sigma,\Xi,\Theta,\Upsilon,*,\diamond,b)$ is a CNbMS, where $a*b=min(a,b)$, $a\diamond b=max(a,b)$ for all $a,b\in[0,1]$. Let $T:\Sigma\to\Sigma$ be such that $T\zeta=\frac{\zeta}{30}$. Then for $k=\frac{2}{3}$,

$$\Xi\left(T\zeta,T\eta,\frac{2\lambda}{3}\right)=\frac{\frac{2\lambda}{3}}{\frac{2\lambda}{3}+|T\zeta-T\eta|}=\frac{\frac{2\lambda}{3}}{\frac{2\lambda}{3}+(|\zeta-\eta|)/_{30}}=\frac{\lambda}{\lambda+\frac{3}{2}\left(\frac{|\zeta-\eta|}{30}\right)}.$$

Now, as $\zeta,\eta\in[0,1]$

$$\left|\frac{\zeta-\eta}{30}\right| \le \left|\frac{\zeta+\eta}{30}\right| \le \frac{1}{3}\left|\frac{3\zeta}{30}+\frac{3\eta}{30}\right| \le \frac{1}{3}\left|\frac{29\zeta}{30}+\frac{29\eta}{30}\right| \le \frac{1}{3}\left(\left|\frac{29\zeta}{30}\right|+\left|\frac{29\eta}{30}\right|\right)$$
$$\Rightarrow 2\left(\frac{3}{2}\right)\left|\frac{\zeta-\eta}{30}\right| \le \left|\frac{29\zeta}{30}\right|+\left|\frac{29\eta}{30}\right|. \qquad (4.6.1)$$

Note that if $a,b,c\ge 0, 2a\le b+c$, then $a\le Max\{b,c\}$. Otherwise $2a>b+c$.
It follows that $\frac{3}{2}\left|\frac{\zeta-\eta}{30}\right| \le \left|\frac{29\zeta}{30}\right|$ or $\frac{3}{2}\left|\frac{\zeta-\eta}{30}\right| \le \left|\frac{29\eta}{30}\right|$.
Without loss of generality assume that $\zeta\ge\eta$, then

$$min\left\{\frac{\lambda}{\lambda+\frac{29\zeta}{30}},\frac{\lambda}{\lambda+\frac{29\eta}{30}}\right\}=\frac{\lambda}{\lambda+\frac{29\zeta}{30}}\text{ and } max\left\{\frac{29\zeta}{30},\frac{29\eta}{30}\right\}=\frac{29\zeta}{30}$$
$$\Rightarrow \frac{3}{2}\left|\frac{\zeta-\eta}{30}\right| \le \left|\frac{29\zeta}{30}\right| \Rightarrow \frac{1}{\frac{3}{2}\left|\frac{\zeta-\eta}{30}\right|} \ge \frac{1}{\frac{29\zeta}{30}} \Rightarrow \frac{\lambda}{\lambda+\frac{3}{2}\left|\frac{\zeta-\eta}{30}\right|} \ge \frac{\lambda}{\lambda+\frac{29\zeta}{30}}$$

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$$\Rightarrow \frac{\frac{2\lambda}{3}}{\frac{2\lambda}{3}+|T\zeta-T\eta|} \geq min\left\{\frac{\lambda}{\lambda+|\zeta-T\zeta|},\frac{\lambda}{\lambda+|\eta-T\eta|}\right\}$$

$$\Rightarrow \ \Xi(T\zeta,T\eta,k\lambda) \geq \Xi(\zeta,T\zeta,\lambda) * \Xi(\eta,T\eta,\lambda).$$

Moreover, $\Theta(T\zeta,T\eta,k\lambda) = \frac{\left|\frac{\zeta-\eta}{30}\right|}{k\lambda+\left|\frac{\zeta-\eta}{30}\right|}$. From inequality (4.6.1) $\left|\frac{\zeta-\eta}{30}\right| \leq \frac{2}{3} max\left\{\frac{29\zeta}{30},\frac{29\eta}{30}\right\}$.

