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Directed Edge-Graceful Labeling of One Point Union of *M*-Copies of Path Graph

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ABSTRACT

Rosa [11] introduced the notion of graceful labelings. The concept of magic, antimagic and conservative labelings have been extended to directed graphs [9]. Bloom and Hsu [3,4,5] extended the notion of graceful labeling to directed graph. In 1985, Lo [10] introduced the notion of edge-graceful graphs. We introduced the concept of edge-graceful labelings to directed graphs and further studied in [15,16,17,18,19,20]. In this paper we investigate directed edge-graceful labeling of one point union of m- copies of path graph.

Keywords: graceful labeling, edge - graceful labeling, directed edge - graceful labeling, directed edge - graceful graphs.

AMS (MOS) Subject Classification: 05C78.

1. INTRODUCTION

All graphs in this paper are finite and directed. Terms not defined here are used in the sense of Harary [8]. The symbols $V(G)$ and $E(G)$ will denote the vertex set and edge set of a graph G. The cardinality of the vertex set is called the order of G denoted by p. The cardinality of the edge set is called the size of G denoted by q. A graph with p vertices and q edges is called a (p, q) graph.

A graph labeling is an assignment of integers to the vertices or edges or both subject to certain conditions. Labeled graphs serve as useful models for a broad range of applications such as coding theory, X-ray crystallography, radar, astronomy, circuit design, communication network addressing, database management etc. [1,2,12,13]. A good account on graceful labeling problems and other types of graph labeling problems can be found in the dynamic survey of J.A. Gallian [6].

A graph G is called a graceful labeling if f is an injection from the vertices of G to the set $\{0, 1, 2, ..., q\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are distinct.

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A graph $G(V, E)$ is said to be edge-graceful if there exists a bijection f from E to $\{1, 2, ..., |E|\}$ such that the induced mapping f^+ from V to $\{0, 1, ..., |V|-1\}$ given by, $f^+(x) = (\Sigma f(xy)) \bmod (|V|)$ taken overall edges xy incident at x is a bijection.

A necessary condition for a graph G with p vertices and q edges to be edge-graceful is $q(q+1) \equiv \frac{p(p+1)}{2} (\bmod\, p)$. Bloom and Hsu [3,4,5] extended the notion of graceful labeling to directed graph. The concept of magic, antimagic and conservative labelings have been extended to directed graphs [9]. In [18], we extended the concept of edge-graceful labelings to directed graphs and further studied in [15,16,17,18,19,20]. In this paper we investigate directed edge-graceful labeling of one point union of m- copies of path graph.

A (p, q) graph G is said to be **directed edge-graceful** if there exists an orientation of G and a labeling f of the arcs A of G with $\{1, 2, ..., q\}$ such that induced mapping g on V defined by, $g(v) = \left[f^+(v) - f^-(v) \right]$ (mod p) is a bijection where, $f^+(v)$ = the sum of the labels of all arcs with head v and $f^-(v)$ = the sum of the labels of all arcs with v as tail.

A graph G is said to be **directed edge-graceful graph** if it has directed edge-graceful labelings. Here, we investigate directed edge-graceful labeling of one point union of m- copies of path graph.

The examples for the comparison of edge-graceful and directed edge - graceful graphs given in [20] and by theorem If a (p, q) graph G is directed edge-graceful then p is odd in [20], it follows that the notion of directed edge-graceful is entirely different from the concept of edge-graceful labeling. Thus in this paper, we identify the graph $S(m, n)$ which belong to directed edge-graceful family.

2. MAIN RESULTS

Definition 2.1

By a graph $\boldsymbol{S(m, n)}$ we mean a graph obtained from one point union of m copies of the path P_n.

Theorem 2.2

The graph $S(m,n)$ $(m \geq 4,\ n \geq 3)$ is directed edge-graceful for all m even and n even.

Proof

Let $V[S(m,n)] = \{v, v_1, v_2, ..., v_{m(n-1)}\}$ be the set of vertices. The edges and their orientation of $S(m,n)$ are as in Fig. 2.1:

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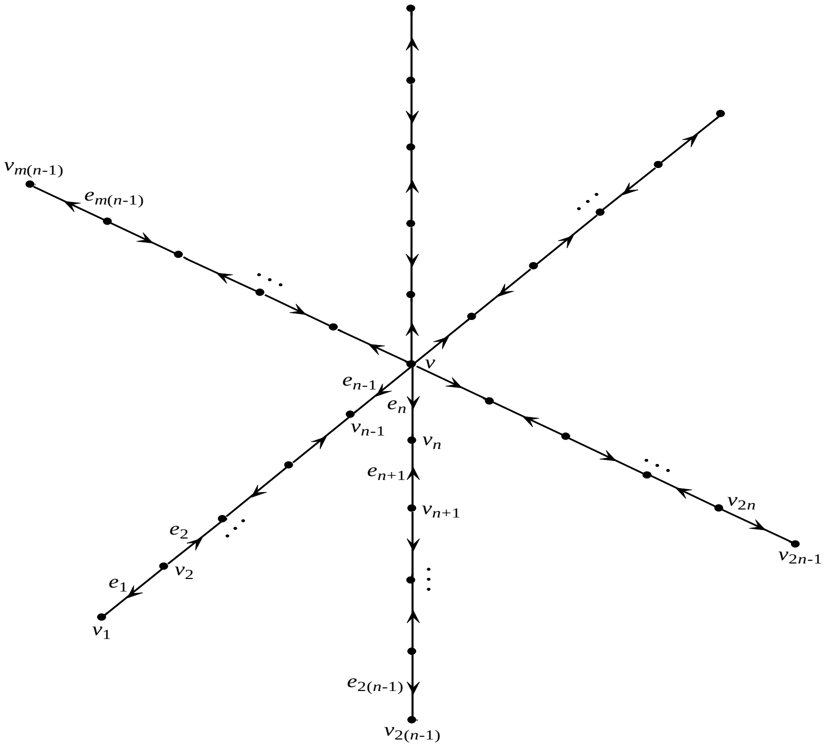

