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On Equitable Edge Coloring of Wheel Graph Families

D.K. Sathiya[1] and S. Senthamilselvi[2*]

[1]Research Scholar, Department of Mathematics,

Vels Institute of Science, Technology and Advanced Studies, Chennai-117.

[2]Department of Mathematics,

Vels Institute of Science, Technology and Advanced Studies, Chennai-117.

*Corresponding Author

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ABSTRACT

An equitable edge coloring of a graph is a proper edge coloring for which the difference between any two color classes is at most one. The minimum cardinality of G for such coloring is called equitable edge chromatic number. In this article, we determine the theorem on equitable edge coloring for sunlet graph, wheel graph and helm graph.

Keywords: Sunlet graph, wheel graph, helm graph and equitable edge coloring.

AMS Subject Classification: 05C15

1. INTRODUCTION

Let us consider all graphs are finite, simple and undirected graph G. The concept of edge coloring introduced by Tait in 1880. Clearly $\chi^{'}(G) \geq \Delta(G)$, where $\Delta(G)$ is the maximum degree of graph G. In 1916, Konig was proved that every bipartite graph can be edge colored with exactly $\Delta(G)$ colors. Xia Zhang and Guizhen Liu [6] prove that the equitable edge-colorings of simple graphs.

In 1949 Shannon proved that every graph can be edge colored with $\leq \frac{3}{2}\Delta(G)$ colors. In 1964, Vizing [5] given the tight bound for edge coloring that $\Delta(G) \leq \chi^{'}(G) \leq \Delta(G)+1$. In 1973, Meyer [3] presented the concept of equitable coloring and equitable chromatic number. After few years, as an extension of equitable coloring, the concept of equitable edge coloring was introduced byHilton and deWerra [1] in 1994. K. Kaliraj [2] proved that equitable edge coloring of some join graphs. Veninstine vivik et.al [4] proved the equitable edge coloring of splitting graph of helm and sunlet graph.

2. PRELIMINARIES

Definition 2.1[5]

The *n*- sunlet graph S_n is obtained by joining *n* pendant edges to all the vertices of the cycle C_n

Definition 2.2[5]

For $n \geq 4$, the wheel W_n is obtained by joining a vertex v_0 to each of the $n-1$ vertices $v_1, v_2, ..., v_{n-1}$ of C_{n-1}.

Definition 2.3[5] The Helm graph H_n is the graph attained by a W_n by adjoining a pendant edge to each vertex of the $n-1$ vertices of the cycle in W_n.

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Lemma 2.4[5]: Let G be a simple graph, then $\chi_e^{'}(G) \geq \Delta(G)$.

3. MAINRESULTS

Theorem3.1.

For any $n \geq 3,$ *the equitable chromatic index for sunlet graph is* $\chi_e^{'}(S_n) = 3.$

Proof.

Let $V(S_n) = \{u_k, v_k : 1 \leq k \leq n\}$ and $E(S_n) = \{e_k, s_k : 1 \leq k \leq n\}$, where the edges $\{e_k : 1 \leq k \leq n\}$ represents the edge $\{v_k v_{k+1(\text{mod } n)} : 1 \leq k \leq n\}$,the edges $\{s_k : 1 \leq k \leq n\}$ represents the edge $\{u_k v_k : 1 \leq k \leq n\}$

Define an edge coloring $c : E(S_n) \to \{1,2,3\}$ as follows. Let us partition the edge set of sunlet graph $E(S_n)$ as follows.

Case (i): $n \equiv 0(\text{mod } 3)$ (i.e) 3, 6, 9...

$$E_1 = \{e_1, e_3, e_7 ...., e_{n-2}\} \cup \{s_3, s_6, ..., s_n\} \tag{3.1}$$

$$E_2 = \{e_2, e_5, e_8 ...., e_{n-1}\} \cup \{s_1, s_4, ..., s_{n-2}\} \tag{3.2}$$

