JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1809-1814
https://publishoa.com
ISSN: 1309-3452

Total Chromatic Number of Middle and Total Graph for Certain Graphs

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Received 2022 April 02**; Revised** 2022 May 20**; Accepted** 2022 June 18**.**

ABSTRACT

A total coloring of a graph G is an assignment of colors to both the vertices and edges of $G,$ such that no two adjacent or incident vertices and edges of G are assigned the same colors. In this paper, we have discussed the total coloring and total chromatic number of middle graph, total graph of twig graph and comb graph.

AMS Subject Classification: 05C15

Keywords and Phrases: Middle graph, total graph, twig graph, comb graph, total coloring and total chromatic number.

1. INTRODUCTION

We start with finite, simple, connected and undirected graphs. Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G)$ and the edge set $\mathrm{E}(G)$, respectively. A coloring of a graph G is an assignment of colors to the vertices or edges or both. A vertex-coloring(edge-coloring) is called proper coloring if no two vertices(edges) receive the color. There are so many different proper colorings such as a-coloring, b-coloring, star coloring, list coloring, harmonious coloring, total coloring etc. In the present work focused on total coloring of graphs.

A total coloring of G, is a function $f: S \to C,$ where $S = V(G) \cup E(G)$ and C is a set of colors to satisfies the given conditions.

(i) no two adjacent vertices receive the same colors

(ii) no two adjacent edges receive the same colors

(iii) no edges and its end vertices receive the same colors

The concept of total coloring was introduced by Behzad [1] and Vizing [16]. Also they have posed the conjecture that for every simple graph G has $\Delta(G) + 1 \leq \chi''(G) \leq \Delta(G) + 2$, where $\Delta(G)$ the maximum degree of G. This conjecture is known as the Total Coloring Conjecture (TCC). The *total chromatic number* $\chi''(G)$ of a graph$G$ is the minimum cardinality k such that G may have a total coloring by k colors. Behzad *et al* [2] proved that the total chromatic number of complete graph K_n. While Yap [17] determined that the total chromatic number of cycle C_n. Rosenfeld [13] and Vijayaditya [15] verified the TCC, for any graph G with maximum degree $\leq$ 3 and Kostochka [8] for maximum degree$\leq$ 5. In Borodin [4] verified The Total Coloring conjecture (TCC) for maximum degree $\geq$ 9 in planar graphs. In recent era, total coloring have been extensively studied in different families of graphs. Mohan et.al [9] given the tight bound of Behzad and Vizing conjecture in Corona product of certain classes of graph. Muthuramakrishnan et.al [10, 11, 12] proved that the total chromatic number of line, middle, total graph of star graph and square graph of bistar graph and also they proved that total chromatic number of middle and total graph of path and sunlet graph. Jayaraman et.al [7] proved that the total chromatic number of middle graph of double star graph and total graph of double star graph. Vaidya et.al [14] has verified TCC for some cycle related graphs. In the present work, we investigate the total chromatic number of middle

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1809-1814
https://publishoa.com
ISSN: 1309-3452

graph, total graph of twig graph and comb graph.

2. PRELIMINARIES

Definition 2.1. The *middle graph*[5] of a graph G, denoted by $M(G)$ is define as follows, the vertex set of $M(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $M(G)$ are adjacent in $M(G)$ in case one of the following condition holds: (i) x, y are in $E(G)$ and x, y is adjacent in G. (ii) x is in $V(G)$, y is in $E(G)$and x, y are incident in G.

Definition 2.2. The *Total graph*[5] of a graph G, denoted by $T(G)$ is define as, the vertex set of $T(G)$ is $V(G) \cup E(G)$. Two vertices x, y in the vertex set of $T(G)$ are adjacent in $T(G)$ in case one of the following condition holds: (i) x, y are in $V(G)$ and x is adjacent to y in G. (ii) x, y are in $E(G)$ and x, y is adjacent in G. (iii) x is in $V(G)$, y is in $E(G)$ and x, y are incident in G.