Fig. 2.1: $S(m,n)$ with orientation

We now label the arcs as follows:

$f((e_i)) = i \qquad\qquad 1 \le i \le m(n-1)$

The computed values of $f^+(v_i)$ and $f^-(v_i)$ are given below:

Case (i): $\frac{m}{2}$ is odd

For $1 \le i \le n-1$, $2n \le i \le 3(n-1)$, $4n-2 \le i \le 5(n-1)$,

$6n-4 \le i \le 7(n-1)$, ..., $\left(\frac{m}{2}-1\right)(n-1)+2 \le i \le \frac{m}{2}(n-1)$

$$f^+(v_i) = \begin{cases} 2i-1 & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

$$f^-(v_i) = \begin{cases} -(2i-1) & i \text{ even} \\ 0 & i \text{ odd.} \end{cases}$$

For $n-1 < i < 2(n-1)$, $3(n-1) < i < 4(n-1)$, $5(n-1) < i < 6(n-1)$, ...,

$\left(\frac{m}{2}-2\right)(n-1) < i < \left(\frac{m}{2}-1\right)(n-1)$

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$$f^{+}(v_i) = \begin{cases} 0 & i \text{ odd} \\ 2i+1 & i \text{ even.} \end{cases}$$

$$f^{-}(v_i) = \begin{cases} -(2i+1) & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

For $i = 2(n-1),\ 4(n-1),\ ...,\ \left(\frac{m}{2}-1\right)(n-1)$ and for

$i = 2(n-1)+1,\ 4(n-1)+1,\ ...,\ \left(\frac{m}{2}-1\right)(n-1)+1$

$$f^{+}(v_i) = i$$

$$f^{-}(v_i) = 0.$$

For $\frac{m}{2}(n-1)+1 \le i < n-1+\frac{m}{2}(n-1)$, $\frac{m}{2}(n-1)+2n-1 \le i < \frac{m(n-1)}{2}+3(n-1)$, ..., $\frac{m(n-1)}{2}+\left(\frac{m}{2}-1\right)(n-1)+1 \le i < m(n-1)$

$$f^{+}(v_i) = \begin{cases} 0 & i \text{ odd} \\ 2i+1 & i \text{ even.} \end{cases}$$

$$f^{-}(v_i) = \begin{cases} -(2i+1) & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

For $\frac{m(n-1)}{2}+n < i \le \frac{m(n-1)}{2}+2(n-1)$, $\frac{m(n-1)}{2}+1+3(n-1) < i \le \frac{m(n-1)}{2}+4(n-1)$, ..., $\frac{m(n-1)}{2}+1+\left(\frac{m}{2}-2\right)(n-1) < i \le \frac{m}{2}(n-1)+\left(\frac{m}{2}-1\right)(n-1)$

$$f^{+}(v_i) = \begin{cases} 2i-1 & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

$$f^{-}(v_i) = \begin{cases} -(2i-1) & i \text{ even} \\ 0 & i \text{ odd.} \end{cases}$$

For $i = 4(n-1), 6(n-1), ..., m(n-1)$ and for

$i = 4(n-1) + 1, 6(n-1)+1, ..., (m-2)(n-1) + 1$

$$f^{+}(v_i) = i$$

$$f^{-}(v_i) = 0.$$

Case (ii): $\frac{m}{2}$ is even

For $1 \le i \le n-1$, $2n \le i \le 3(n-1)$, $4n-2 \le i \le 5(n-1)$,

$6(n-4) \le i \le 7(n-1)$, ..., $\left(\frac{m}{2}-2\right)(n-1)+2 \le i \le \left(\frac{m}{2}-1\right)(n-1)$

$$f^+(v_i) = \begin{cases} 2i-1 & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

$$f^-(v_i) = \begin{cases} -(2i-1) & i \text{ even} \\ 0 & i \text{ odd.} \end{cases}$$

For $n-1 < i < 2(n-1)$, $3(n-1) < i < 4(n-1)$, $5(n-1) < i < 6(n-1)$, ...,

$\left(\frac{m}{2}-1\right)(n-1) < i < \frac{m}{2}(n-1)$

$$f^+(v_i) = \begin{cases} 0 & i \text{ odd} \\ 2i+1 & i \text{ even.} \end{cases}$$

$$f^-(v_i) = \begin{cases} -(2i+1) & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

For $i = 2(n-1)$, $4(n-1)$, $6(n-1)$, ..., $\frac{m}{2}(n-1)$ and for

$i = 2(n-1)+1$, $4(n-1)+1$, $6(n-1)+1$, ..., $\left(\frac{m}{2}-2\right)(n-1)+1$

$$f^+(v_i) = i$$

$$f^-(v_i) = 0.$$

For $\frac{m}{2}(n-1)+2 \le i \le \left(\frac{m}{2}+1\right)(n-1)$, $\left(\frac{m}{2}+2\right)(n-1)+2 \le i \le \left(\frac{m}{2}+3\right)(n-1)$, $\left(\frac{m}{2}+4\right)(n-1)+2 \le i \le \left(\frac{m}{2}+5\right)(n-1)$, ..., $\frac{m}{2}(n-1)+\left(\frac{m}{2}-2\right)(n-1)+2 \le i \le \frac{m}{2}(n-1)+\left(\frac{m}{2}-1\right)(n-1)$

$$f^+(v_i) = \begin{cases} 0 & i \text{ odd} \\ 2i+1 & i \text{ even.} \end{cases}$$

$$f^-(v_i) = \begin{cases} -(2i+1) & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

For $\left(\frac{m}{2}+1\right)(n-1)+1 \le i < \left(\frac{m}{2}+2\right)(n-1)$, $\left(\frac{m}{2}+3\right)(n-1)+1 \le i < \left(\frac{m}{2}+4\right)(n-1)$,

$\left(\frac{m}{2}+5\right)(n-1)+1 \le i < \left(\frac{m}{2}+6\right)(n-1)$, ..., $\frac{m}{2}(n-1)+1+\left(\frac{m}{2}-1\right)(n-1) \le i < m(n-1)$

