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# PARTIAL SUMS FOR SUBCLASS OF HARMONIC UNIVALENT FUNCTIONS

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**Abstract:** In this paper we determine the lower bounds of harmonic univalent functions to its partial sums for the class $L_H(\beta)$.

**Keywords and Phrases:** Univalent Function, Harmonic Functions.

**2010 Mathematics Subject Classification:** Primary 30C45

## 1 Introduction

A continuous complex valued function defined in a simply connected domain D is said to be harmonic in D if both u and v are real harmonic in D, the harmonic function has a unique representation $f = h + \overline{g}$ ,where $h$ and $g$ are analytic and co analytic part of $f$ respectively. The Jacobian of $f = h + \overline{g}$ is given by $J_f(z) = |h^{'}(z)|^2 - |g^{'}(z)|^2$. The mapping $z \rightarrow f(z)$ is orientation preserving and locally $1 - 1$ if $D$ if and only if $J_f(z) > 0$ in $D$ see [1].

Let $\mathcal{H}$ denote the family of normalized functions $f = h + \overline{g}$ that are harmonic , orientation preserving and univalent in the open unit disk $\Delta = \{z : |z| < 1\}$ [2, 3] where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad , g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{1}$$

here $z^{'} = \frac{\partial}{\partial\theta}(z = re^{i\theta}), f^{'}(z) = \frac{\partial}{\partial\theta}(f(z)) = \frac{\partial}{\partial\theta}(f(re^{i\theta}), 0 \leq r < 1, \theta \in \mathbb{R}$.

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see [8] For $0 \leq \beta < 1$, we consider the subclass $L_H(\beta)$ of harmonic univalent functions $f = h + \bar{g}$ satisfying the condition

$$Re\left\{(1+e^{i\alpha})\left(\frac{z^2 f''(z) - zf'(z)}{f(z)}\right) - e^{i\alpha}\right\} \geq \beta$$

Further let $L_{\bar{H}}(\beta)$ denote the subclass of $L_H(\beta)$ consisting of functions such that

$$h(z) = z - \sum_{n=2}^{\infty}|a_n|z^n \quad , g(z) = \sum_{n=1}^{\infty}|b_n|z^n \tag{2}$$

Initially we recall certain standard results of sub-classes of $\mathcal{S}$ and sufficient conditions for a function $f \in \mathcal{S}$ to be in these subclasses. The sufficient condition for the function $f = h + \bar{g}$ given by (2) to be in $L_{\overline{H}}(\beta)$ is that

$$\sum_{k=1}^{\infty}\frac{2k^2-1-\beta}{1-\beta}|a_k| + \sum_{k=1}^{\infty}\frac{2k^2+1+\beta}{1-\beta}|b_k| \leq 2. \tag{3}$$

Several authors studied the partial sums of analytic univalent functions [5, 6], the analogous results on partial sums on harmonic univalent functions were studied by Saurabh Porwal [4], Saurabh Porwal and Kaushal Kishore Dixit [7] In this paper we extend the results to the class $L_H(\beta)$. Now, we let the sequence of partial sums of the functions of the form (1) with $b_1 = 0$ are

$$f_m(z) = z + \sum_{k=2}^{m} a_k z^k + \sum_{k=2}^{\infty}\overline{b_k z^k}.$$
$$f_n(z) = z + \sum_{k=2}^{\infty} a_k z^k + \sum_{k=2}^{n}\overline{b_k z^k}.$$
$$f_{m,n}(z) = z + \sum_{k=2}^{m} a_k z^k + \sum_{k=2}^{n}\overline{b_k z^k}.$$

In this paper, we determine the lower bounds for $Re\left\{\frac{f(z)}{f_m(z)}\right\}$, $Re\left\{\frac{f_m(z)}{f(z)}\right\}$, $Re\left\{\frac{f'(z)}{f'_m(z)}\right\}$, $Re\left\{\frac{f'_m(z)}{f'(z)}\right\}$, $Re\left\{\frac{f(z)}{f_n(z)}\right\}$, $Re\left\{\frac{f_n(z)}{f(z)}\right\}$, $Re\left\{\frac{f(z)}{f_{m,n}(z)}\right\}$, $Re\left\{\frac{f_{m,n}(z)}{f(z)}\right\}$, $Re\left\{\frac{f'(z)}{f'_{m,n}(z)}\right\}$, $Re\left\{\frac{f'_{m,n}(z)}{f'(z)}\right\}$. Where $f'(z) = \frac{\partial}{\partial\theta}f(re^{i\theta})$.

