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# A Note on Equitable Edge Coloring of Graphs

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**ABSTRACT**

An equitable edge coloring of a graph is a proper edge coloring for which the difference between any two color classes is at most one. The minimum cardinality of G for such coloring is called equitable edge chromatic number. In this article, we determine the theorem on equitable edge coloring for Gear graph $G_n$, Double star graph $K_{1,n,n}$ and Fan graph $F_{1,n}$ respectively.

**Keywords**: Gear graph, double star graph, fan graph and equitable edge coloring.

**AMS Subject Classification**: 05C15

## 1. INTRODUCTION

Let us consider all graphs are finite, simple and undirected graph G. The concept of edge coloring introduced by Tait in 1880. Clearly $\chi'(G) \geq \Delta(G)$, where $\Delta(G)$ is the maximum degree of graph G. In 1916, Konig was proved that every bipartite graph can be edge colored with exactly $\Delta(G)$ colors. Xia Zhang and Guizhen Liu [6] prove that the equitable edge-colorings of simple graphs.

In 1949 Shannon proved that every graph can be edge colored with $\leq \frac{3}{2}\Delta(G)$ colors. In 1964, Vizing [5] given the tight bound for edge coloring that $\Delta(G) \leq \chi'(G) \leq \Delta(G)+1$. In 1973, Meyer [3] presented the concept of equitable coloring and equitable chromatic number. After few years, as an extension of equitable coloring, the concept of equitable edge coloring was introduced byHilton and deWerra [1] in 1994. K. Kaliraj [2] proved that equitable edge coloring of some join graphs. Veninstinevivik et.al [4] proved the equitable edge coloring of splitting graph of helm and sunlet graph.

## 2. PRELIMINARIES

**Definition 2.1:** The Gear graph $G_n$ is obtained from a wheel graph $W_n$ by insert a new vertex to each edge of the $n-1$ cycle in $W_n$.

**Definition 2.2:** The double star graph $K_{1,n,n}$ is a tree obtained from the star $K_{1,n}$ by adding a new pendent edge of the exiting *n* vertices. It has *2n+1* vertices and *2n* edges.

**Definition 2.3:** The Fan graph $F_{1,n}$ is defined as the graph join $P_n + K_1$, where $K_1$ is the complete graph with one vertex and $P_n$ is the path graph on *n* vertices.

**Lemma 2.4[5]:** Let G be a simple graph, then $\chi_e'(G) \geq \Delta(G)$.

## 3. MAINRESULTS

**Theorem 3.1**

*For any* $n \geq 4$, *the equitable chromatic index for gear graph is* $\chi_e^{'}(G_n) = n-1.$

***Proof.*** Let $V(G_n) = \{v_0\} \cup \{u_k : 1 \leq k \leq n-1\} \cup \{v_k : 1 \leq k \leq n-1\}$ and

Let $E(G_n) = \{e_k : 1 \leq k \leq n-1\} \cup \{f_k : 1 \leq k \leq n-1\} \cup \{s_k : 1 \leq k \leq n-1\}$, where the edges $\{e_k : 1 \leq k \leq n-1\}$ represents the edge $\{v_0 v_k : 1 \leq k \leq n-1\}$, the edges $\{f_k : 1 \leq k \leq n-1\}$ represents the edge $\{v_k u_k : 1 \leq k \leq n-1\}$ and the edges $\{s_k : 1 \leq k \leq n-1\}$ represents the edge $\{u_k v_{k+1(\text{mod n-1})} : 1 \leq k \leq n-1\}$

An edge coloring is define as $c : E(G_n) \to \{1, 2, 3, ..., n-1\}$ as follows. Let us partition the edge set for gear graph $E(S_n)$ as follows.