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$$f^+(v_i) = \begin{cases} 2i-1 & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

$$f^-(v_i) = \begin{cases} -(2i-1) & i \text{ even} \\ 0 & i \text{ odd.} \end{cases}$$

For $i = \frac{m}{2}(n-1)+1, \left(\frac{m}{2}+2\right)(n-1), \left(\frac{m}{2}+4\right)(n-1), ..., m(n-1)$

and for $i = \left(\frac{m}{2}+2\right)(n-1)+1, \left(\frac{m}{2}+4\right)(n-1)+1, ..., (m-2)(n-1)+1$

$$f^+(v_i) = i \quad ; \quad f^-(v_i) = 0.$$

$$f^+(v) = 0 \quad ; \quad f^-(v) = -\frac{m}{2}\left[m(n-1)+1\right]$$

Then the induced vertex labels are:

Case (i) $\frac{m}{2}$ is odd

For $1 \le i \le n-1$, $2n \le i \le 3(n-1)$, $4n-2 \le i \le 5(n-1)$,

$6n-4 \le i \le 7(n-1)$, ..., $\left(\frac{m}{2}-1\right)(n-1)+2 \le i \le \frac{m}{2}(n-1)$

$$g(v_i) = \begin{cases} 2i-1 & i \text{ odd} \\ m(n-1)-2i+2 & i \text{ even.} \end{cases}$$

For $n-1 < i < 2(n-1)$, $3(n-1) < i < 4(n-1)$, $5(n-1) < i < 6(n-1)$, ...,

$\left(\frac{m}{2}-2\right)(n-1) < i < \left(\frac{m}{2}-1\right)(n-1)$

$$g(v_i) = \begin{cases} m(n-1)-2i & i \text{ odd} \\ 2i+1 & i \text{ even.} \end{cases}$$

For $i = 2(n-1), 4(n-1), ..., \left(\frac{m}{2}-1\right)(n-1)$

$$g(v_i) = i$$

For $i = 2(n-1)+1, 4(n-1)+1, ..., \left(\frac{m}{2}-1\right)(n-1)+1$

$$g(v_i) = i$$

For $\frac{m}{2}(n-1)+1 \le i < n-1+\frac{m}{2}(n-1)$, $\frac{m}{2}(n-1)+2n-1 \le i < \frac{m(n-1)}{2}+3(n-1)$, ...,

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$\frac{m(n-1)}{2}+\left(\frac{m}{2}-1\right)(n-1)+1\le i<m(n-1)$

$$g(v_i) = \begin{cases} m(n-1)-2\left(i-\frac{m}{2}(n-1)+1\right)+3 & i \text{ odd} \\ 2\left(i-\frac{m}{2}(n-1)-1\right)+2 & i \text{ even.} \end{cases}$$

For $\frac{m(n-1)}{2}+n<i\le\frac{m(n-1)}{2}+2(n-1)$, $\frac{m(n-1)}{2}+1+3(n-1)<i\le\frac{m(n-1)}{2}+4(n-1)$, ...,

$\frac{m}{2}(n-1)+1+\left(\frac{m}{2}-2\right)(n-1)<i\le\frac{m}{2}(n-1)+\left(\frac{m}{2}-1\right)(n-1)$

$$g(v_i) = \begin{cases} 2\left(i-\frac{m}{2}(n-1)-1\right) & i \text{ odd} \\ m(n-1)-2\left(i-\frac{m}{2}(n-1)-1\right)+1 & i \text{ even.} \end{cases}$$

For $i = 4(n-1), 6(n-1), ..., m(n-1)$

$$g(v_i) = i$$

For $i = 4(n-1)+1, 6(n-1)+1, ..., (m-2)(n-1)+1$

$$g(v_i) = i$$

Case (ii): $\frac{m}{2}$ is even

For $1\le i\le n-1$, $2n\le i\le 3(n-1)$, $4n-2\le i\le 5(n-1)$,

$6(n-4)\le i\le 7(n-1)$, ..., $\left(\frac{m}{2}-2\right)(n-1)+2\le i\le\left(\frac{m}{2}-1\right)(n-1)$

$$g(v_i) = \begin{cases} 2i-1 & i \text{ odd} \\ m(n-1)-2i+2 & i \text{ even.} \end{cases}$$

For $n-1<i<2(n-1)$, $3(n-1)<i<4(n-1)$, $5(n-1)<i<6(n-1)$, ...,

$\left(\frac{m}{2}-1\right)(n-1)<i<\frac{m}{2}(n-1)$

$$g(v_i) = \begin{cases} m(n-1)-2i & i \text{ odd} \\ 2i+1 & i \text{ even.} \end{cases}$$

For $i = 2(n-1), 4(n-1), 6(n-1), ..., \frac{m}{2}(n-1)$

$$g(v_i) = i$$

For $i = 2(n-1)+1,\ 4(n-1)+1,\ 6(n-1)+1,\ \ldots,\ \left(\frac{m}{2}-2\right)(n-1)+1$

$$g(v_i) = i$$

For $\frac{m}{2}(n-1)+2 \le i \le \left(\frac{m}{2}+1\right)(n-1)$, $\left(\frac{m}{2}+2\right)(n-1)+2 \le i \le \left(\frac{m}{2}+3\right)(n-1)$, $\left(\frac{m}{2}+4\right)(n-1)+2 \le i \le \left(\frac{m}{2}+5\right)(n-1)$, ..., $\frac{m}{2}(n-1)+\left(\frac{m}{2}-2\right)(n-1)+2 \le i \le \frac{m}{2}(n-1)+\left(\frac{m}{2}-1\right)(n-1)$

$$g(v_i) = \begin{cases} m(n-1)-2\left(i-\frac{m}{2}(n-1)\right)+3 & i \text{ even} \\ 2\left(i-\frac{m}{2}(n-1)-1\right) & i \text{ odd.} \end{cases}$$