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## 2 Main Results

In this section we obtain the lower bound for $Re\left\{\frac{f(z)}{f_m(z)}\right\}$.

Theorem 2.1: If $f$ of the form (1) with $b_1 = 0$, satisfying condition (2), then

$$Re\left\{\frac{f(z)}{f_m(z)}\right\} \geq \frac{2(m+1)^2-2}{2(m+1)^2-1-\beta} \quad (z \in \Delta). \tag{4}$$

The result (4) is sharp with the function given by

$$f(z) = z + \frac{1-\beta}{2(m+1)^2-1-\beta} z^{m+1}. \tag{5}$$

Proof 2.1: We may write

$$\frac{1+w(z)}{1-w(z)} = \frac{2(m+1)^2-1-\beta}{1-\beta}\left[\frac{f(re^{i\theta})}{f_m(re^{i\theta})} - \frac{2(m+1)^2-2}{2(m+1)^2-1-\beta}\right]$$

$$= \frac{1+\sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{\infty} \overline{b_k} r^{k-1} e^{i(k-1)\theta} + \frac{2(m+1)^2-1-\beta}{1-\beta}\left[\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta}\right]}{1+\sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{\infty} \overline{b_k} r^{k-1} e^{i(k-1)\theta}}$$

so that
$w(z)$

$$= \frac{\frac{2(m+1)^2-1-\beta}{1-\beta}\left[\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta}\right]}{2+2\left(\sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{\infty} \overline{b_k} r^{k-1} e^{i(k-1)\theta}\right) + \frac{2(m+1)^2-1-\beta}{1-\beta}\left(\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta}\right)}$$

Then
$|w(z)|$

$$= \frac{\frac{2(m+1)^2-1-\beta}{1-\beta}\left[\sum_{k=m+1}^{\infty}|a_k|\right]}{2-2\left(\sum_{k=2}^{m}|a_k| + \sum_{k=2}^{\infty}|b_k|\right) - \frac{2(m+1)^2-1-\beta}{1-\beta}\left(\sum_{k=m+1}^{\infty}|a_k|\right)}.$$

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The last expression is bounded by 1, if and only if

$$\sum_{k=2}^{m}|a_k|+\sum_{k=2}^{\infty}|b_k|+\frac{2(m+1)^2-1-\beta}{1-\beta}\left(\sum_{k=m+1}^{\infty}|a_k|\right)\leq 1. \tag{6}$$

It suffices to show that LHS of (6) is bounded above by $\sum_{k=2}^{\infty}\frac{2k^2-1-\beta}{1-\beta}|a_k|+\sum_{k=2}^{\infty}\frac{2k^2+1+\beta}{1-\beta}|b_k|$, which is equivalent to

$$\sum_{k=2}^{m}\frac{k^2-1}{1-\beta}|a_k|+\sum_{k=2}^{\infty}\frac{k^2+\beta}{1-\beta}|b_k+\sum_{k=m+1}^{\infty}\frac{k^2-(m+1)^2}{1-\beta}|a_k|\geq 0.$$

In this section we obtain the lower bound for $Re\left\{\frac{f_m(z)}{f(z)}\right\}$.

Theorem 2.2: If $f$ of the form (1.1) with $b_1=0$, satisfying condition (1.3), then

$$Re\left\{\frac{f_m(z)}{f(z)}\right\}\geq\frac{2(m+1)^2-1-\beta}{2(m+1)^2-2\beta},\quad (z\in\Delta). \tag{7}$$

The result (7) is sharp with the function (5).