$$E_3 = \{e_3, e_6, e_7 ...., e_n\} \cup \{s_2, s_5, ..., s_{n-1}\} \tag{3.3}$$

From the equation (3.1) to (3.3), clearly the sunlet graph S_n is equitable edge colored with 3 colors. Also we observe that the color classes E_1, E_2 and E_3 are independent sets of S_n and its satisfies the inequiality $\| E_i | - | E_j \| \leq 1,$ for $i \neq j$. Hence $\chi_e^{'}(S_n) \leq 3$. Since $\Delta = 3$ and $\chi_e^{'}(S_n) \geq \Delta = 3$ Therefore $\chi_e^{'}(S_n) = 3.$ When $n = 3, 6, 9, \ldots$, i.e consider $n = 6,$ for which the color classes $| E_1 |=| E_2 |=| E_3 |= 4$ and which implies that $\| E_1 | - | E_2 \| \leq 1.$ Thus, it is equitable edge colored with 3 colors. Therefore $\chi_e^{'}(S_6) \leq 3.$ The maximum degree of sunlet graph is 3($\Delta = 3$) and by lemma 2.4, $\chi_e^{'}(S_6) \geq \Delta = 3$. Hence $\chi_e^{'}(S_6) = 3.$

Case (ii): $n \equiv 1 \ (\text{mod } 3)$

$$E_1 = \{e_1, e_4, e_7 ...., e_{n-3}\} \cup \{s_3, s_6, ..., s_{n-1}\} \cup \{s_n\} \tag{3.4}$$

$$E_2 = \{e_2, e_5, e_8 ...., e_{n-2}\} \cup \{e_n\} \cup \{s_4, s_7, ..., s_{n-3}\} \tag{3.5}$$

$$E_3 = \{e_3, e_6, e_9 ...., e_{n-1}\} \cup \{s_1, s_2\} \cup \{s_5, s_8, ..., s_{n-2}\} \tag{3.6}$$

From the equation (3.4) to (3.6), clearly the sunlet graph S_n is equitable edge colored with 3 colors. Also we observe that the color classes E_1, E_2 and E_3 are independent sets of S_n and its satisfies the inequiality $||E_i| - |E_j|| \leq 1$, for $i \neq j$. Hence $\chi_e'(S_n) \leq 3$. Since $\Delta = 3$ and $\chi_e'(S_n) \geq \Delta = 3$ Therefore $\chi_e'(S_n) = 3.$ For example, in the case(ii) when $n \equiv 1 \pmod 3$, i.e consider $n = 10,$ for which the color classes $|E_1| = |E_3| = 7$ and $|E_2| = 6$, which implies that $||E_i| - |E_j|| \leq 1$. Thus, it is equitable edge colored with 3 colors. So that $\chi_e'(S_{10}) \leq 3.$ The maximum degree of sunlet graph is 3($\Delta = 3$) and by lemma 2.4, $\chi_e'(S_{10}) \geq \Delta = 3.$ Hence $\chi_e'(S_{10}) = 3.$

Case (iii): $n \equiv 2 \pmod 3$

$$E_1 = \{e_1, e_4, e_7 ...., e_{n-1}\} \bigcup \{s_3, s_6, ..., s_{n-2}\} \tag{3.7}$$

$$E_2 = \{e_2, e_5, e_8 ...., e_{n-3}\} \bigcup \{e_n\} \bigcup \{s_4, s_7, s_{10} ...., s_{n-1}\} \tag{3.8}$$

$$E_3 = \{e_3, e_6, e_9 ...., e_{n-2}\} \bigcup \{s_1, s_2\} \bigcup \{s_5, s_8, ..., s_n\} \tag{3.9}$$

From the equation (3.7) to (3.9), clearly the sunlet graph S_n is equitable edge colored with 3 colors. Also we observe that the color classes E_1, E_2 and E_3 are independent sets of S_n and its satisfies the inequiality $||E_i| - |E_j|| \leq 1$, for each (i, j) . Hence $\chi_e'(S_n) \leq 3$. Since $\Delta = 3$ and $\chi_e'(S_n) \geq \Delta = 3$ Therefore $\chi_e'(S_n) = 3.$ For example, in the case(iii) when $n \equiv 2 \pmod 3$, i.e consider $n = 11,$ for which the color classes $|E_1| = |E_2| = 7$ and $|E_3| = 8,$ which implies that $||E_1| - |E_2|| \leq 1.$ Thus $\chi_e'(S_{11}) \leq 3.$ The maximum degree of sunlet graph is 3($\Delta = 3$) and by using the lemma 2.4, $\chi_e'(S_{11}) \geq \Delta = 3$. Hence $\chi_e'(S_{11}) = 3.$