For $\left(\frac{m}{2}+1\right)(n-1)+1 \le i < \left(\frac{m}{2}+2\right)(n-1)$, $\left(\frac{m}{2}+3\right)(n-1)+1 \le i < \left(\frac{m}{2}+4\right)(n-1)$, $\left(\frac{m}{2}+5\right)(n-1)+1 \le i < \left(\frac{m}{2}+6\right)(n-1)$, ..., $\frac{m}{2}(n-1)+1+\left(\frac{m}{2}-1\right)(n-1) \le i < m(n-1)$

$$g(v_i) = \begin{cases} m(n-1)-2\left(i-\frac{m}{2}(n-1)\right)+1 & i \text{ odd} \\ 2\left(i-\frac{m}{2}(n-1)\right) & i \text{ even.} \end{cases}$$

For $i = \frac{m}{2}(n-1)+1,\ \left(\frac{m}{2}+2\right)(n-1),\ \left(\frac{m}{2}+4\right)(n-1),\ \ldots,\ m(n-1)$

$$g(v_i) = i$$

For $i = \left(\frac{m}{2}+2\right)(n-1)+1,\ \left(\frac{m}{2}+4\right)(n-1)+1,\ \ldots,\ (m-2)(n-1)+1$

$$g(v_i) = i$$

Clearly, it follows that all the vertex labels are distinct and ranges between 0 to $p-1$. Thus, g is a bijection. Hence, the graph $S(m,n)$ $(m \ge 4,\ n \ge 3)$ is directed edge - graceful for m even and n even. The directed edge - graceful labeling of $S(6,6)$ and $S(8,6)$ are given in Fig. 2.2 and Fig. 2.3 respectively.

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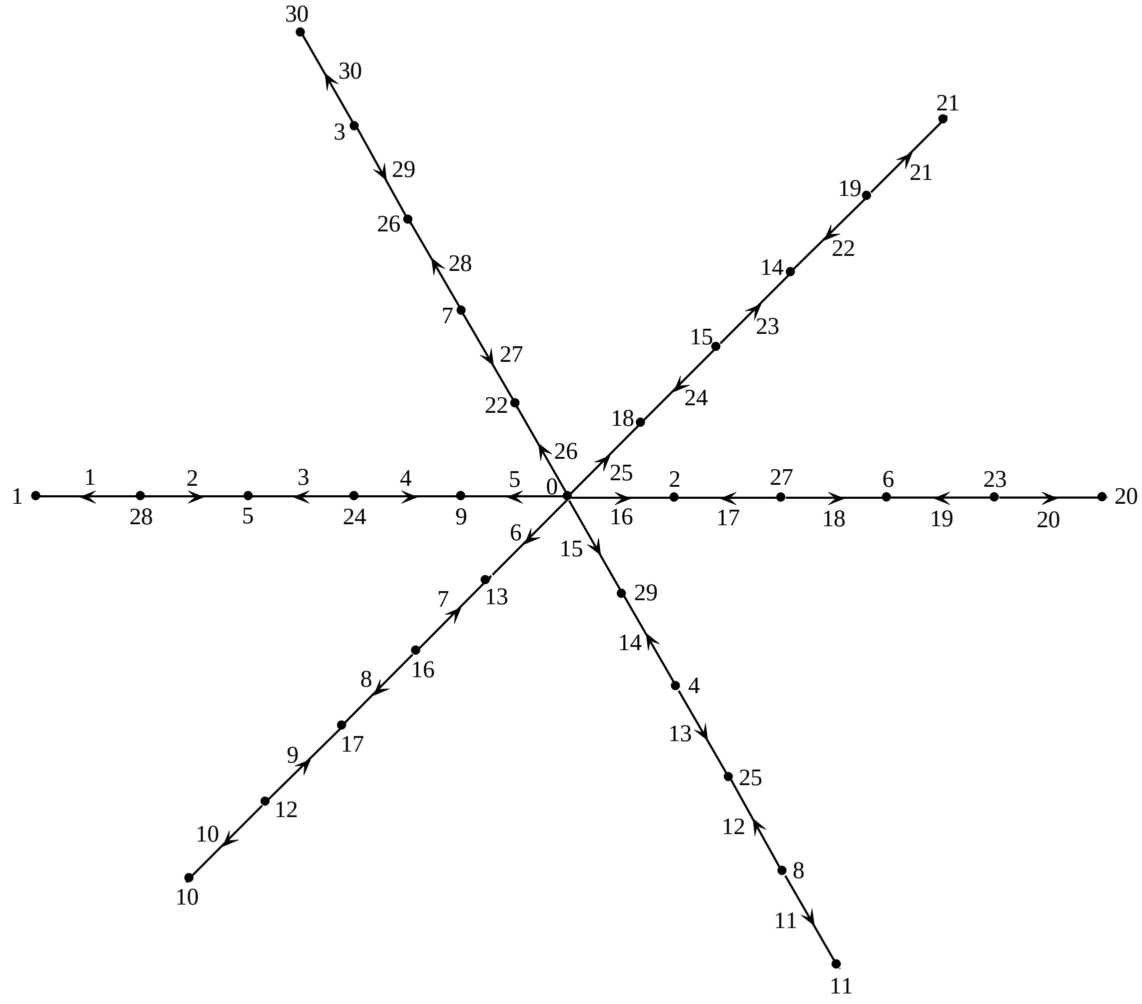