Proof 2.2: We may write

$$\frac{1+w(z)}{1-w(z)}=\frac{2(m+1)^2-2\beta}{1-\beta}\left[\frac{f_m(re^{i\theta})}{f(re^{i\theta})}-\frac{2(m+1)^2-1-\beta}{2(m+1)^2-2\beta}\right]$$
$$=\frac{1+\sum_{k=2}^{m}a_kr^{k-1}e^{i(k-1)\theta}+\sum_{k=2}^{\infty}\overline{b_k}r^{k-1}e^{i(k-1)\theta}+\frac{2(m+1)^2-2\beta}{1-\beta}\left[\sum_{k=m+1}^{\infty}a_kr^{k-1}e^{i(k-1)\theta}\right]}{1+\sum_{k=2}^{m}a_kr^{k-1}e^{i(k-1)\theta}+\sum_{k=2}^{\infty}\overline{b_k}r^{k-1}e^{i(k-1)\theta}}$$

so that
$w(z)$

$$=\frac{\frac{2(m+1)^2-2\beta}{1-\beta}\left[\sum_{k=m+1}^{\infty}a_kr^{k-1}e^{i(k-1)\theta}\right]}{2+2\left(\sum_{k=2}^{m}a_kr^{k-1}e^{i(k-1)\theta}+\sum_{k=2}^{\infty}\overline{b_k}r^{k-1}e^{i(k-1)\theta}\right)+\frac{2(m+1)^2-2\beta}{1-\beta}\left(\sum_{k=m+1}^{\infty}a_kr^{k-1}e^{i(k-1)\theta}\right)}$$

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Then
$|w(z)|$

$$= \frac{\frac{2(m+1)^2-2\beta}{1-\beta}\left[\sum_{k=m+1}^{\infty}|a_k|\right]}{2-2\left(\sum_{k=2}^{m}|a_k|+\sum_{k=2}^{\infty}|b_k|\right)-\frac{2(m+1)^2-2\beta}{1-\beta}\left(\sum_{k=m+1}^{\infty}|a_k|\right)}.$$

The last expression is bounded by 1, if and only if

$$\sum_{k=2}^{m}|a_k|+\sum_{k=2}^{\infty}|b_k|+\frac{2(m+1)^2-2\beta}{1-\beta}\left(\sum_{k=m+1}^{\infty}|a_k|\right)\leq 1. \tag{8}$$

It suffices to show that LHS of (8) is bounded above by
$\sum_{k=2}^{\infty}\frac{2k^2-1-\beta}{1-\beta}|a_k|+\sum_{k=2}^{\infty}\frac{2k^2+1+\beta}{1-\beta}|b_k|$, which is equivalent to

$$\sum_{k=2}^{m}\frac{k^2-1}{1-\beta}|a_k|+\sum_{k=2}^{\infty}\frac{k^2+\beta}{1-\beta}|b_k+\sum_{k=m+1}^{\infty}\frac{k^2-(m+1)^2}{1-\beta}|a_k|\geq 0.$$

Theorem 2.3: If $f$ of the form (1) with $b_1=0$, satisfying condition (2), then

$$Re\left\{\frac{f'(z)}{f_m'(z)}\right\}\geq\frac{2(m+1)^2-m(1-\beta)-2}{2(m+1)^2-1-\beta},\quad (z\in\Delta). \tag{9}$$

The result (9) is sharp with the function (5).

Proof 2.3: We may write

$$\frac{1+w(z)}{1-w(z)}=\frac{2(m+1)^2-1-\beta}{(m+1)(1-\beta)}\left[\frac{f'(re^{i\theta})}{f_m'(re^{i\theta})}-\frac{2(m+1)^2-m(1-\beta)-2}{2(m+1)^2-1-\beta}\right]$$

$$=\frac{1+\sum_{k=2}^{m}a_kr^{k-1}e^{i(k-1)\theta}-\sum_{k=2}^{\infty}\overline{b_k}r^{k-1}e^{i(k-1)\theta}+\frac{2(m+1)^2-1-\beta}{(m+1)(1-\beta)}\left[\sum_{k=m+1}^{\infty}a_kr^{k-1}e^{i(k-1)\theta}\right]}{1+\sum_{k=2}^{m}a_kr^{k-1}e^{i(k-1)\theta}-\sum_{k=2}^{\infty}\overline{b_k}r^{k-1}e^{i(k-1)\theta}}$$

The result follows by the same technique as used in the previous theorem.