$$E_1 = \{e_1, f_6, s_1\} \tag{3.1}$$

$$E_2 = \{e_2, f_1, s_2\} \tag{3.2}$$

$$E_3 = \{e_3, f_2, s_3\} \tag{3.3}$$

$$E_4 = \{e_4, f_3, s_4\} \tag{3.4}$$

$$E_5 = \{e_5, f_4, s_5\} \tag{3.5}$$

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$$E_{n-4} = \{e_{n-4}, f_{n-3}, s_{n-4}\} \tag{3.6}$$

$$E_{n-3} = \{e_{n-3}, f_{n-4}, s_{n-3}\} \tag{3.7}$$

$$E_{n-2} = \{e_{n-2}, f_{n-3}, s_{n-2}\} \tag{3.8}$$

$$E_{n-1} = \{e_{n-1}, f_{n-2}, s_{n-1}\} \tag{3.9}$$

From the equation (3.1) to (3.9), clearly the gear graph $G_n$ is equitableedge colored with $n-1$ colors. Also observe that the color classes are independent sets of $G_n$, the cardinality of the color classes $|E_1| = |E_2| = |E_3| ..... = |E_{n-2}| = |E_{n-1}| = 3$ and its satisfies the inequiality $||E_i| - |E_j|| \leq 1,$ for $i \neq j$. Hence $\chi_e^{'}(G_n) \leq n-1$. Since $\Delta = n-1$ and by lemma 2.4, $\chi_e^{'}(G_n) \geq \Delta = n-1.$ Therefore $\chi_e^{'}(G_n) = n-1.$ For example, consider the gear graph with $n = 8,$ vertices, such that the color classes $|E_1| = |E_2| = |E_3| ..... = |E_7| = 3$ and which

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implies that $\| E_i | - | E_j \| \leq 1$, for $i \neq j$. Such that the equitable edge colored with $7$ colors. Thus $\chi_e^{'}(G_8) \leq 7$. The maximum degree of gear graph is $7$ ($\Delta = 7$) and $\chi_e^{'}(G_8) \geq \Delta = 7$. Hence $\chi_e^{'}(G_8) = 7$.

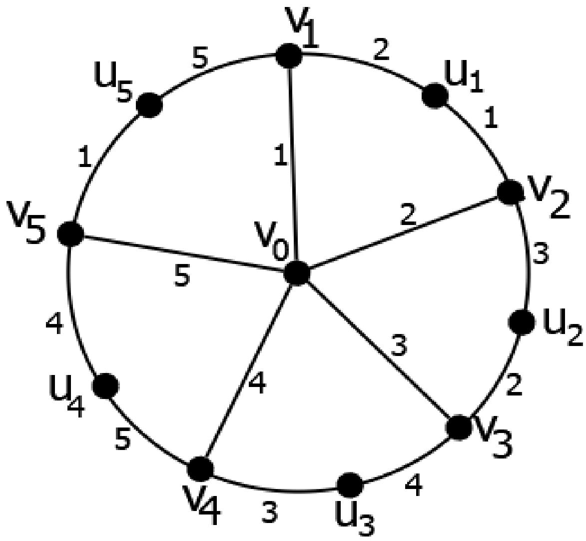


Figure 3: Equitable edge coloring of gear graph

**Theorem 3.2**

*For any positive integer, the equitable chromatic index for double star graph is* $\chi_e^{'}(K_{1,n,n}) = n$.

***Proof.*** Let $V(K_{1,n,n}) = \{v_0\} \cup \{v_k : 1 \leq k \leq n\} \cup \{u_k : 1 \leq k \leq n\}$ and

Let $E(K_{1,n,n}) = \{e_k : 1 \leq k \leq n\} \cup \{f_k : 1 \leq k \leq n\}$, where the edges $\{e_k : 1 \leq k \leq n\}$ represents the edge $\{v_0 v_k : 1 \leq k \leq n\}$, the edges $\{f_k : 1 \leq k \leq n\}$ represents the edge $\{v_k u_k : 1 \leq k \leq n\}$ respectively.