Fig. 2.2: $S(6,6)$ with directed edge-graceful labeling

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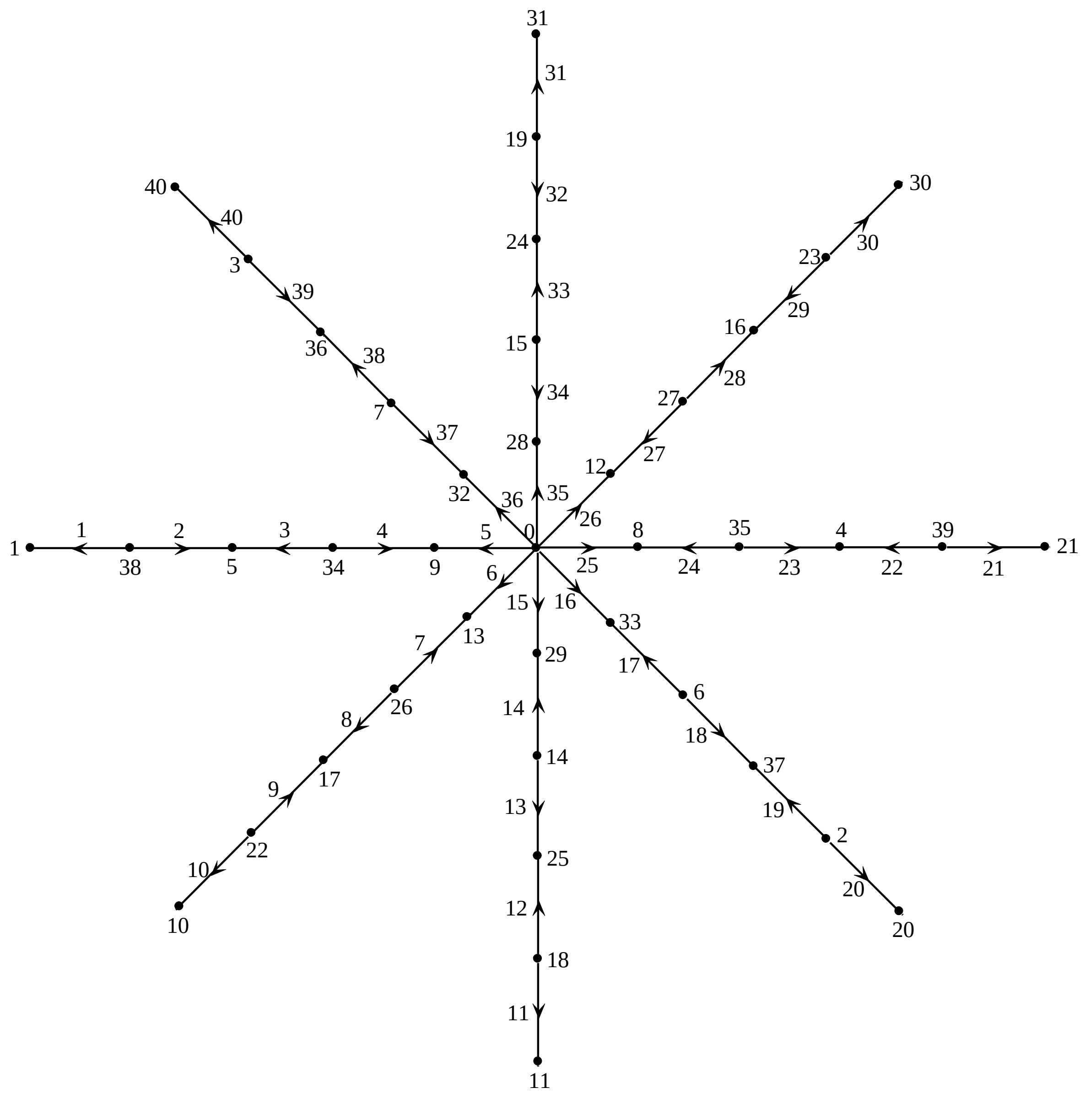

Fig. 2.3: $S(8,6)$ with directed edge-graceful labeling

Theorem 2.3

The graph $S(m,n)$ $(m \geq 4,\ n \geq 3)$ is directed edge-graceful for all m even and n odd.

Proof

Let $V[S(m,n)] = \{v, v_1, v_2, ..., v_{m(n-1)}\}$ be the set of vertices. The edges and their orientation of $S(m,n)$ are as in Fig. 2.4:

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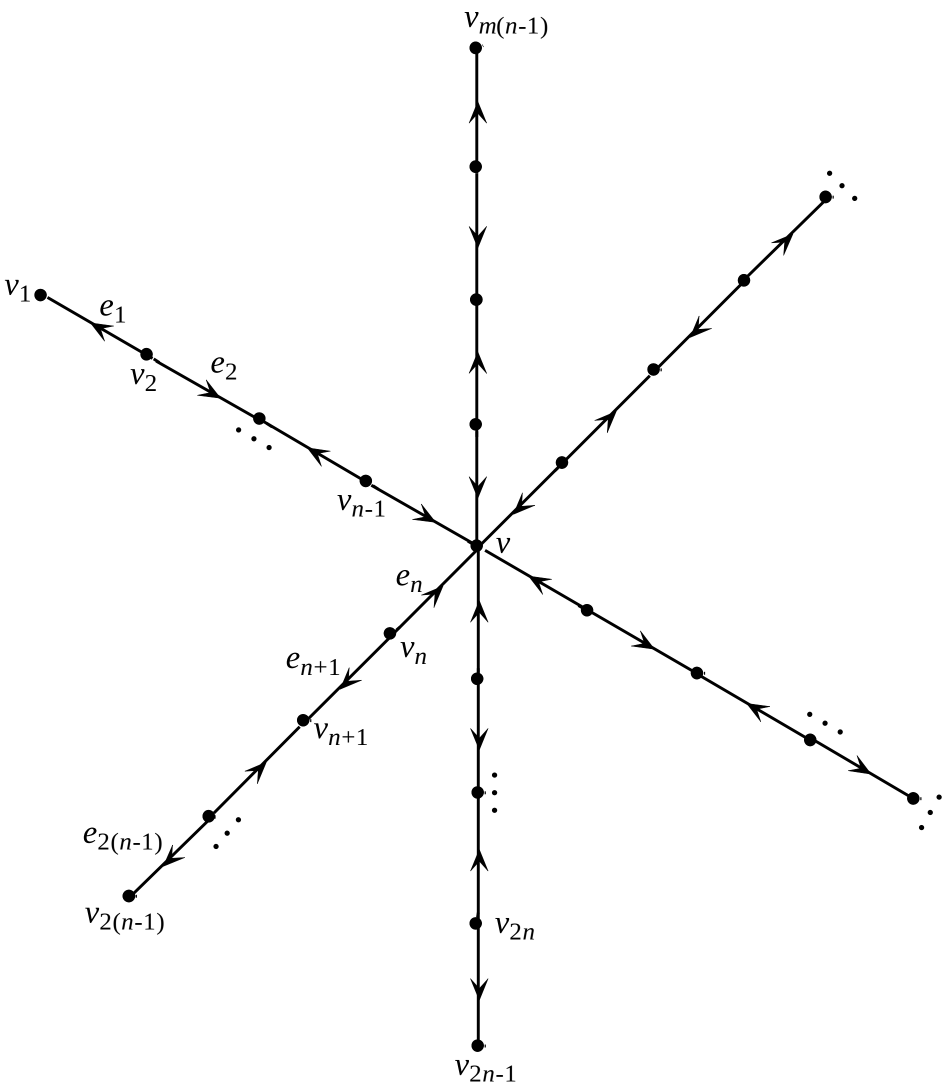