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Theorem 2.4: If $f$ of the form (1) with $b_1 = 0$, satisfying condition (2), then

$$Re\left\{\frac{f_m'(z)}{f'(z)}\right\} \geq \frac{2(m+1)^2 - 1 - \beta}{2(m+1)^2 + m(1-\beta) - 2\beta}, \quad (z \in \Delta). \tag{10}$$

The result (10) is sharp with the function (5).

Proof 2.4: We may write

$$\frac{1+w(z)}{1-w(z)} = \frac{2(m+1)^2 - 2\beta}{1-\beta}\left[\frac{f_m(re^{i\theta})}{f(re^{i\theta})} - \frac{2(m+1)^2 - 1 - \beta}{2(m+1)^2 - 2\beta}\right]$$

$$= \frac{1 + \sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{\infty} \overline{b_k} r^{k-1} e^{i(k-1)\theta} + \frac{2(m+1)^2 - 2\beta}{1-\beta}\left[\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta}\right]}{1 + \sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{\infty} \overline{b_k} r^{k-1} e^{i(k-1)\theta}}$$

Proceeding exactly as in the proof of Theorem 2.3, we get the required result.

Theorem 2.5: If $f$ of the form (1) with $b_1 = 0$, satisfying condition (2), then

$$Re\left\{\frac{f(z)}{f_n(z)}\right\} \geq \frac{2(n+1)^2 + 2\beta}{2(n+1)^2 + 1 + \beta}, \quad (z \in \Delta). \tag{11}$$

The result(11) is sharp with the function

$$f(z) = z + \frac{1-\beta}{2(n+1)^2 + 1 + \beta}\overline{z}^{n+1}. \tag{12}$$

Proof 2.5: We may write

$$\frac{1+w(z)}{1-w(z)} = \frac{2(n+1)^2 + 1 + \beta}{1-\beta}\left[\frac{f(re^{i\theta})}{f_n(re^{i\theta})} - \frac{2(n+1)^2 + 2\beta}{2(n+1)^2 + 1 + \beta}\right]$$

$$= \frac{1 + \sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{\infty} \overline{b_k} r^{k-1} e^{i(k-1)\theta} + \frac{2(n+1)^2 + 1 + \beta}{1-\beta}\left[\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta}\right]}{1 + \sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{\infty} \overline{b_k} r^{k-1} e^{i(k-1)\theta}}$$

The details of the proof are similar to the proof of the Theorem 2.1 and is omitted here.

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Theorem 2.6: If $f$ of the form (1) with $b_1 = 0$, satisfying condition (2), then

$$Re\left\{\frac{f_n(z)}{f(z)}\right\} \geq \frac{2(n+1)^2 + 2\beta}{2(n+1)^2 + 1 + \beta}, \quad (z \in \Delta). \tag{13}$$

The result (13) is sharp with the function (12).

Proof 2.6: We may write

$$\frac{1+w(z)}{1-w(z)} = \frac{2(n+1)^2+2}{1-\beta}\left[\frac{f(re^{i\theta})}{f_n(re^{i\theta})} - \frac{2(n+1)^2+1+\beta}{2(n+1)^2+2}\right]$$

$$= \frac{1 + \sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{\infty} \overline{b_k} r^{k-1} e^{i(k-1)\theta} + \frac{2(n+1)^2+1+\beta}{2(n+1)^2+2}\left[\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta}\right]}{1 + \sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{\infty} \overline{b_k} r^{k-1} e^{i(k-1)\theta}}$$

The details of the proof are similar to the proof of the Theorem 2.5 and is omitted here.

We next determine bounds for $Re\left\{\frac{f(z)}{f_{m,n}(z)}\right\}$.