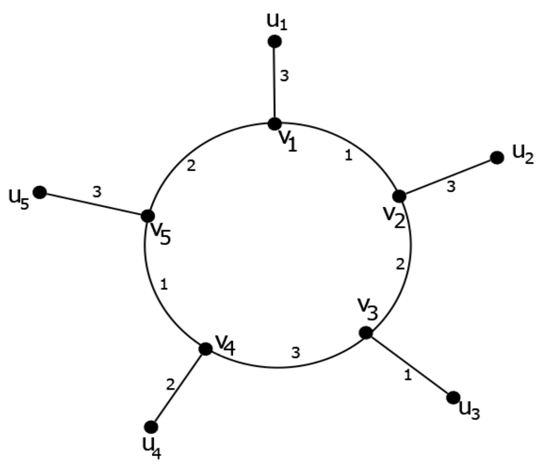

Figure 1: Equitable edge coloring of sunlet graph with 5 vertices

Theorem 3.2

For any $n \geq 4,$ *the equitable chromatic index for wheel graph is* $\chi_e'(w_n) = n - 1.$

Proof. Let $V(W_n) = \{v_0\} \cup \{v_k : 1 \le k \le n-1\}$ and

Let $E(W_n) = \{g_k : 1 \le k \le n-1\} \cup \{s_k : 1 \le k \le n-1\}$, where the edges $\{g_k : 1 \le k \le n-1\}$ represents the edge $\{v_0 v_k : 1 \le k \le n-1\}$ the edges $\{s_k : 1 \le k \le n\}$ represents the edge $\{v_k v_{k+1} : 1 \le k \le n-1\}$

Construct an edge coloring $c : E(W_n) \to \{1, 2, 3, ..., n-1\}$ as follows. Let us partition the edge set for wheel graph $E(W_n)$ as follows.

$$E_1 = \{g_1, s_2\} \tag{3.10}$$

$$E_2 = \{g_2, s_3\} \tag{3.11}$$

$$E_3 = \{g_3, s_4\} \tag{3.12}$$

$$E_4 = \{g_4, s_5\} \tag{3.13}$$

$$E_5 = \{g_5, s_6\} \tag{3.14}$$

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$$E_{n-4} = \{g_{n-4}, s_{n-3}\} \tag{3.15}$$

$$E_{n-3} = \{g_{n-3}, s_{n-2}\} \tag{3.16}$$

$$E_{n-2} = \{g_{n-2}, s_{n-1}\} \tag{3.17}$$

$$E_{n-1} = \{g_{n-1}, s_n\} \tag{3.18}$$

From the equation (3.10) to (3.18), clearly the wheel graph W_n is equitableedgecolored with $n-1$ colors. Also observe that color classes $E_1, E_2, ..., E_{n-1}$ are independent sets of W_n, the cardinality of the color classes $|E_1| = |E_2| = |E_3| ..... = |E_{n-2}| = |E_{n-1}| = 2$ and its satisfies the inequiality $||E_i| - |E_j|| \le 1,$ for $i \ne j$. Hence $\chi_e^{'}(W_n) \le n-1$. Since $\Delta = n-1$ and $\chi_e^{'}(W_n) \ge n-1$. Therefore $\chi_e^{'}(W_n) = n-1$. For example, consider $n = 8$, vertices, such that the color classes $|E_1| = |E_2| = |E_3| ..... = |E_7| = 2$ and which implies that $||E_1| - |E_2|| \le 1$. Thus,

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an equitable edge colored with 7 colors and so that $\chi_e^{'}(W_8) \le 7.$ The maximum degree of wheel graph is 7 ($\Delta = 7$) and by lemma 2.4, $\chi_e^{'}(W_8) \ge \Delta = 7.$ Hence $\chi_e^{'}(W_8) = 7.$

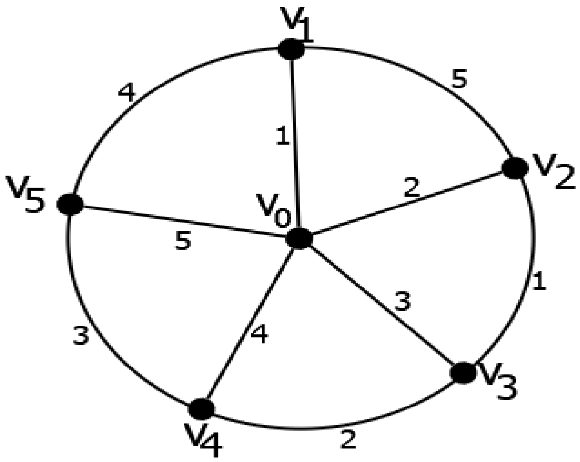

Figure 2: Equitable edge coloring of wheel graph with 6 vertices

Theorem 3.3

For any $n \ge 4,$ *the equitable chromatic index for helm graph is* $\chi_e^{'}(H_n) = n-1.$