Fig. 2.4: $S(m,n)$ with orientation

We now label the arcs as follows:

$f((e_i)) = i \qquad\qquad 1 \le i \le m(n-1)$

The computed values of $f^+(v_i)$ and $f^-(v_i)$ are given below:

Case (i): $\frac{m}{2}$ is odd

For $1 \le i \le n-1$, $2n \le i \le 3(n-1)$, $4n-2 \le i \le 5(n-1)$,

$6n-4 \le i \le 7(n-1)$, ..., $\left(\frac{m}{2}-1\right)(n-1)+2 \le i \le \frac{m}{2}(n-1)$

$$f^+(v_i) = \begin{cases} 2i-1 & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

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$$f^-(v_i) = \begin{cases} -(2i-1) & i \text{ even} \\ 0 & i \text{ odd.} \end{cases}$$

For $n-1 < i < 2(n-1)$, $3(n-1) < i < 4(n-1)$, $5(n-1) < i < 6(n-1)$, ...,

$\left(\frac{m}{2}-2\right)(n-1) < i < \left(\frac{m}{2}-1\right)(n-1)$

$$f^+(v_i) = \begin{cases} 0 & i \text{ odd} \\ 2i+1 & i \text{ even.} \end{cases}$$

$$f^-(v_i) = \begin{cases} -(2i+1) & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

For $i = 2(n-1), 4(n-1), 6(n-1), ..., \left(\frac{m}{2}-1\right)(n-1)$ and for

$i = 2(n-1)+1, 4(n-1)+1, ..., \left(\frac{m}{2}-1\right)(n-1)+1$

$$f^+(v_i) = i$$

$$f^-(v_i) = 0$$

For $\frac{m}{2}(n-1)+1 \le i < n-1+\frac{m}{2}(n-1)$,

$\frac{m}{2}(n-1)+2n-1 \le i < \frac{m(n-1)}{2}+3(n-1)$, ...,

$\frac{m(n-1)}{2}+\left(\frac{m}{2}-1\right)(n-1)+1 \le i < m(n-1)$

$$f^+(v_i) = \begin{cases} 0 & i \text{ odd} \\ 2i+1 & i \text{ even.} \end{cases}$$

$$f^-(v_i) = \begin{cases} -(2i+1) & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

For $\frac{m(n-1)}{2}+n < i \le \frac{m(n-1)}{2}+2(n-1)$,

$\frac{m(n-1)}{2}+1+3(n-1) < i \le \frac{m(n-1)}{2}+4(n-1)$, ...,

$\frac{m}{2}(n-1)+1+\left(\frac{m}{2}-2\right)(n-1) < i \le \frac{m}{2}(n-1)+\left(\frac{m}{2}-1\right)(n-1)$

$$f^+(v_i) = \begin{cases} 2i-1 & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

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$$f^-(v_i) = \begin{cases} -(2i-1) & i \text{ even} \\ 0 & i \text{ odd.} \end{cases}$$

For $i = 4(n-1),\ 6(n-1), \ldots,\ m(n-1)$ and for

$i = 4(n-1)+1,\ 6(n-1)+1, \ldots,\ (m-2)(n-1)+1$

$$f^+(v_i) = i \quad ; \quad f^-(v_i) = 0$$

$$f^+(v) = \frac{m}{2}\big(m(n-1)+1\big) \quad ; \quad f^-(v) = 0$$

Case (ii): $\frac{m}{2}$ is even

For $1 \le i \le n-1$, $2n \le i \le 3(n-1)$, $4n-2 \le i \le 5(n-1)$, $6n-4 \le i \le 7(n-1)$,

$8n-6 \le i \le 9(n-1)$, ..., $\left(\frac{m}{2}-2\right)(n-1)+2 \le i \le \left(\frac{m}{2}-1\right)(n-1)$,

$$f^+(v_i) = \begin{cases} 2i-1 & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

$$f^-(v_i) = \begin{cases} -(2i-1) & i \text{ even} \\ 0 & i \text{ odd.} \end{cases}$$

For $n-1 < i < 2(n-1)$, $3(n-1) < i < 4(n-1)$, $5(n-1) < i < 6(n-1)$, ...,

$\left(\frac{m}{2}-1\right)(n-1) < i < \frac{m}{2}(n-1)$

$$f^+(v_i) = \begin{cases} 0 & i \text{ odd} \\ 2i+1 & i \text{ even.} \end{cases}$$

$$f^-(v_i) = \begin{cases} -(2i+1) & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

For $i = 2(n-1), 4(n-1), 6(n-1), \ldots, \frac{m}{2}(n-1)$ and for

$i = 2(n-1)+1, 4(n-1)+1, \ldots, \left(\frac{m}{2}-2\right)(n-1)+1$

$$f^+(v_i) = i$$

$$f^-(v_i) = 0$$

For $\frac{m}{2}(n-1)+2 \le i \le \left(\frac{m}{2}+1\right)(n-1)$, $\left(\frac{m}{2}+2\right)(n-1)+2 \le i \le \left(\frac{m}{2}+3\right)(n-1)$,