As it is assumed that $\zeta \geq \eta$, therefore,

$$\left|\frac{\zeta-\eta}{30}\right| \leq \frac{2}{3}\left(\frac{29\zeta}{30}\right),$$

$$\frac{3}{2}\left(\left|\frac{\zeta-\eta}{30}\right|\right) \leq \frac{29\zeta}{30}$$

$$1+\frac{\lambda}{\frac{3}{2}\left(\left|\frac{\zeta-\eta}{30}\right|\right)} \geq 1+\frac{\lambda}{\frac{29\zeta}{30}}$$

$$\frac{\frac{3}{2}\left(\left|\frac{\zeta-\eta}{30}\right|\right)+\lambda}{\frac{3}{2}\left(\left|\frac{\zeta-\eta}{30}\right|\right)} \geq \frac{\frac{29\zeta}{30}+\lambda}{\frac{29\zeta}{30}}$$

$$\frac{\frac{2\lambda}{3}+|T\zeta-T\eta|}{|T\zeta-T\eta|} \geq \frac{\lambda+|\zeta-T\zeta|}{|\zeta-T\zeta|}$$

$$\frac{|T\zeta-T\eta|}{\frac{2\lambda}{3}+|T\zeta-T\eta|} \leq \frac{|\zeta-T\zeta|}{\lambda+|\zeta-T\zeta|}$$

$\Theta(T\zeta,T\eta,k\lambda) \leq \Theta(\zeta,T\zeta,\lambda) \diamond \Theta(\eta,T\eta,\lambda)$. Also,

$$\Upsilon(T\zeta,T\eta,k\lambda) = \frac{|T\zeta-T\eta|}{\frac{2\lambda}{3}} = \frac{\frac{1}{30}|\zeta-\eta|}{\frac{2\lambda}{3}}.$$

Now, as $\zeta,\eta \in [0,1]$

$$\left|\frac{\zeta-\eta}{30}\right| \leq \left|\frac{\zeta+\eta}{30}\right| \leq \frac{1}{3}\left|\frac{3\zeta}{30}+\frac{3\eta}{30}\right| \leq \frac{1}{3}\left|\frac{29\zeta}{30}+\frac{29\eta}{30}\right| \leq \frac{1}{3}\left(\left|\frac{29\zeta}{30}\right|+\left|\frac{29\eta}{30}\right|\right)$$

$$\Rightarrow \ 2\left(\frac{3}{2}\right)\left|\frac{\zeta-\eta}{30}\right| \leq \left|\frac{29\zeta}{30}\right|+\left|\frac{29\eta}{30}\right|. \tag{4.6.2}$$

From inequality (4.6.1)

$$\left|\frac{\zeta-\eta}{30}\right| \leq \frac{2}{3} max\left\{\frac{29\zeta}{30},\frac{29\eta}{30}\right\}.$$

As it is assumed that $\zeta \geq \eta$, therefore,

$$\left|\frac{\zeta-\eta}{30}\right| \leq \frac{2}{3}\left(\frac{29\zeta}{30}\right)$$

$$\frac{3}{2}\left(\left|\frac{\zeta-\eta}{30}\right|\right) \leq \frac{29\zeta}{30}$$

$$\frac{\frac{3}{2}\left(\left|\frac{\zeta-\eta}{30}\right|\right)}{\lambda} \leq \frac{\frac{29\zeta}{30}}{\lambda}$$

$$\frac{\left|\frac{\zeta-\eta}{30}\right|}{\frac{2\lambda}{3}} \leq \frac{\frac{29\zeta}{30}}{\lambda}, \frac{|T\zeta-T\eta|}{\frac{2\lambda}{3}} \leq \frac{|\zeta-T\zeta|}{\lambda}.$$

$\Upsilon(T\zeta,T\eta,k\lambda) \leq \Upsilon(\zeta,T\zeta,\lambda) \diamond \Upsilon(\eta,T\eta,\lambda)$.