Theorem 2.7: If $f$ of the form (1) with $b_1 = 0$, satisfying condition (2), then

$$(i) \quad Re\left\{\frac{f(z)}{f_{m,n}(z)}\right\} \geq \frac{2(m+1)^2 - 2}{2(m+1)^2 - 1 - \beta}, \quad (z \in \Delta). \tag{14}$$

$$(ii) \quad Re\left\{\frac{f(z)}{f_{m,n}(z)}\right\} \geq \frac{2(n+1)^2 + 2\beta}{2(n+1)^2 + 1 + \beta}, \quad (z \in \Delta). \tag{15}$$

The results (14) and (15) are sharp with the functions given by (4) and (12) respectively.

Proof 2.7: For prove part (i), We write

$$\frac{1+w(z)}{1-w(z)} = \frac{2(m+1)^2-1-\beta}{1-\beta}\left[\frac{f(re^{i\theta})}{f_{m,n}(re^{i\theta})} - \frac{2(m+1)^2-2}{2(m+1)^2-1-\beta}\right]$$

$$1 + \sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{n} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}$$

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$$= \frac{+\frac{2(m+1)^2-2}{2(m+1)^2-1-\beta}\left[\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=n+1}^{\infty} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}\right]}{1+\sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{\infty} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}}$$

so that

$$w(z) = \frac{\frac{2(m+1)^2-2}{2(m+1)^2-1-\beta}\left[\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=n+1}^{\infty} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}\right]}{2+2\left(\sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{n} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}\right)}$$

$$+\frac{2(m+1)^2-2}{2(m+1)^2-1-\beta}\left(\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=n+1}^{\infty} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}\right)$$

Then

$$|w(z)| \leq \frac{\frac{2(m+1)^2-2}{2(m+1)^2-1-\beta}\left[\sum_{k=m+1}^{\infty}|a_k| + \sum_{k=n+1}^{\infty}|b_k|\right]}{2-2\left(\sum_{k=2}^{m}|a_k| + \sum_{k=2}^{n}|b_k|\right) - \frac{2(m+1)^2-2}{2(m+1)^2-1-\beta}\left[\sum_{k=m+1}^{\infty}|a_k| + \sum_{k=n+1}^{\infty}|b_k|\right]}$$

The last expression is bounded by 1 if and only if

$$\sum_{k=2}^{m}|a_k| + \sum_{k=2}^{n}|b_k| - \frac{2(m+1)^2-2}{2(m+1)^2-1-\beta}\left(\sum_{k=m+1}^{\infty}|a_k| + \sum_{k=n+1}^{\infty}|b_k|\right) \leq 1. \tag{16}$$

It suffices to show that LHS of (16) is bounded by $\sum_{k=2}^{\infty} \frac{2k^2-1-\beta}{1-\beta}|a_k| + \sum_{k=2}^{\infty} \frac{2k^2+1+\beta}{1-\beta}|b_k|$ , which is equivalent

$$\sum_{k=2}^{m} \frac{k^2-1}{1-\beta}|a_k| + \sum_{k=2}^{\infty} \frac{k^2+\beta}{1-\beta}|b_k + \sum_{k=m+1}^{\infty} \frac{k^2-(m+1)^2}{1-\beta}|a_k| + \sum_{k=n+1}^{\infty} \frac{k^2-(m+1)^2+1+\beta}{1-\beta}|b_k| \geq 0.$$

To prove part (ii), we write

$$\frac{1+w(z)}{1-w(z)} = \frac{2(n+1)^2+1+\beta}{1-\beta}\left[\frac{f(re^{i\theta})}{f_{m,n}(re^{i\theta})} - \frac{2(n+1)^2+2\beta}{2(n+1)^2+1+\beta}\right]$$

$$= \frac{\begin{array}{c} 1+\sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta}+\sum_{k=2}^{n} \overline{b_k} r^{k-1} e^{-i(k+1)\theta} \\ +\frac{2(n+1)^2+1+\beta}{1-\beta}\left[\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta}+\sum_{k=n+1}^{\infty} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}\right]\end{array}}{1+\sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta}+\sum_{k=2}^{\infty} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}}$$

so that

$$w(z)=\frac{\frac{2(n+1)^2+1+\beta}{1-\beta}\left[\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta}+\sum_{k=n+1}^{\infty} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}\right]}{\begin{array}{c} 2+2\left(\sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta}+\sum_{k=2}^{n} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}\right) \\ +\frac{2(n+1)^2+1+\beta}{1-\beta}\left(\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta}+\sum_{k=n+1}^{\infty} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}\right)\end{array}}$$