Define an edge coloring $c : E(K_{1,n,n}) \rightarrow \{1, 2, 3, ..., 2n\}$ as follows. Let us partition the edge set for double star graph $E(K_{1,n,n})$ as follows.

$$E_1 = \{e_1, f_n\} \quad (3.10)$$

$$E_2 = \{e_2, f_1\} \quad (3.11)$$

$$E_3 = \{e_3, f_2\} \quad (3.12)$$

$$E_4 = \{e_4, f_3\} \quad (3.13)$$

$$E_5 = \{e_5, f_4\} \quad (3.14)$$

$$E_6 = \{e_6, f_5\} \quad (3.15)$$

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$$E_{n-2} = \{e_{n-2}, f_{n-3}\} \qquad (3.16)$$

$$E_{n-1} = \{e_{n-1}, f_{n-2}\} \qquad (3.17)$$

$$E_n = \{e_n, f_{n-1}\} \qquad (3.18)$$

From the equation (3.10) to (3.18), clearly the double star graph $K_{1,n,n}$ is equitable edge colored with $n$ colors. Also we observe that the color classes are independent sets of $K_{1,n,n}$, the cardinality of the color classes $|E_1|=|E_2|=|E_3|.....=|E_{n-1}|=|E_n|=2$ and its satisfies the inequiality $||E_i|-|E_j||\le 1,$ for $i \ne j$. Therefore $\chi_e^{'}(K_{1,n,n}) \le n$. We know that $\Delta = n$ and $\chi_e^{'}(K_{1,n,n}) \ge \Delta = n.$ Hence $\chi_e^{'}(K_{1,n,n}) = n.$ For example, Consider $n = 6,$ vertices, the color classes are $|E_1|=|E_2|=|E_3|.....=|E_7|=2$ and which implies that $||E_1|-|E_2||\le 1.$ Thus, equitable edge colored with $6$ colors. Therefore $\chi_e^{'}(K_{1,6,6}) \le 6$. The maximum degree of double star graph is $6.$ Hence $\chi_e^{'}(K_{1,6,6}) = 6.$

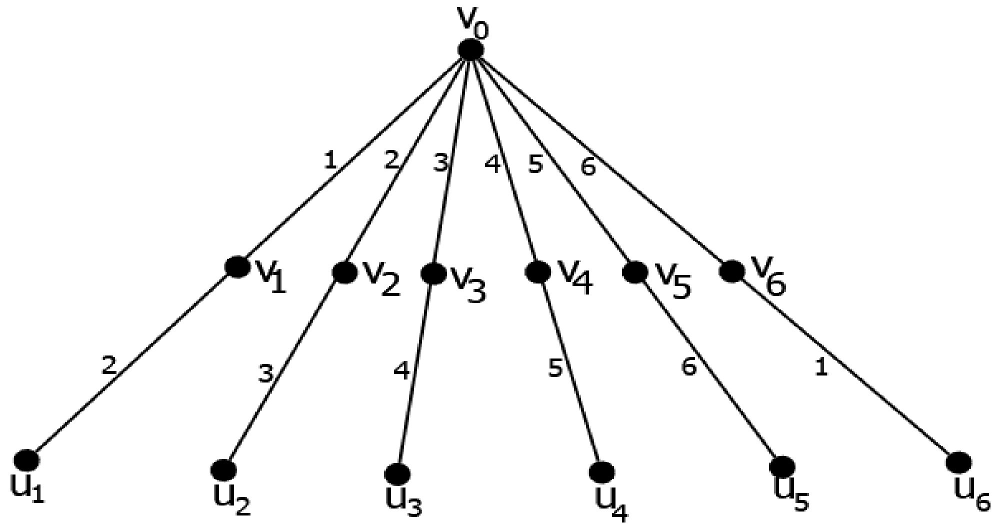


Figure 2: Equitable edge coloring of double star graph

**Theorem 3.3**

*For any positive integer, the equitable chromatic index for fan graph is* $\chi_e^{'}(F_{1,n}) = n.$