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Hence T satisfies the contractive conditions of Theorem (4.1.3) to obtain a fixed point.

Theorem: 4.7

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond, b)$ is a CNbMS with $*$ t-norm and $\diamond$ conorm defined as $a * b = min\{a, b\}$, $a \diamond b = max\{a, b\}$, $\Xi(\zeta, \eta, .)$ is strictly increasing, $\Theta(\zeta, \eta, .)$ and $\Upsilon(\zeta, \eta, .)$ are strictly decreasing functions respectively. Let $\mathcal{A}: \Sigma \to \Sigma$ be a self mapping on Σ. If for all $\zeta, \eta \in \Sigma$, $0 < k < 1/_{2b}$, $\mathcal{A}$ satisfies the following conditions:

$\Xi(\mathcal{A}\zeta, \mathcal{A}\eta, k\lambda) \geq \Xi(\zeta, \mathcal{A}\eta, \lambda) * \Xi(\eta, \mathcal{A}\zeta, \lambda)$, (4.7.1)

$\Theta(\mathcal{A}\zeta, \mathcal{A}\eta, k\lambda) \leq \Theta(\zeta, \mathcal{A}\eta, \lambda) \diamond \Theta(\eta, \mathcal{A}\zeta, \lambda)$, (4.7.2)

$\Upsilon(\mathcal{A}\zeta, \mathcal{A}\eta, k\lambda) \leq \Upsilon(\zeta, \mathcal{A}\eta, \lambda) \diamond \Upsilon(\eta, \mathcal{A}\zeta, \lambda)$. (4.7.3)

Where $\lambda > 0$. Then $\mathcal{A}$ has a unique fixed point.

Proof:

Let $\zeta_0 \in \Sigma$ be an arbitrary point such that $\zeta_n = \mathcal{A}\zeta_{n-1}$ is a sequence in Σ.

$$\begin{aligned}\Xi(\zeta_n, \zeta_{n+1}, k\lambda) &= \Xi(\mathcal{A}\zeta_{n-1}, \mathcal{A}\zeta_n, k\lambda)\\ &\geq \Xi(\zeta_{n-1}, \mathcal{A}\zeta_n, \lambda) * \Xi(\zeta_n, \mathcal{A}\zeta_{n-1}, \lambda)\\ &= \Xi(\zeta_{n-1}, \zeta_{n+1}, \lambda) * \Xi(\zeta_n, \zeta_n, \lambda).\end{aligned}$$

Since $\Xi(\zeta_n, \zeta_n, \lambda) = 1$, $\Xi(\zeta_n, \zeta_{n+1}, k\lambda) \geq \Xi(\zeta_{n-1}, \zeta_{n+1}, \lambda)$.

By using (f) of definition (3.1), we have

$\Xi(\zeta_n, \zeta_{n+1}, k\lambda) \geq \Xi\left(\zeta_{n-1}, \zeta_n, \lambda/_{2b}\right) * \Xi\left(\zeta_n, \zeta_{n+1}, \lambda/_{2b}\right)$.

Since $\Xi(\zeta, \eta, .)$ is strictly increasing function and $k\lambda < \lambda/_{2b}$, if $min\left\{\Xi\left(\zeta_{n-1}, \zeta_n, \lambda/_{2b}\right), \Xi\left(\zeta_n, \zeta_{n+1}, \lambda/_{2b}\right)\right\} = \Xi\left(\zeta_n, \zeta_{n+1}, \lambda/_{2b}\right)$ then we reach to a contradiction $\Xi(\zeta_n, \zeta_{n+1}, k\lambda) \geq \Xi\left(\zeta_n, \zeta_{n+1}, \lambda/_{2b}\right)$.

Therefore, $\Xi(\zeta_n, \zeta_{n+1}, k\lambda) \geq \Xi\left(\zeta_{n-1}, \zeta_n, \lambda/_{2b}\right)$, Continuing this process, we have

$\Xi(\zeta_n, \zeta_{n+1}, k\lambda) \geq \Xi\left(\zeta_0, \zeta_1, \lambda/_{(2b)^n k^{n-1}}\right)$.