Definition 2.3. The graph obtained from a path by attaching exactly two pendant edges to each internal vertex of the path is called *Twig graph* and it is denoted by T_n.

Definition 2.4. The graph obtained by joining a single pendent edge to each vertex of a path is called *Comb graph* [6] and it is denoted by P_n^+.

3. Main Results

Theorem 3.1. Let $M(T_n)$ be the middle graph of twig graph. Then the total chromatic number of middle graph of twig graph is $\Delta(M(T_n)) + 1$.

Proof. Let $V(T_n) = \{u_k: 1 \leq k \leq n-2\} \cup \{v_k: 1 \leq k \leq n\} \cup \{w_k: 1 \leq k \leq n-2\}$ and $E(T_n) = \{u_k v_{k+1}: 1 \leq k \leq n-2\} \cup \{v_k v_{k+1}: 1 \leq k \leq n-1\} \cup \{w_k v_{k+1}: 1 \leq k \leq n-2\}$. Now we construct the middle graph of twig graph, each edge $\{u_k v_{k+1}: 1 \leq k \leq n-2\}$, $\{v_k v_{k+1}: 1 \leq k \leq n-1\}$ and $\{w_k v_{k+1}: 1 \leq k \leq n-2\}$ are subdivided by the vertices $\{u'_k: 1 \leq k \leq n-2\}$, $\{v'_k: 1 \leq k \leq n-1\}$ and $\{w'_k: 1 \leq k \leq n-2\}$ in $M(T_n)$. The vertex set and the edge set of middle graph of twig is given by

$$V(M(T_n)) = \{u_k, u'_k : 1 \leq k \leq n-2\} \cup \{v_k, w_k, v'_k, w'_k : 1 \leq k \leq n\}.$$

$$E(M(T_n)) = \{u'_k v'_{k+1},\ v'_k w'_k, u_k u'_k, w_k w'_k, u'_k v'_k, u'_k w'_k, v_{k+1} u'_k,\ v'_k v'_{k+1}, v_k v'_{k+1}, v_{k+1} w'_k,$$
$$v'_k w'_{k+1} : 1 \leq k \leq n-2\} \cup \{v_k v'_k: 1 \leq k \leq n-1\}.$$

We define the total coloring f, such that $f : S \rightarrow C$, where $S = V(M(T_n)) \cup E(M(T_n))$ and C is the set of colors $C = \{1,2, \ldots, 9\}$. Assign total coloring to these vertices and edges as follows.

For $1 \leq k \leq n$

$$f(v_k) = \begin{cases} 1; & if\ k \equiv 1\ (mod\ 3) \\ 3; & if\ k \equiv 2\ (mod\ 3) \\ 2; & if\ k \equiv 0\ (mod\ 3) \end{cases}$$

For $1 \leq k \leq n-2$

$$f(u_k) = 3, \quad f(w_k) = 3, \quad f(u'_k) = 9, \quad f(w'_k) = 5,$$

$$f(u_k u'_k) = 2, \quad f(u'_k v'_{k+1}) = 5, \quad f(u'_k v'_k) = 4,$$

$$f(u'_k w'_k) = 1, \quad f(u'_k v_{k+1}) = 6, \quad f(w'_k v_{k+1}) = 8,$$

$$f(v'_k w'_k) = 6, \quad f(v'_{k+1} w'_k) = 7, \quad f(w_k w'_k) = 2$$

For $1 \leq k \leq n-1$

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https://publishoa.com
ISSN: 1309-3452

$$f(v'_k) = \begin{cases} 2, & if\ k \equiv 1\ (mod\ 3) \\ 1, & if\ k \equiv 2\ (mod\ 3) \\ 3, & if\ k \equiv 0\ (mod\ 3) \end{cases}$$

$$f(v'_k v'_{k+1}) = \begin{cases} 8, & if\ k\ is\ odd \\ 9, & if\ k\ is\ even \end{cases}$$

$$f(v_k v'_k) = \begin{cases} 3, & if\ k \equiv 1\ (mod\ 3) \\ 2, & if\ k \equiv 2\ (mod\ 3) \\ 1, & if\ k \equiv 0\ (mod\ 3) \end{cases}$$

$$f(v'_k v_{k+1}) = \begin{cases} 1, & if\ k \equiv 1\ (mod\ 3) \\ 3, & if\ k \equiv 2\ (mod\ 3) \\ 2, & if\ k \equiv 0\ (mod\ 3) \end{cases}$$