Then

$$|w(z)| \leq \frac{\frac{2(n+1)^2+1+\beta}{1-\beta}\left[\sum_{k=m+1}^{\infty}|a_k|+\sum_{k=n+1}^{\infty}|b_k|\right]}{2-2\left(\sum_{k=2}^{m}|a_k|+\sum_{k=2}^{n}|b_k|\right)-\frac{2(n+1)^2+1+\beta}{1-\beta}\left[\sum_{k=m+1}^{\infty}|a_k|+\sum_{k=n+1}^{\infty}|b_k|\right]}$$

The last expression is bounded by 1 if and only if

$$\sum_{k=2}^{m}|a_k|+\sum_{k=2}^{n}|b_k|-\frac{2(n+1)^2+1+\beta}{1-\beta}\left(\sum_{k=m+1}^{\infty}|a_k|+\sum_{k=n+1}^{\infty}|b_k|\right) \leq 1. \tag{17}$$

It suffices to show that LHS of (17) is bounded by $\sum_{k=2}^{\infty} \frac{2k^2-1-\beta}{1-\beta}|a_k|+\sum_{k=2}^{\infty} \frac{2k^2+1+\beta}{1-\beta}|b_k|$ , which is equivalent

$$\sum_{k=2}^{m} \frac{k^2-1}{1-\beta}|a_k|+\sum_{k=2}^{\infty} \frac{k^2+\beta}{1-\beta}|b_k+\sum_{k=m+1}^{\infty} \frac{k^2-(m+1)^2}{1-\beta}|a_k|+\sum_{k=n+1}^{\infty} \frac{k^2-(n+1)^2}{1-\beta}|b_k| \geq 0.$$

and the proof is complete.

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We next determine bounds for $Re\left\{\frac{f_{m,n}(z)}{f(z)}\right\}$.

Theorem 2.8: If $f$ of the form (1) with $b_1 = 0$, satisfying condition (2), then

$$(i) \quad Re\left\{\frac{f_{m,n}(z)}{f(z)}\right\} \geq \frac{2(m+1)^2-1-\beta}{2(m+1)^2-2\beta}, \quad (z \in \Delta). \tag{18}$$

$$(ii) \quad Re\left\{\frac{f_{m,n}(z)}{f(z)}\right\} \geq \frac{2(n+1)^2+2\beta}{2(n+1)^2+1+\beta}, \quad (z \in \Delta). \tag{19}$$

The results (18) and (19) are sharp with the functions given by (4) and (12) respectively.

Proof 2.8: For prove part (i), We write

$$\frac{1+w(z)}{1-w(z)} = \frac{2(m+1)^2-2\beta}{1-\beta}\left[\frac{f_{m,n}(re^{i\theta})}{f(re^{i\theta})} - \frac{2(m+1)^2-1-\beta}{2(m+1)^2-2\beta}\right]$$

$$1+\sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{n} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}$$

$$= \frac{+\frac{2(m+1)^2-1-\beta}{2(m+1)^2-2\beta}\left[\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=n+1}^{\infty} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}\right]}{1+\sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{\infty} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}}$$

so that

$$w(z) = \frac{\frac{2(m+1)^2-1-\beta}{2(m+1)^2-2\beta}\left[\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=n+1}^{\infty} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}\right]}{2+2\left(\sum_{k=2}^{m} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=2}^{n} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}\right)}$$

$$+\frac{2(m+1)^2-1-\beta}{2(m+1)^2-2\beta}\left(\sum_{k=m+1}^{\infty} a_k r^{k-1} e^{i(k-1)\theta} + \sum_{k=n+1}^{\infty} \overline{b_k} r^{k-1} e^{-i(k+1)\theta}\right)$$