Clearly, $1 \geq \Xi(\zeta_n, \zeta_{n+1}, k\lambda) \geq \Xi\left(\zeta_0, \zeta_1, \lambda/_{(2b)^n k^{n-1}}\right) \to 1$, when $n \to \infty$.

Thus, $\lim_{n \to \infty} \Xi(\zeta_n, \zeta_{n+1}, k\lambda) = 1$. Moreover,

$$\begin{aligned}\Theta(\zeta_n, \zeta_{n+1}, k\lambda) &= \Theta(\mathcal{A}\zeta_{n-1}, \mathcal{A}\zeta_n, k\lambda)\\ &\leq \Theta(\zeta_{n-1}, \mathcal{A}\zeta_n, \lambda) \diamond \Theta(\zeta_n, \mathcal{A}\zeta_{n-1}, \lambda)\\ &= \Theta(\zeta_{n-1}, \zeta_{n+1}, \lambda) \diamond \Theta(\zeta_n, \zeta_n, \lambda).\end{aligned}$$

Since $\Theta(\zeta_n, \zeta_n, \lambda) = 0$, $\Theta(\zeta_n, \zeta_{n+1}, k\lambda) \leq \Theta(\zeta_{n-1}, \zeta_{n+1}, \lambda)$. By using (k) of definition (3.1), we have

$\Theta(\zeta_n, \zeta_{n+1}, k\lambda) \leq \Theta\left(\zeta_{n-1}, \zeta_n, \lambda/_{2b}\right) \diamond \Theta\left(\zeta_n, \zeta_{n+1}, \lambda/_{2b}\right)$.

Since $\Theta(\zeta, \eta, .)$ is strictly decreasing function and $k\lambda < \lambda/_{2b}$, by the same argument $\Theta(\zeta_n, \zeta_{n+1}, k\lambda) \leq \Theta\left(\zeta_n, \zeta_{n+1}, \lambda/_{2b}\right)$ is not possible.

Therefore, $\Theta(\zeta_n, \zeta_{n+1}, k\lambda) \leq \Theta\left(\zeta_{n-1}, \zeta_n, \lambda/_{2b}\right)$, continuing this process, we have

$\Theta(\zeta_n, \zeta_{n+1}, k\lambda) \leq \Theta\left(\zeta_0, \zeta_1, \lambda/_{(2b)^n k^{n-1}}\right)$.

Clearly, $0 \leq \Theta(\zeta_n, \zeta_{n+1}, k\lambda) \leq \Theta\left(\zeta_0, \zeta_1, \lambda/_{(2b)^n k^{n-1}}\right) \to 0$, when $n \to \infty$.

Thus, $\lim_{n \to \infty} \Theta(\zeta_n, \zeta_{n+1}, k\lambda) = 0$.

Moreover, $\Upsilon(\zeta_n, \zeta_{n+1}, k\lambda) = \Upsilon(\mathcal{A}\zeta_{n-1}, \mathcal{A}\zeta_n, k\lambda) \leq \Upsilon(\zeta_{n-1}, \mathcal{A}\zeta_n, \lambda) \diamond \Upsilon(\zeta_n, \mathcal{A}\zeta_{n-1}, \lambda)$

$= \Upsilon(\zeta_{n-1}, \zeta_{n+1}, \lambda) \diamond \Upsilon(\zeta_n, \zeta_n, \lambda)$.

Since $\Upsilon(\zeta_n, \zeta_n, \lambda) = 0$, $\Upsilon(\zeta_n, \zeta_{n+1}, k\lambda) \leq \Upsilon(\zeta_{n-1}, \zeta_{n+1}, \lambda)$.

By using (p) of definition (3.1), we have

$\Upsilon(\zeta_n, \zeta_{n+1}, k\lambda) \leq \Upsilon\left(\zeta_{n-1}, \zeta_n, \lambda/_{2b}\right) \diamond \Upsilon\left(\zeta_n, \zeta_{n+1}, \lambda/_{2b}\right)$.