***Proof.*** Let $V(F_{1,n}) = \{v_0\} \cup \{v_k : 1 \le k \le n\}$ and

Let $E(F_{1,n}) = \{g_k : 1 \le k \le n\} \cup \{h_k : 1 \le k \le n-1\}$, where the edges $\{g_k : 1 \le k \le n\}$ represents the edge $\{v_0 v_k : 1 \le k \le n\}$, the edges $\{h_k : 1 \le k \le n-1\}$ represents the edge $\{v_k v_{k+1} : 1 \le k \le n-1\}$ respectively.

An edge coloring is define $c : E(F_{1,n}) \to \{1,2,3,...,n\}$ as follows. Let us partition the edge set for fan graph $F_{1,n}$ as

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follows.

$$E_1 = \{g_1, h_{n-1}\} \tag{3.19}$$

$$E_2 = \{g_2\} \tag{3.20}$$

$$E_3 = \{g_3, h_1\} \tag{3.21}$$

$$E_4 = \{g_4, h_2\} \tag{3.22}$$

$$E_5 = \{g_5, h_3\} \tag{3.23}$$

$$E_6 = \{g_6, h_4\} \tag{3.24}$$

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$$E_{n-2} = \{g_{n-2}, h_{n-4}\} \tag{3.25}$$

$$E_{n-1} = \{g_{n-1}, h_{n-3}\} \tag{3.26}$$

$$E_n = \{g_n, h_{n-1}\} \tag{3.27}$$

From the equation (3.19) to (3.27), clearly the fan graph $F_{1,n}$ is equitable edgecolored with $n$ colors. Also we observe that the color classes $E_1, E_2, ...., E_n$ are independent sets of $F_{1,n}$, the cardinality of the color classes $|E_1|=|E_3|.....=|E_{n-1}|=|E_n|=2$ and $|E_2|=1$, its satisfies the inequiality $||E_i|-|E_j||\leq 1$, for $i \neq j$. Hence $\chi_e^{'}(F_{1,n}) \leq n$. We know that $\Delta = n$ and by lemma 2.5 $\chi_e^{'}(F_{1,n}) \geq \Delta = n.$ Therefore $\chi_e^{'}(F_{1,n}) = n.$ For example, consider the fan graphwith $n = 6,$ vertices, suchthat the color classes $|E_1|=|E_3|.....=|E_6|=2$ and $|E_2|=1$, which implies that $||E_1|-|E_2||\leq 1.$ Thus, the equitable edge colored with $6$ colors. Therefore $\chi_e^{'}(F_{1,6}) \leq 6$ and $\chi_e^{'}(F_{1,6}) \geq \Delta = 6$. Hence $\chi_e^{'}(F_{1,6}) = 6.$

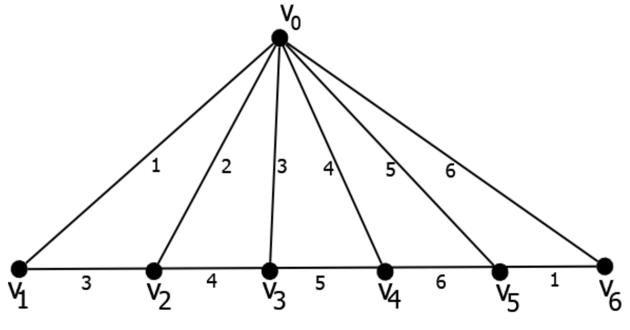


Figure 3: Equitable edge coloring coloring of fan graph

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## 4. CONCLUSION

In this article, we determined the equitable chromatic index of Gear, Double star and Fan graph. The proofs establish an optimal solution to the equitable edge coloring of these graph families. The field of equitable edge coloring of graphs is broad open. It would be further interesting to determine the bounds of equitable edge coloring of various families of graphs.

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