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Then

$$|w(z)|\leq \frac{\frac{2(m+1)^2-1-\beta}{2(m+1)^2-2\beta}\left[\sum_{k=m+1}^{\infty}|a_k|+\sum_{k=n+1}^{\infty}|b_k|\right]}{2-2\left(\sum_{k=2}^{m}|a_k|+\sum_{k=2}^{n}|b_k|\right)-\frac{2(m+1)^2-1-\beta}{2(m+1)^2-2\beta}\left[\sum_{k=m+1}^{\infty}|a_k|+\sum_{k=n+1}^{\infty}|b_k|\right]}$$

The last expression is bounded by 1 if and only if

$$\sum_{k=2}^{m}|a_k|+\sum_{k=2}^{n}|b_k|-\frac{2(m+1)^2-2}{2(m+1)^2-1-\beta}\left(\sum_{k=m+1}^{\infty}|a_k|+\sum_{k=n+1}^{\infty}|b_k|\right)\leq 1. \qquad (20)$$

It suffices to show that LHS of (20) is bounded by
$\sum_{k=2}^{\infty}\frac{2k^2-1-\beta}{1-\beta}|a_k|+\sum_{k=2}^{\infty}\frac{2k^2+1+\beta}{1-\beta}|b_k|$ , which is equivalent

$$\sum_{k=2}^{m}\frac{k^2-1}{1-\beta}|a_k|+\sum_{k=2}^{\infty}\frac{k^2+\beta}{1-\beta}|b_k+\sum_{k=m+1}^{\infty}\frac{k^2-(m+1)^2}{1-\beta}|a_k|+\sum_{k=n+1}^{\infty}\frac{k^2-(m+1)^2+1+\beta}{1-\beta}|b_k|\geq 0.$$

To prove part (ii), we write

$$\frac{1+w(z)}{1-w(z)}=\frac{2(n+1)^2+2}{1-\beta}\left[\frac{f(re^{i\theta})}{f_{m,n}(re^{i\theta})}-\frac{2(n+1)^2+1+\beta}{2(n+1)^2+2}\right]$$

$$=\frac{1+\sum_{k=2}^{m}a_kr^{k-1}e^{i(k-1)\theta}+\sum_{k=2}^{n}\overline{b_k}r^{k-1}e^{-i(k+1)\theta}+\frac{2(n+1)^2+2}{1-\beta}\left[\sum_{k=m+1}^{\infty}a_kr^{k-1}e^{i(k-1)\theta}+\sum_{k=n+1}^{\infty}\overline{b_k}r^{k-1}e^{-i(k+1)\theta}\right]}{1+\sum_{k=2}^{m}a_kr^{k-1}e^{i(k-1)\theta}+\sum_{k=2}^{\infty}\overline{b_k}r^{k-1}e^{-i(k+1)\theta}}$$

so that

$$|w(z)|\leq \frac{\frac{2(n+1)^2+2}{1-\beta}\left[\sum_{k=m+1}^{\infty}|a_k|+\sum_{k=n+1}^{\infty}|b_k|\right]}{2-2\left(\sum_{k=2}^{m}|a_k|+\sum_{k=2}^{n}|b_k|\right)-\frac{2(n+1)^2+2\beta}{1-\beta}\left[\sum_{k=m+1}^{\infty}|a_k|+\sum_{k=n+1}^{\infty}|b_k|\right]}$$

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The last expression is bounded by 1 if and only if

$$\sum_{k=2}^{m}|a_k|+\sum_{k=2}^{n}|b_k|+\frac{2(n+1)^2+1+\beta}{1-\beta}\Bigg(\sum_{k=m+1}^{\infty}|a_k|+\sum_{k=n+1}^{\infty}|b_k|\Bigg)\leq 1. \qquad (21)$$