Since $\Upsilon(\zeta, \eta, .)$ is strictly decreasing function and $k\lambda < \lambda/_{2b}$, by the same argument $\Upsilon(\zeta_n, \zeta_{n+1}, k\lambda) \leq \Upsilon\left(\zeta_n, \zeta_{n+1}, \lambda/_{2b}\right)$ is not possible.

Therefore, $\Upsilon(\zeta_n, \zeta_{n+1}, k\lambda) \leq \Upsilon\left(\zeta_{n-1}, \zeta_n, \lambda/_{2b}\right)$, continuing this process, we have

$\Upsilon(\zeta_n, \zeta_{n+1}, k\lambda) \leq \Upsilon\left(\zeta_0, \zeta_1, \lambda/_{(2b)^n k^{n-1}}\right)$.

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Clearly, $0 \le \Upsilon(\zeta_n, \zeta_{n+1}, k\lambda) \le \Upsilon\left(\zeta_0, \zeta_1, \lambda/_{(2b)^n k^{n-1}}\right) \to 0$ when $n \to \infty$.

Thus, $\lim_{n \to \infty} \Upsilon(\zeta_n, \zeta_{n+1}, k\lambda) = 0$.

Let $\alpha_n(\lambda) = \Xi(\zeta_n, \zeta_{n+1}, \lambda)$, $\beta_n(\lambda) = \Theta(\zeta_n, \zeta_{n+1}, \lambda)$ and $\gamma_n(\lambda) = \Upsilon(\zeta_n, \zeta_{n+1}, \lambda)$ for all $n \in \mathbb{N}\cup\{0\}$ and $\lambda > 0$. Clearly, $\lim_{\lambda \to \infty} \alpha_n(\lambda) = 1$, $\lim_{\lambda \to \infty} \beta_n(\lambda) = 0$ and $\lim_{\lambda \to \infty} \gamma_n(\lambda) = 0$. Next, we show that the sequence $\{\zeta_n\}$ is a Cauchy sequence.

If it is not then there exists $0 < \epsilon < 1$ and the sequences $\{u(n)\}, \{v(n)\}$ such that for every $n \in \mathbb{N}\cup\{0\}$, $\lambda > 0$, $u(n) > v(n) \ge n$,

$\Xi(\zeta_{u(n)}, \zeta_{v(n)}, \lambda) \le 1 - \epsilon$, $\Theta(\zeta_{u(n)}, \zeta_{v(n)}, \lambda) \ge \epsilon$ and $\Upsilon(\zeta_{u(n)}, \zeta_{v(n)}, \lambda) \ge \epsilon$ and

$\Xi(\zeta_{u(n)-1}, \zeta_{v(n)-1}, \lambda) > 1 - \epsilon$, $\Xi(\zeta_{u(n)-1}, \zeta_{v(n)}, \lambda) > 1 - \epsilon$,

$\Theta(\zeta_{u(n)-1}, \zeta_{v(n)-1}, \lambda) < \epsilon$, $\Theta(\zeta_{u(n)-1}, \zeta_{v(n)}, \lambda) < \epsilon$ and

$\Upsilon(\zeta_{u(n)-1}, \zeta_{v(n)-1}, \lambda) < \epsilon$, $\Upsilon(\zeta_{u(n)-1}, \zeta_{v(n)}, \lambda) < \epsilon$.