It is clear that the above rule of total coloring, the graph $M(T_n)$ is properly total colored with 7colors. Hence the total chromatic number of the middle graph of twig graph $M(T_n)$, $\chi''(M(T_n) = \Delta(M(T_n)) + 1$.

Theorem 3.2. Let $M(P_n^+)$ be the middle graph comb graph. Then the total chromatic number of middle graph of comp graph is $\Delta(M(P_n^+)) + 1$.

Proof. Let $V(P_n^+) = \{u_k, v_k : 1 \le k \le n\}$ and $E(P_n^+) = \{u_k v_k : 1 \le k \le n\} \cup \{v_k v_{k+1} : 1 \le k \le n - 1\}$. Now we construct the middle graph of comb graph, each edge $\{u_k v_k : 1 \le k \le n\}$ and $\{v_k v_{k+1} : 1 \le k \le n - 1\}$ are subdivided by the vertices $\{u'_k : 1 \le k \le n\}$ and $\{v'_k : 1 \le k \le n - 1\}$ in $M(P_n^+)$. The vertex set and the edge set of middle graph of comb graph is given by

$$V(M(P_n^+)) = \{u_k, v_k, u'_k : 1 \le k \le n\} \cup \{v'_k : 1 \le k \le n - 1\}$$

$$E(M(P_n^+)) = \{v_k v_{k+1}, v'_k u_{k+1}, u'_k v'_k : 1 \le k \le n - 1\} \cup \{v'_k v'_{k+1} : 1 \le k \le n - 2\} \cup \{v_k u'_k,\ u_k u'_k : 1 \le k \le n\}.$$

We define the total coloring f, such that $f : S \to C$, where $S = V(M(P_n^+)) \cup E(M(P_n^+))$ and the set of colors C = {1, 2, 3, 4, 5, 6, 7}. Assign total coloring to these vertices and edges as follows.

For $1 \le k \le n$

$$f(v_k) = \begin{cases} 1, & if\ k\ is\ odd \\ 2, & if\ k\ is\ even \end{cases}$$

$$f(u_k) = 6,\ f(u'_k) = 3,\ f(v_k u'_k) = 2,\ f(u_k u'_k) = 5,$$

For $1 \le k \le n - 1$

$$f(v'_k) = \begin{cases} 4, & if\ k\ is\ odd \\ 6, & if\ k\ is\ even \end{cases}$$

$$f(v'_k u_{k+1}) = 6,\ f(u'_k v'_k) = 7,\ f(v_k v'_k) = 3,\ f(v'_k v_{k+1}) = 5,$$

For $1 \le k \le n - 2$

$$f(v'_k v'_{k+1}) = \begin{cases} 2, & if\ k\ is\ odd \\ 1, & if\ k\ is\ even \end{cases}$$

Using the above pattern of total coloring, the graph $M(P_n^+)$ is properly total colored with 7 colors. Hence the total chromatic number of the middle graph of comb graph $M(P_n^+)$, $\chi''(M(P_n^+)) = \Delta(M(P_n^+)) + 1$.

Theorem 3.3. Let $T(T_n)$ be the total coloring of twig graph. Then the total chromatic number of total graph of twig graph is $\Delta(T(T_n)) + 1$.

Proof. Let $V(T_n) = \{u_k, w_k : 1 \le k \le n-2\} \cup \{v_k : 1 \le k \le n\}$ and

$E(T_n) = \{u_k u_{k+1},\ w_k v_{k+1} : 1 \le k \le n-2\} \cup \{v_k v_{k+1} : 1 \le k \le n-1\}$.