Proof. Let $V(H_n) = \{v_0\} \cup \{v_k : 1 \le k \le n-1\} \cup \{u_k : 1 \le k \le n-1\}$ and

Let $E(H_n) = \{e_k : 1 \le k \le n-1\} \cup \{f_k : 1 \le k \le n-1\} \cup \{s_k : 1 \le k \le n-1\}$, where the edges $\{e_k : 1 \le k \le n-1\}$ represents the edge $\{v_0 v_k : 1 \le k \le n-1\}$, the edges $\{f_k : 1 \le k \le n-1\}$ represents the edge $\{v_k v_{k+1 (\text{mod n-1})} : 1 \le k \le n-1\}$ and the edges $\{s_k : 1 \le k \le n-1\}$ represents the edge $\{v_k u_k : 1 \le k \le n-1\}$

By construction an edge coloring $c : E(H_n) \to \{1, 2, 3, ..., n-1\}$ as follows. Let us partition the edge set for helm graph $E(H_n)$ as follows.

$$E_1 = \{e_1, f_2, s_5\} \tag{3.19}$$

$$E_2 = \{e_2, f_3, s_1\} \tag{3.20}$$

$$E_3 = \{e_3, f_4, s_2\} \tag{3.21}$$

$$E_4 = \{e_4, f_5, s_3\} \tag{3.22}$$

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$$E_5 = \{e_5, f_1, s_4\} \tag{3.23}$$

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$$E_{n-5} = \{e_{n-5}, f_{n-4}, s_{n-1}\} \tag{3.24}$$

$$E_{n-4} = \{e_{n-4}, f_{n-3}, s_{n-5}\} \tag{3.25}$$

$$E_{n-3} = \{e_{n-3}, f_{n-2}, s_{n-4}\} \tag{3.26}$$

$$E_{n-2} = \{e_{n-2}, f_{n-1}, s_{n-3}\} \tag{3.27}$$

$$E_{n-1} = \{e_{n-1}, f_{n-5}, s_{n-2}\} \tag{3.28}$$

From the equation (3.19) to (3.28), clearly the helm graph H_n is equitable edge colored with $n-1$ colors. Also observe that the color classes independent sets of H_n, the cardinality of the color classes $|E_1|=|E_2|=|E_3|.....=|E_{n-2}|=|E_{n-1}|=3$ and its satisfies the inequiality $||E_i|-|E_j||\le 1,$ for any (i, j). Hence $\chi_e^{'}(H_n) \le n-1$. Since $\Delta = n-1$and $\chi_e^{'}(H_n) \ge \Delta = n-1.$ Therefore $\chi_e^{'}(H_n) = n-1.$ For example, consider the helm $n=8,$ vertices, the color classes $|E_1|=|E_2|=|E_3|.....=|E_7|=3$ and which implies that $||E_1|-|E_2||\le 1.$ So that the equitable edge colored with 7 colors. So that $\chi_e^{'}(H_8) \le 7.$ The maximum degree of helm graph is 7 ($\Delta = 7$) and by lemma 2.4, it follows that $\chi_e^{'}(H_8) \ge \Delta = 7$. Hence $\chi_e^{'}(H_8) = 7.$

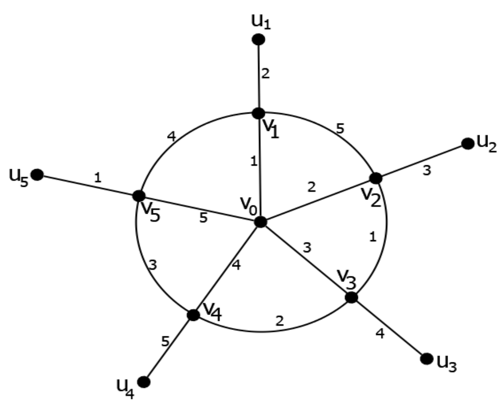

Figure 3: Equitable edge coloring of helm graph with 6 vertices

4. CONCLUSION

In this article, we determined the equitable chromatic index of sunlet, wheel, helm graph. The proofs establish an optimal solution to the equitable edge coloring of these graph families. The field of equitable edge coloring of graphs is broad open. It would be further interesting to determine the bounds of equitable edge coloring of various families of graphs.

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