Now, $1 - \epsilon \ge \Xi(\zeta_{u(n)}, \zeta_{v(n)}, \lambda) \ge \Xi\left(\zeta_{u(n)-1}, \zeta_{u(n)}, \lambda/_{2b}\right) * \Xi\left(\zeta_{u(n)-1}, \zeta_{v(n)}, \lambda/_{2b}\right)$

$\ge \alpha_{u(n)-1}\left(\lambda/_{2b}\right) * (1 - \epsilon)$,

$\epsilon \le \Theta(\zeta_{u(n)}, \zeta_{v(n)}, \lambda) \le \Theta\left(\zeta_{u(n)-1}, \zeta_{u(n)}, \lambda/_{2b}\right) \diamond \Theta\left(\zeta_{u(n)-1}, \zeta_{v(n)}, \lambda/_{2b}\right)$

$\le \beta_{u(n)-1}\left(\lambda/_{2b}\right) \diamond \epsilon$ and

$\epsilon \le \Upsilon(\zeta_{u(n)}, \zeta_{v(n)}, \lambda) \le \Upsilon\left(\zeta_{u(n)-1}, \zeta_{u(n)}, \lambda/_{2b}\right) \diamond \Upsilon\left(\zeta_{u(n)-1}, \zeta_{v(n)}, \lambda/_{2b}\right)$

$\le \gamma_{u(n)-1}\left(\lambda/_{2b}\right) \diamond \epsilon$.

Since $\alpha_{u(n)-1}\left(\lambda/_{2b}\right) \to 1$, $\beta_{u(n)-1}\left(\lambda/_{2b}\right) \to 0$ and $\gamma_{u(n)-1}\left(\lambda/_{2b}\right) \to 0$ for every λ, it follows that

$1 - \epsilon \ge \Xi(\zeta_{u(n)}, \zeta_{v(n)}, \lambda) > 1 - \epsilon$, $\epsilon \le \Theta(\zeta_{u(n)}, \zeta_{v(n)}, \lambda) < \epsilon$ and $\epsilon \le \Upsilon(\zeta_{u(n)}, \zeta_{v(n)}, \lambda) < \epsilon$.

Clearly, this leads to a contradiction. Hence, ζ_n is a Cauchy sequence in Σ.

Since Σ is complete so there exists $\eta \in \Sigma$ such that $\lim_{n \to \infty} \zeta_n = \eta$.

Assume that $\eta \ne \mathcal{A}\eta$, then there exists $\lambda > 0$ such that $\Xi(\eta, \mathcal{A}\eta, \lambda) \ne 1$ or $\Theta(\eta, \mathcal{A}\eta, \lambda) \ne 0$ and $\Upsilon(\eta, \mathcal{A}\eta, \lambda) \ne 0$. For this $\lambda > 0$,

$\Xi(\mathcal{A}\zeta_n, \mathcal{A}\eta, k\lambda) \ge \Xi(\zeta_n, \mathcal{A}\eta, \lambda) * \Xi(\eta, \mathcal{A}\zeta_n, \lambda)$, by inequality (4.7.1).

That is $\Xi(\zeta_{n+1}, \mathcal{A}\eta, k\lambda) \ge \Xi(\zeta_n, \mathcal{A}\eta, \lambda) * \Xi(\eta, \mathcal{A}\zeta_n, \lambda)$.

In limiting case as $n \to \infty$, $\Xi(\eta, \mathcal{A}\eta, k\lambda) \ge \Xi(\eta, \mathcal{A}\eta, \lambda) * \Xi(\eta, \mathcal{A}\eta, \lambda) = \Xi(\eta, \mathcal{A}\eta, \lambda)$.

As $\Xi(\eta, \mathcal{A}\eta, \lambda) \ne 1$, the above inequality yields a contradiction to the fact that $\Xi(\zeta, \eta, .)$ is strictly increasing. Moreover, $\Theta(\mathcal{A}\zeta_n, \mathcal{A}\eta, k\lambda) \le \Theta(\zeta_n, \mathcal{A}\eta, \lambda) \diamond \Theta(\eta, \mathcal{A}\zeta_n, \lambda)$, by inequality (4.7..2).

That is $\Theta(\zeta_{n+1}, \mathcal{A}\eta, k\lambda) \le \Theta(\zeta_n, \mathcal{A}\eta, \lambda) \diamond \Theta(\eta, \mathcal{A}\zeta_n, \lambda)$.