It suffices to show that LHS of (21) is bounded by
$\sum_{k=2}^{\infty}\frac{2k^2-1-\beta}{1-\beta}|a_k|+\sum_{k=2}^{\infty}\frac{2k^2+1+\beta}{1-\beta}|b_k|$ , which is equivalent

$$\sum_{k=2}^{m}\frac{k^2-1}{1-\beta}|a_k|+\sum_{k=2}^{\infty}\frac{k^2+\beta}{1-\beta}|b_k+\sum_{k=m+1}^{\infty}\frac{k^2-(m+1)^2}{1-\beta}|a_k|+\sum_{k=n+1}^{\infty}\frac{k^2-(n+1)^2}{1-\beta}|b_k|\geq 0.$$

and the proof is complete.

Theorem 2.9: If $f$ of the form (1) with $b_1=0$, satisfying condition (2), then

$$Re\left\{\frac{f'(z)}{f'_{m,n}(z)}\right\}\geq\frac{2(m+1)^2-2}{2(m+1)^2-1-\beta},\quad (z\in\Delta). \qquad (22)$$

The result (22) is sharp with the function given by (4).

Proof 2.9: We write

$$\frac{1+w(z)}{1-w(z)}=\frac{2(m+1)^2-1-\beta}{(m+1)(1-\beta)}\left[\frac{f'(re^{i\theta})}{f'_{m,n}(re^{i\theta})}-\frac{2(m+1)^2-2}{2(m+1)^2-1-\beta}\right]$$

$$=\frac{1+\sum_{k=2}^{m}ka_kr^{k-1}e^{i(k-1)\theta}+\sum_{k=2}^{n}k\overline{b_k}r^{k-1}e^{-i(k+1)\theta}+\frac{2(m+1)^2-1-\beta}{(m+1)(1-\beta)}\left[\sum_{k=m+1}^{\infty}ka_kr^{k-1}e^{i(k-1)\theta}+\sum_{k=n+1}^{\infty}k\overline{b_k}r^{k-1}e^{-i(k+1)\theta}\right]}{1+\sum_{k=2}^{m}ka_kr^{k-1}e^{i(k-1)\theta}+\sum_{k=2}^{\infty}k\overline{b_k}r^{k-1}e^{-i(k+1)\theta}}$$

so that

$$|w(z)|\leq\frac{\frac{2(m+1)^2-1-\beta}{(m+1)(1-\beta)}\left[\sum_{k=m+1}^{\infty}k|a_k|+\sum_{k=n+1}^{\infty}k|b_k|\right]}{2-2\left(\sum_{k=2}^{m}k|a_k|+\sum_{k=2}^{n}k|b_k|\right)-\frac{2(m+1)^2-1-\beta}{(m+1)(1-\beta)}\left[\sum_{k=m+1}^{\infty}k|a_k|+\sum_{k=n+1}^{\infty}k|b_k|\right]}$$

The last inequality is equivalent to

$$\sum_{k=2}^{m} k|a_k| + \sum_{k=2}^{n} k|b_k| + \frac{2(m+1)^2-1-\beta}{(m+1)(1-\beta)}\left(\sum_{k=m+1}^{\infty} k|a_k| + \sum_{k=n+1}^{\infty} k|b_k|\right) \leq 1. \qquad (23)$$

Since the LHS of (23) is bounded above by
$\sum_{k=2}^{\infty} \frac{2k^2-1-\beta}{1-\beta}|a_k| + \sum_{k=2}^{\infty} \frac{2k^2+1+\beta}{1-\beta}|b_k|$ , and hence the proof is complete.

Theorem 2.10: If $f$ of the form (1) with $b_1 = 0$, satisfying condition (2), then

$$Re\left\{\frac{f'_{m,n}(z)}{f'(z)}\right\} \geq \frac{2(m+1)^2-1-\beta}{2(m+1)^2-1-\beta+(m+1)(1-\beta)}, \quad (z \in \Delta) \qquad (24)$$

The result (24) is sharp with the function given by (4).

Proof 2.10: The proof of the result is akin to the proof of that of Theorem 2.9 and hence omitted.

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