$\left(\frac{m}{2}+4\right)(n-1)+2 \le i \le \left(\frac{m}{2}+5\right)(n-1)$, ..., $\frac{m}{2}(n-1)+\left(\frac{m}{2}-2\right)(n-1)+2 \le i \le \frac{m}{2}(n-1)+\left(\frac{m}{2}-1\right)(n-1)$

$$f^{+}(v_i) = \begin{cases} 2i-1 & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

$$f^{-}(v_i) = \begin{cases} -(2i-1) & i \text{ even} \\ 0 & i \text{ odd.} \end{cases}$$

For $\left(\frac{m}{2}+1\right)(n-1)+1 \le i < \left(\frac{m}{2}+2\right)(n-1)$, $\left(\frac{m}{2}+3\right)(n-1)+1 \le i < \left(\frac{m}{2}+4\right)(n-1)$, ...,

$\frac{m}{2}(n-1)+1+\left(\frac{m}{2}-1\right)(n-1) \le i < m(n-1)$

$$f^{+}(v_i) = \begin{cases} 0 & i \text{ odd} \\ 2i+1 & i \text{ even.} \end{cases}$$

$$f^{-}(v_i) = \begin{cases} -(2i+1) & i \text{ odd} \\ 0 & i \text{ even.} \end{cases}$$

For $i = \frac{m}{2}(n-1)+1$, $\left(\frac{m}{2}+2\right)(n-1)$, $\left(\frac{m}{2}+4\right)(n-1)$, ..., $m(n-1)$

and for $i = \left(\frac{m}{2}+2\right)(n-1)+1$, $\left(\frac{m}{2}+4\right)(n-1)+1$, ..., $(m-2)(n-1)+1$

$$f^{+}(v_i) = i \quad ; \quad f^{-}(v_i) = 0$$

$$f^{+}(v) = \frac{m}{2}\left(m(n-1)+1\right) \quad ; \quad f^{-}(v) = 0$$

Then the induced vertex labels are:

Case (i): $\frac{m}{2}$ is odd

For $1 \le i \le n-1$, $2n \le i \le 3(n-1)$, $4n-2 \le i \le 5(n-1)$,

$6n-4 \le i \le 7(n-1)$, ..., $\left(\frac{m}{2}-1\right)(n-1)+2 \le i \le \frac{m}{2}(n-1)$

$$g(v_i) = \begin{cases} 2i-1 & i \text{ odd} \\ m(n-1)-2i+2 & i \text{ even.} \end{cases}$$

For $n-1 < i < 2(n-1)$, $3(n-1) < i < 4(n-1)$, $5(n-1) < i < 6(n-1)$, ...,

$\left(\frac{m}{2}-2\right)(n-1) < i < \left(\frac{m}{2}-1\right)(n-1)$

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$$g(v_i) = \begin{cases} m(n-1)-2i & i \text{ odd} \\ 2i+1 & i \text{ even.} \end{cases}$$

For $i = 2(n-1), 4(n-1), 6(n-1), \ldots, \left(\frac{m}{2}-1\right)(n-1)$

$$g(v_i) = i$$

For $i = 2(n-1)+1, 4(n-1)+1, \ldots, \left(\frac{m}{2}-1\right)(n-1)+1$

$$g(v_i) = i$$

For $\frac{m}{2}(n-1)+1 \le i < n-1+\frac{m}{2}(n-1)$,

$\frac{m}{2}(n-1)+2n-1 \le i < \frac{m(n-1)}{2}+3(n-1)$, ...,

$\frac{m(n-1)}{2}+\left(\frac{m}{2}-1\right)(n-1)+1 \le i < m(n-1)$

$$g(v_i) = \begin{cases} m(n-1)-2\left(i-\frac{m}{2}(n-1)+1\right)+3 & i \text{ odd} \\ 2\left(i-\frac{m}{2}(n-1)-1\right)+2 & i \text{ even.} \end{cases}$$

For $\frac{m(n-1)}{2}+n < i \le \frac{m(n-1)}{2}+2(n-1)$,

$\frac{m(n-1)}{2}+1+3(n-1) < i \le \frac{m(n-1)}{2}+4(n-1)$, ...,

$\frac{m}{2}(n-1)+1+\left(\frac{m}{2}-2\right)(n-1) < i \le \frac{m}{2}(n-1)+\left(\frac{m}{2}-1\right)(n-1)$

$$g(v_i) = \begin{cases} 2\left(i-\frac{m}{2}(n-1)-1\right) & i \text{ odd} \\ m(n-1)-2\left(i-\frac{m}{2}(n-1)-1\right)+1 & i \text{ even.} \end{cases}$$

For $i = 4(n-1),\ 6(n-1), \ldots,\ m(n-1)$

$$g(v_i) = i$$

For $i = 4(n-1)+1,\ 6(n-1)+1, \ldots, (m-2)(n-1)+1$

$$g(v_i) = i$$

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Case (ii): $\frac{m}{2}$ is even

For $1 \le i \le n-1$, $2n \le i \le 3(n-1)$, $4n-2 \le i \le 5(n-1)$, $6n-4 \le i \le 7(n-1)$,

$8n-6 \le i \le 9(n-1)$, ..., $\left(\frac{m}{2}-2\right)(n-1)+2 \le i \le \left(\frac{m}{2}-1\right)(n-1)$,

$$g(v_i) = \begin{cases} 2i-1 & i \text{ odd} \\ m(n-1)-2i+2 & i \text{ even.} \end{cases}$$

For $n-1 < i < 2(n-1)$, $3(n-1) < i < 4(n-1)$, $5(n-1) < i < 6(n-1)$, ...,

$\left(\frac{m}{2}-1\right)(n-1) < i < \frac{m}{2}(n-1)$

$$g(v_i) = \begin{cases} m(n-1)-2i & i \text{ odd} \\ 2i+1 & i \text{ even.} \end{cases}$$

For $i = 2(n-1)$, $4(n-1)$, $6(n-1)$, ..., $\frac{m}{2}(n-1)$

$$g(v_i) = i$$

For $i = 2(n-1)+1$, $4(n-1)+1$, ..., $\left(\frac{m}{2}-2\right)(n-1)+1$

$$g(v_i) = i$$

For $\frac{m}{2}(n-1)+2 \le i \le \left(\frac{m}{2}+1\right)(n-1)$, $\left(\frac{m}{2}+2\right)(n-1)+2 \le i \le \left(\frac{m}{2}+3\right)(n-1)$,