In limiting case as $n \to \infty$, $\Theta(\eta, \mathcal{A}\eta, k\lambda) \le \Theta(\eta, \mathcal{A}\eta, \lambda) \diamond \Theta(\eta, \mathcal{A}\eta, \lambda) = \Theta(\eta, \mathcal{A}\eta, \lambda)$.

As $\Theta(\eta, \mathcal{A}\eta, \lambda) \ne 0$, the above inequality yields a contradiction to the fact that $\Theta(\zeta, \eta, .)$ is strictly decreasing. Moreover, $\Upsilon(\mathcal{A}\zeta_n, \mathcal{A}\eta, k\lambda) \le \Upsilon(\zeta_n, \mathcal{A}\eta, \lambda) \diamond \Upsilon(\eta, \mathcal{A}\zeta_n, \lambda)$, by inequality (4.7.3).

That is $\Upsilon(\zeta_{n+1}, \mathcal{A}\eta, k\lambda) \le \Upsilon(\zeta_n, \mathcal{A}\eta, \lambda) \diamond \Upsilon(\eta, \mathcal{A}\zeta_n, \lambda)$.

In limiting case as $n \to \infty$, $\Upsilon(\eta, \mathcal{A}\eta, k\lambda) \le \Upsilon(\eta, \mathcal{A}\eta, \lambda) \diamond \Upsilon(\eta, \mathcal{A}\eta, \lambda) = \Upsilon(\eta, \mathcal{A}\eta, \lambda)$.

As $\Upsilon(\eta, \mathcal{A}\eta, \lambda) \ne 0$, the above inequality yields a contradiction to the fact that $\Upsilon(\zeta, \eta, .)$ is strictly decreasing. Hence $\eta = \mathcal{A}\eta$.

For uniqueness, let η, θ be two fixed points of $\mathcal{A}$. So, $\eta = \mathcal{A}\eta$ and $\theta = \mathcal{A}\theta$.

Then $\Xi(\eta, \mathcal{A}\eta, \lambda) = 1, \Xi(\theta, \mathcal{A}\theta, k\lambda) = 1, \Theta(\eta, \mathcal{A}\eta, \lambda) = 0, \Theta(\theta, \mathcal{A}\theta, k\lambda) = 0$ and

$\Upsilon(\eta, \mathcal{A}\eta, \lambda) = 0, \Upsilon(\theta, \mathcal{A}\theta, k\lambda) = 0$.

Now, $1 \ge \Xi(\eta, \theta, \lambda) = \Xi(\mathcal{A}\eta, \mathcal{A}\theta, \lambda) \ge \Xi\left(\eta, \mathcal{A}\theta, \lambda/_k\right) * \Xi\left(\theta, \mathcal{A}\eta, \lambda/_k\right)$

$= \Xi\left(\eta, \theta, \lambda/_k\right) * \Xi\left(\theta, \eta, \lambda/_k\right) = \Xi\left(\eta, \theta, \lambda/_k\right) = \Xi\left(\mathcal{A}\eta, \mathcal{A}\theta, \lambda/_k\right)$

$\ge \Xi\left(\eta, \mathcal{A}\theta, \lambda/_{k^2}\right) * \Xi\left(\theta, \mathcal{A}\eta, \lambda/_{k^2}\right) \ge \cdots \ge \Xi\left(\eta, \theta, \lambda/_{k^n}\right) \to 1$ as $n \to \infty$

$0 \le \Theta(\eta, \theta, \lambda) = \Theta(\mathcal{A}\eta, \mathcal{A}\theta, \lambda) \le \Theta\left(\eta, \mathcal{A}\theta, \lambda/_k\right) \diamond \Theta\left(\theta, \mathcal{A}\eta, \lambda/_k\right)$

$= \Theta\left(\eta, \theta, \lambda/_k\right) \diamond \Theta\left(\theta, \eta, \lambda/_k\right) = \Theta\left(\eta, \theta, \lambda/_k\right) = \Theta\left(\mathcal{A}\eta, \mathcal{A}\theta, \lambda/_k\right)$

