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# Matrices over Non-Commutative Rings as Sums of Fifth Powers

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**ABSTRACT**
Let R be non-commutative ring with unity and $n \geq p \geq 2$, p prime. S. A. Katre, Deepa Krishnamurthi proved that an $n \times n$ matrix over R is the sum of $p^{th}$ powers if and only if its trace can be written as a sum of $p^{th}$ powers and commutators modulo pR. This extends the results of L. N.Vaserstein (p = 2) and S. A. Katre, Kshipra Wadikar (p = 3). Also S. A. Katre, Deepa Krishnamurthi obtained necessary and sufficient conditions for a matrix over R to be written as a sum of fourth powers when $n \geq 2$. In this paper, we obtain necessary and sufficient conditions for a matrix over R to be written as a sum of fifth powers when $n \geq 3$.
**Keywords :-** Matrices, non-commutative rings, trace, sums of powers, Waring's problem

## 1. INTRODUCTION

Carlitz showed as a solution to a problem proposed in Canadian Mathematical Bulletin that every $2 \times 2$ integer matrix is a sum of at most 3 squares (see [1]). Initial work related to integer matrices and matrices over commutative rings as sums of squares can be found in [3, 8]. Wadikar and Katre [10] proved that every integer matrix is a sum of four cubes. Richman [6] studied Waring's problem for matrices over commutative rings as sums of kth powers.Katre and Garge [4] gave generalized trace condition for a matrix over a commutative ringto be a sum of kth powers.All our rings are associative. By a non-commutative ring, we mean a ring with unity which is not necessarily commutative.

In this paper, R will be a non-commutative ring, and $M_n(R)$ will denote the ring of $n \times n$ matrices over R. For a non-commutative ring R,Vaserstein proved that a matrix of size $n \geq 2$ over R is a sum of squares if and only if itstrace is a sum of squares modulo 2R (see [9]). Recently, Katre and Wadikar proved that amatrix of size $n \geq 2$ over R is a sum of cubes if and only if its trace is a sum of cubes andcommutators modulo 3R (see [5]). In the context of Waring's problem formatrices, S. A. Katre, Deepa Krishnamurthi obtained a result for $p^{th}$ powers when $n \geq p \geq 2$, p prime and obtained an analogue of this result for fourth powers for $n \geq 2$(see [7]). For both these results, they used the following general trace condition for a matrix over a non-commutative ring to be a sum of kth powers ([5], Theorem 3.2).Theorem (Katre, Wadikar): Let n, $k \geq 2$ be integers and $A \in M_n(R)$. A is a sum of kth powers of matrices in $M_n(R)$ if and only if trace(A) is a sum of traces of kth powers ofmatrices in $M_n(R)$.

In this paper, we obtain result for fifth powers for $n \geq 3$. For this result, we use the above general trace condition for a matrix over a non-commutative ring to be a sum of kth powers ([5], Theorem 3.2).

## 2. NOTATIONS

$E_{ij}$: The $n \times n$ matrix whose $(i, j)^{th}$ entry is 1 and other entries are 0.

$E'_{ij}$: The $p \times p$ matrix whose $(i, j)^{th}$ entry is 1 and other entries are 0.

$C(a_1, a_2, \ldots, a_k) = a_1 a_2 \cdots a_k + a_2 \cdots a_k a_1 + \cdots + a_k a_1 a_2 \cdots a_{k-1}$ where $a_1, a_2, \ldots, a_k \in R$, is called a cyclic sum of length k and $[x, y] = xy - yx$ is called the commutator of x and y.

Note that $-C(a_1, a_2, \ldots, a_k) = C(-a_1, a_2, \ldots, a_k)$ is a cyclic sum and $-[x, y] = [-x, y]$ commutator.

## 3. MAIN RESULT

In the case of $p^{th}$ powers, we required to show in our proof that a cyclic sum $C(a_1, a_2, \ldots, a_p)$ is in $T_p$. For this, we showed that $C(a_1, a_2, \ldots, a_p) = trace(F^p)$, where F is a $p \times p$ matrix. Because of this our proof required $n \geq p$. We shall see in the next section that for fifth powers we can make use of the five entries in a $3 \times 3$ matrix to show that $C(a, b, c, d, e) \in T_5$. This will give us a criterion for $A \in M_n(R)$ to be a sum of fifth powers for $n \geq 3$.

The following theorem gives a non-commutative version of Theorem 2.3, 2.6 in [2].

**Theorem:** Let $n \geq 3$ be an integer and let $T_5 = T_{\{5,n\}}$ be set of those elements of R that can be expressed as sums of traces of fifth powers of $n \times n$ matrices over R. For a, b, c, d, $e \in R$, Let C(a, b, c, d, e)= abcde + bcdea + cdeab + deabc + eabcd. Then

(i) For a, b, c, d, $e \in R$, $C(a, b, c, d, e) \in T_5$. Also $5a, a^5 \in T_5$.