Now we construct the middle graph of twig graph, each edge $\{u_k v_{k+1} : 1 \le k \le n-2\}, \{v_k v_{k+1} : 1 \le k \le n-1\}$ and $\{w_k v_{k+1} : 1 \le k \le n-2\}$ are subdivided by the vertices $\{u'_k : 1 \le k \le n-2\}$, $\{v'_k : 1 \le k \le n-1\}$ and $\{w'_k : 1 \le k \le n-2\}$ in $M(T_n)$.The vertex set and the edge set of middle graph of twig is given by

$$V\big(T(T_n)\big) = \{u_k, u'_k : 1 \le k \le n-2\} \cup \{v_k, w_k, v'_k, w'_k : 1 \le k \le n\}.$$

$$E\big(T(T_n)\big) = \{u'_k v'_{k+1},\ v'_k w'_k, u_k u'_k, w_k w'_k, u'_k v'_k, u'_k w'_k, v_{k+1} u'_k,\ v'_k v'_{k+1}, v_k v'_{k+1},\ v_{k+1} w'_k,$$
$$v'_k w'_{k+1}, u_k v_{k+1} : 1 \le k \le n-2\} \cup \{v_k v'_k, v'_k v_{k+1}, v_k v_{k+1} : 1 \le k \le n-1\}.$$

We define the total coloring f, such that $f : S \to C$, where $S = V(T(T_n)) \cup E(T(T_n))$ and C set of colors $C = \{1, 2, \ldots, 9\}$. Assign total coloring to these vertices and edges as follows.

For $1 \le k \le n$

$$f(v_k) = \begin{cases} 1, & if\ k \equiv 1\ (mod\ 3) \\ 3, & if\ k \equiv 2\ (mod\ 3) \\ 2, & if\ k \equiv 0\ (mod\ 3) \end{cases}$$

For $1 \le k \le n-2$

$$f(w_k) = 7, \quad f(u'_k) = 8, \quad f(u_k) = 9,\ f(w'_k) = 6,$$

$$f(u_k u'_k) = 7,\ f(u'_k v'_{k+1}) = 5,\ f(u'_k v'_k) = 4,$$

$$f(u'_k w'_k) = 3,\ f(u'_k v_{k+1}) = 6,\ f(v_{k+1} w'_k) = 7,$$

$$f(v'_k w'_k) = 8,\ f(v'_{k+1} w'_k) = 9,\ f(w_k w'_k) = 5,$$

$$f(u_k v_{k+1}) = 8,\ f(v_{k+1} w_k) = 9,$$

$$f(v'_k v'_{k+1}) = \begin{cases} 6, & if\ k\ is\ odd \\ 7, & if\ k\ is\ even \end{cases}$$

For $1 \le k \le n-1$

$$f(v'_k) = \begin{cases} 2, & if\ k \equiv 1\ (mod\ 3) \\ 1, & if\ k \equiv 2\ (mod\ 3) \\ 3, & if\ k \equiv 0\ (mod\ 3) \end{cases}$$

$$f(v_k v_{k+1}) = \begin{cases} 4, & if\ k\ is\ odd \\ 5, & if\ k\ is\ even \end{cases}$$

$$f(v_k v'_k) = \begin{cases} 3, & if\ k \equiv 1\ (mod\ 3) \\ 2, & if\ k \equiv 2\ (mod\ 3) \\ 1, & if\ k \equiv 0\ (mod\ 3) \end{cases}$$

$$f(v'_k v_{k+1}) = \begin{cases} 1, & if\ k \equiv 1\ (mod\ 3) \\ 3, & if\ k \equiv 2\ (mod\ 3) \\ 2, & if\ k \equiv 0\ (mod\ 3) \end{cases}$$

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1809-1814
https://publishoa.com
ISSN: 1309-3452

It is clear that the above procedure of total coloring, the graph $T(T_n)$ is properly total colored with 9 colors. Hence the total chromatic number of the total graph of twig graph $T(T_n)$, $\chi''(T(T_n) = \Delta(T(T_n)) + 1$.