$\le \Theta\left(\eta, \mathcal{A}\theta, \lambda/_{k^2}\right) \diamond \Theta\left(\theta, \mathcal{A}\eta, \lambda/_{k^2}\right) \le \cdots \le \Theta\left(\eta, \theta, \lambda/_{k^n}\right) \to 0$ as $n \to \infty$ and

$0 \le \Upsilon(\eta, \theta, \lambda) = \Upsilon(\mathcal{A}\eta, \mathcal{A}\theta, \lambda) \le \Upsilon\left(\eta, \mathcal{A}\theta, \lambda/_k\right) \diamond \Upsilon\left(\theta, \mathcal{A}\eta, \lambda/_k\right)$

$= \Upsilon\left(\eta, \theta, {}^{\lambda}/_{k}\right) \diamond \Upsilon\left(\theta, \eta, {}^{\lambda}/_{k}\right) = \Upsilon\left(\eta, \theta, {}^{\lambda}/_{k}\right) = \Upsilon\left(\mathcal{A}\eta, \mathcal{A}\theta, {}^{\lambda}/_{k}\right)$
$\leq \Upsilon\left(\eta, \mathcal{A}\theta, {}^{\lambda}/_{k^2}\right) \diamond \Upsilon\left(\theta, \mathcal{A}\eta, {}^{\lambda}/_{k^2}\right) \leq \cdots \leq \Upsilon\left(\eta, \theta, {}^{\lambda}/_{k^n}\right) \to 0$ as $n \to \infty$.
Now from (d), (i) and (n) of definition (3.1), we have $\eta = \theta$.

Corollary: 4.8

Let (Σ, d) be a complete metric space and $\mathcal{A}: \Sigma \to \Sigma$ be a map which satisfies the following condition for all $\zeta, \eta \in \Sigma$ and $0 < k < 1$:

$$d(\mathcal{A}\zeta, \mathcal{A}\eta) \leq \frac{k}{2}[d(\zeta, \mathcal{A}\eta) + d(\eta, \mathcal{A}\zeta)] \quad (4.8.1)$$

Then $\mathcal{A}$ has unique fixed point in Σ.

Theorem: 4.9

Let $(\Sigma, \Xi, \Theta, \Upsilon, *, \diamond, b)$ be a CFbMS. Let $\mathcal{T}: \Sigma \to \Sigma$ be a mapping on Σ. If for $\zeta, \eta \in \Sigma$ and $\lambda > 0$, any one of the following condition is satisfied:

(i) $\Xi(\mathcal{T}\zeta, \mathcal{T}\eta, \mathrm{k}\lambda) \geq \Xi(\zeta, \eta, \lambda)$ for $0 < k < 1$ and $*$ is any continuous t-norm;

(ii) $\Xi(\mathcal{T}\zeta, \mathcal{T}\eta, \mathrm{k}\lambda) \geq \Xi(\zeta, \mathcal{T}\zeta, \lambda) * \Xi(\eta, \mathcal{T}\eta, \lambda)$, for $0 < k < 1$, $\lambda > 0$, $a * b = min\{a, b\}$, for all $a, b \in [0,1]$ and $\Xi(\zeta, \eta, .)$ is strictly increasing function;

(iii) $\Xi(\mathcal{T}\zeta, \mathcal{T}\eta, \mathrm{k}\lambda) \geq \Xi(\zeta, \mathcal{T}\eta, \lambda) * \Xi(\eta, \mathcal{T}\zeta, \lambda)$ for $0 < k < {}^{1}/_{2b}$, $\lambda > 0$, $a * b = min\{a, b\}$ for all $a, b \in [0,1]$ and $\Xi(\zeta, \eta, .)$ is strictly increasing function;

Then $\mathcal{T}$ has a unique fixed point in Σ.

Proof: (i), (ii) and (iii) are respectively special cases of Theorem (4.1.1), (4.1.2) and (4.2.1)

Conclusion:

In this paper, we have proved results on Complete Neutrosophic b-Metric Space.

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