(ii) $T_5 = \{\sum_{j=1}^{q} C(a_j, b_j, c_j, d_j, e_j) + \sum_{j=1}^{l} g_j^5 / a_j$, $b_j, c_j, d_j, e_j, g_j \in R$, $q, l \geq 1\}$.

(iii) $T_5 = \{\sum_{j=1}^{q} (a_j b_j - b_j a_j) + \sum_{j=1}^{l} c_j^5 + 5r$ / $a_j$, $b_j, c_j, r \in R$, $q, l \geq 1\}$.

(iv) A matrix $A \in M_n(R)$ is a sum of fifth powers if and only if trace(A) is a sum of fifth powers and commutators modulo 5R.

(v) A matrix A in $M_n(R)$ is a sum of fifth powers if and only if trace(A) = $x_0^5 + 5x_1^5 +$ a sum of commutators where$x_0, x_1 \in R$.

**Proof:**

(i) For the $3 \times 3$ matrix $E'_{ij}$, and the zero matrix$O_{n-3}$of order n-3, let a, b, c, d, $e \in R$,

$$N_1 = \begin{bmatrix} a & 0 & b \\ e & d & 0 \\ 0 & c & 0 \end{bmatrix}, N_2 = \begin{bmatrix} a & 0 & b \\ -e & 0 & 0 \\ 0 & c & 0 \end{bmatrix}, N_3 = \begin{bmatrix} 0 & 0 & b \\ -e & d & 0 \\ 0 & c & 0 \end{bmatrix}, N_4 = \begin{bmatrix} -a & 0 & 0 \\ 0 & -a & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$N_5 = \begin{bmatrix} -d & 0 & 0 \\ 0 & -d & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We have, trace $\sum_{i=1}^{5} N_i^5$= [C(a, b, c, d, e) + $a^5$ + $d^5$ + (bceaa + aabce + eaabc + ceaab + abcea) + (bcdde + ebcdd + ddebc + cddeb + debcd)] + [$a^5$ - bceaa - aabce - eaabc - ceaab - abcea] + [$d^5$ -

bcdde - ebcdd - ddebc - cddeb - debcd] - $2a^5$ - $2d^5$ = C(a, b, c, d, e).

Hence, C(a, b, c, d, e) ∈ $T_5$. Also C(a, 1, 1, 1, 1) = 5a, hence 5a ∈ $T_5$. Also$a^5$= trace($aE^5_{11}$)∈ $T_5$.

(ii) From (i), C($a_j$, $b_j$, $c_j$, $d_j$, $e_j$) ∈ $T_5$, also $g^5_j$ ∈ $T_5$. Thus, every element of R.H.S. of (ii) ∈ $T_5$.

Conversely,for A ∈ $M_n(R)$, trace of $A^5$ is sum of fifth powers of diagonal entries and entries

of the type C(a, b, c, d, e), so $T_5$ ⊆ R.H.S of (ii).

(iii) By (i), every term in the elements of R.H.S. of (iii) ∈ $T_5$, so R.H.S. of (iii) ⊆ $T_5$and

conversely by (ii), $T_5$ ⊆R.H.S. of (iii).

(iv) A matrix A ∈$M_n(R)$ is a sum of fifth powers if and only if trace of A is a sum oftraces of fifth powers of matrices in $M_n(R)$ if and only if, by (iii), trace(A) is a sum of fith powers andcommutators modulo 5R.

(v) By (iv), A in $M_n(R)$ is sum of fifth powers if and only if trace(A) is a sumof fifth powers

and sum of commutators modulo 5R.

Now Consider, $a^5 + b^5 = (a+ b)^5 - (a^4b^1a^0 + a^3b^1a^1 + a^2b^1a^2 + a^1b^1a^3 + a^0b^1a^4) - (b^4a^1b^0 + b^3a^1b^1 + b^2a^1b^2 + b^1a^1b^3 + b^0a^1b^4) - (b^3a^2b^0 + b^2a^2b^1 + b^1a^2b^2 + b^0a^2b^3 + b^4a^2b^4) - (a^3b^2a^0 + a^2b^2a^1 + a^1b^2a^2 + a^0b^2a^3 + a^4b^2a^4)$

$= (a + b)^5 - (a^4b^1a^0 + a^0b^1a^4 + a^3b^1a^1 + a^1b^1a^3 + a^2b^1a^2) - (b^4a^1b^0 + b^0a^1b^4 + b^3a^1b^1 + b^1a^1b^3 + b^2a^1b^2) - (b^3a^2b^0 + b^0a^2b^3 + b^2a^2b^1 + b^1a^2b^2 + b^4a^2b^4) - (a^3b^2a^0 + a^0b^2a^3 + a^2b^2a^1 + a^1b^2a^2 + a^4b^2a^4)$

$= (a + b)^5$- cyclic sums

Since every cyclic sum is sum of commutators modulo 5R, we get $a^5 + b^5 = (a + b)^5$+ sum of commutators modulo 5R.

Note: $T_5$ is independent of n for n ≥3.

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