Theorem 3.4. The total chromatic number of total graph of comp graph is $\Delta(T(P_n^+)) + 1$.

Proof. Let $V(P_n^+) = \{u_k, v_k : 1 \le k \le n\}$ and $E(P_n^+) = \{u_k v_k : 1 \le k \le n\} \cup \{v_k v_{k+1} : 1 \le k \le n - 1\}$. Now we construct the total graph of comb graph, each edge $\{u_k v_k : 1 \le k \le n\}$ and $\{v_k v_{k+1} : 1 \le k \le n - 1\}$ are subdivided by the vertices $\{u'_k : 1 \le k \le n\}$ and $\{v'_k : 1 \le k \le n - 1\}$ in $T(P_n^+)$. The vertex set and the edge set of total graph of comb graph is given by

$$V(T(P_n^+)) = \{u_k, v_k, u'_k : 1 \le k \le n\} \cup \{v'_k : 1 \le k \le n - 1\}$$

$$E(T(P_n^+)) = \{v_k v_{k+1}, v'_k u_{k+1}, v'_k v_{k+1}, u'_k v'_k : 1 \le k \le n - 1\} \cup$$
$$\{v'_k v'_{k+1} : 1 \le k \le n - 2\} \cup \{v_k u'_k, u_k u'_k, u_k v_k : 1 \le k \le n\}.$$

We define the total coloring f, such that $f : V(T(P_n^+)) \cup E(T(P_n^+)) \to C$, where C is the set of colors $C = \{1, 2, \dots, 7\}$. Assign total coloring to these vertices and edges as follows.

For $1 \le k \le n$

$$f(v_k) = \begin{cases} 1, & if\ k \equiv 1\ (mod\ 3) \\ 3, & if\ k \equiv 2\ (mod\ 3) \\ 2, & if\ k \equiv 0\ (mod\ 3) \end{cases}$$

$$f(u_k) = 4, \quad f(u'_k) = 7, \quad f(v_k u'_k) = 6,$$

$$f(u_k u'_k) = 3, \quad f(u_k v_k) = 7.$$

For $1 \le k \le n - 1$

$$f(v_k u'_k) = \begin{cases} 3, & if\ k \equiv 1\ (mod\ 3) \\ 2, & if\ k \equiv 2\ (mod\ 3) \\ 1, & if\ k \equiv 0\ (mod\ 3) \end{cases}$$

$$f(v'_k) = \begin{cases} 2, & if\ k \equiv 1\ (mod\ 3) \\ 1, & if\ k \equiv 2\ (mod\ 3) \\ 3, & if\ k \equiv 0\ (mod\ 3) \end{cases}$$

$$f(v'_k v_{k+1}) = \begin{cases} 1, & if\ k \equiv 1\ (mod\ 3) \\ 3, & if\ k \equiv 2\ (mod\ 3) \\ 2, & if\ k \equiv 0\ (mod\ 3) \end{cases}$$

$$f(v_k v_{k+1}) = \begin{cases} 4, & if\ k\ is\ odd \\ 5, & if\ k\ is\ even \end{cases}$$

$$f(u'_k v'_k) = 4, \quad f(v'_k u'_{k+1}) = 5,$$

$$f(v'_k v'_{k+1}) = \begin{cases} 6, & if\ k\ is\ odd \\ 7, & if\ k\ is\ even \end{cases}$$

It is clear that the above rule of total coloring, the graph $T(P_n^+)$ is properly total colored with 5

colors. Hence the total chromatic number of the total graph of comb graph $T(P_n^+)$, $\chi''(T(P_n^+)) = \Delta(T(P_n^+)) + 1$.

JOURNAL OF ALGEBRAIC STATISTICS
Volume 13, No. 3, 2022, p. 1809-1814
https://publishoa.com
ISSN: 1309-3452

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