$\left(\frac{m}{2}+4\right)(n-1)+2 \le i \le \left(\frac{m}{2}+5\right)(n-1)$, ..., $\frac{m}{2}(n-1)+\left(\frac{m}{2}-2\right)(n-1)+2 \le i \le \frac{m}{2}(n-1)+\left(\frac{m}{2}-1\right)(n-1)$

$$g(v_i) = \begin{cases} 2\left(i-\frac{m}{2}(n-1)-1\right) & i \text{ odd} \\ m(n-1)-2\left(i-\left(\frac{m}{2}(n-1)\right)\right)+3 & i \text{ even.} \end{cases}$$

For $\left(\frac{m}{2}+1\right)(n-1)+1 \le i < \left(\frac{m}{2}+2\right)(n-1)$, $\left(\frac{m}{2}+3\right)(n-1)+1 \le i < \left(\frac{m}{2}+4\right)(n-1)$, ...,

$\frac{m}{2}(n-1)+1+\left(\frac{m}{2}-1\right)(n-1) \le i < m(n-1)$

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$$g(v_i) = \begin{cases} m(n-1) - 2\left(i - \frac{m}{2}(n-1)\right) + 1 & i \text{ odd} \\ 2\left(i - \frac{m}{2}(n-1)\right) & i \text{ even.} \end{cases}$$

For $i = \frac{m}{2}(n-1)+1$, $\left(\frac{m}{2}+2\right)(n-1)$, $\left(\frac{m}{2}+4\right)(n-1)$, ..., $m(n-1)$

$$g(v_i) = i$$

For $\left(\frac{m}{2}+2\right)(n-1)+1$, $\left(\frac{m}{2}+4\right)(n-1)+1$, ..., $(m-2)(n-1)+1$

$$g(v_i) = i$$

Clearly, it follows that all the vertex labels are distinct and ranges betweens 0 to $p - 1$. Thus, g is a bijection. Hence, the graph $S(m,n)$ $(m \geq 4,\ n \geq 3)$ is directed edge - graceful for all m even and n odd. The directed edge - graceful labeling of $S(8,\ 3)$ and $S(8,\ 5)$ are given in Fig. 2.5 and Fig.2.6 respectively.

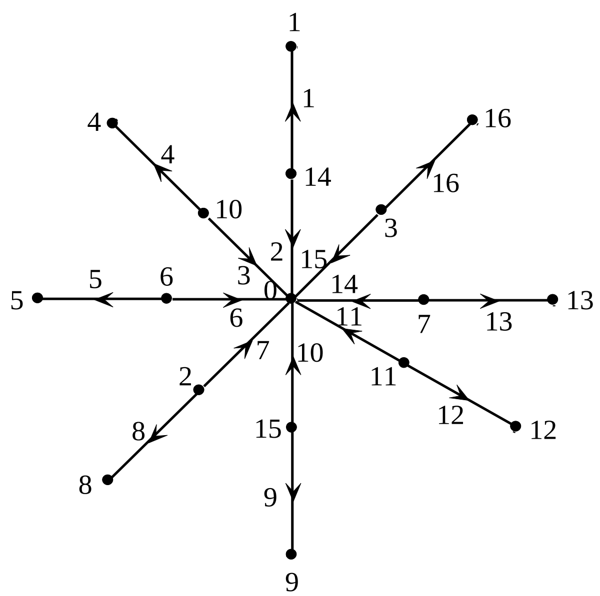

Fig. 2.5: $S(8,3)$ with directed edge-graceful labeling

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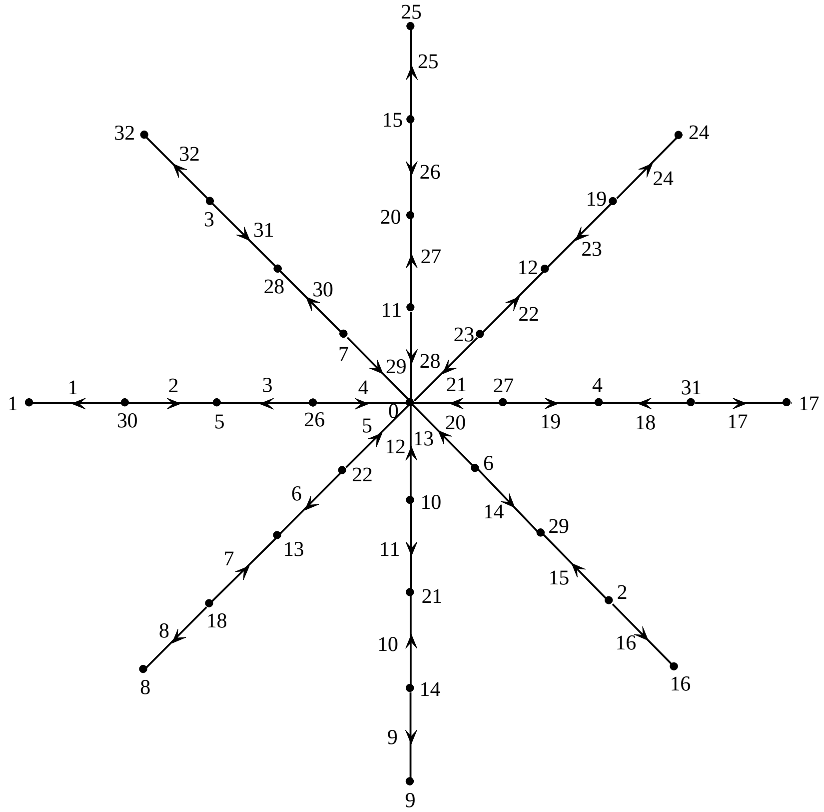

Fig. 2.6: $S(8,5)$ **with directed edge-graceful labeling**

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