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Prime Radicals in Right Duo Seminearrings

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Abstract

In this paper the notions of radical, prime radical and quasi prime radical are extended to seminearrings and the results analogue to those on prime radical and quasi prime radical are presented. We also obtained the class of right duo seminearrings in various characteristic in terms of ideals.

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1. Introduction

B. Van Rootselaar, V.G. Van Hoorn introduced a seminearring, by combining the concepts of algebraic structures say nearring and semiring, in 1962 [1, 9]. If $(S,+)$ is a semigroup and $(S,.)$ is another semigroup, then the triplet $(S,+,.)$ with double binary operations called addition and multiplication with the right(left) distributive law referred to as right(left) seminearring. Semigroup mapping, linear sequential machines, secure communication system, and more uses of seminearrings may be found in [2]. The whole collection of semigroup mappings $(\Gamma,+)$ with absorbance zero, which refers to point-by-point mapping addition and composition of the form $\mathfrak{M}(\Gamma)$ [7]. The set of all mappings $\mathfrak{M}(\Gamma)$ is a natural example of seminearrings [8].

In the study of nearrings, the concept of prime ideals and related radicals is crucial. The correlation between different forms of prime radicals and prime ideals in nearrings was totally demonstrated by Birkenmeier, Heatherly, and Lee [6]. In this paper, we define various prime ideals of seminearrings and obtain the interrelations among them.

2. Preliminaries

This section compiles all of the seminearring theory terms that we utilize in our research.

A subset I (non empty) of a seminearring $(S,+,.)$ is called a left (right) ideal of S if: $(i) x+y \in I$ for all $x,y \in I$, $(ii) a.x \in I (x.a \in I)$ for all $x \in I$ and $a \in S$. If an ideal I is called a two-sided ideal if it is a left and right ideal of S [4, 5]. An ideal I of S is called a proper ideal of S if I is a proper subset of S. A seminearring S is called simple, if it has no proper ideals and it is called semi-simple if every ideal of S is idempotent. We define the radical $\sqrt{A}$ of A to be $\{a \in S / a^k \in A$ for some integer k greater or equal to 1$\}$ for $A \subseteq S$ and clearly, $A \subseteq \sqrt{A}$. A seminearring S is P-semi-simple if $\sqrt{S}=0$. An ideal I of S is called (i) nilpotent, when $I^m=0$ for some integer m which greater than or equals 1 (ii) nil ideal if for every $a \in I$ is nilpotent (iii) idempotent, if $I^2=I$ [3]. If every right (left) ideal of S is two sided,

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is said to be right (left) duo [3, 10]. When an ideal P of S is prime, $AB \subseteq P \Rightarrow A \subseteq B$ or $B \subseteq P$ for ideals A, B. When ideals A, B of S is irreducible, $P = A \cap B$ gives either $P = A$ or $P = B$. An ideal is totally irreducible if $P = \cup A_\alpha$ implies $P = A_\alpha$ for $\alpha \in I$ of S. A subset of S is denoted by K has atleast one element is referred to as i) M-system if $K \cap B \neq \varnothing$, $K \cap A \neq \varnothing$ (A, B ideals of S). ii) QM-system, if $K \cap A \neq \varnothing$ implies $K \cap A^2 \neq \varnothing$ (A, an ideal of S). Any subseminearring T of S is clearly seen to be an M-system and every M-system is clearly a QM-system. In general, an M-system need not be a seminearring. $r(A) = \cap \{P_i | P_i\ a\ prime\ ideal\ containing\ A\}, qr(A) = \cap \{Q_i | Q_i\ a\ quasi - prime\ ideal\ containing\ A\}$.

3. Main Results

An ideal P of S is referred to as quasi-prime, if for any ideal A of S, $A^2 \subseteq P$ implies $A \subseteq P$ and we call the seminearring S as quasi-prime ideal seminearring, if every ideal of S is quasi-prime. Clearly every prime ideal is quasi-prime. The converse, however, is not true, as illustrated by the following example.

Example 3.1 For any integers $n > 0$ and r such that $1 < r \leq n$ the ideal $I = X_1^{a_1} X_2^{a_2} \dots X_n^{a_n}$ $(X_1, X_2, \dots X_n)$ is a prime ideal, in the polynomial seminearring $\mathrm{R}[\mathrm{X}_1, \mathrm{X}_2, \dots \mathrm{X}_\mathrm{n}]$, which is not quasi prime.

Example 3.2 Another example is let D be an integral domain with atleast two maximal ideals. Let R be the quotient seminearring of D, X an indeterminate over K and let $I = D + XR[X]$. Then XI is not prime ideal but that is quasi prime ideal.

While $\mathrm{r(A)}$ and $\mathrm{qr(A)}$ are ideals of S, $\sqrt{\mathrm{A}}$ need not be an ideal of S as shown by the following example.

Example 3.3 Consider the seminearring S and take the ideal $6Z$. Since the $\sqrt{mZ} = rZ$ where $r = \Pi_{p|m}$ and p is a prime number. Here $\sqrt{6Z} = 6Z$ but $6Z$ is not a prime ideal since 6 is not prime.

Lemma 3.1 Any intersection of prime (quasi-prime) ideals of R is quasi-prime.

Proof. Let I and J be two ideals. The intersection of two ideals I and J always an ideal, regardless of whether I and J are prime or not. If x, y in $I \cap J$ gives $x + y$ in I and $x + y$ in J and hence $x + y$ in $I \cap J$.

Similarly if $x \in I \cap J$ and $rx \in I$, $rx \in J$. Since I and J are ideals, hence $rx \in I \cap J$. If R is commutative then this is enough to show that $I \cap J$ is an ideal and if R is not commutative then use the same argument for the right multiplication. (for non-commutative right ideal assuming that ideal means two sided ideal).

Lemma 3.2 An ideal P in S is prime iff it is QP and irreducible.

Proof. Clearly P be a both prime and irreducible. To prove the converse, assume that an ideal P is quasi-prime and irreducible. Let $AB \subseteq P$ (A, B ideals of S). If we set, $C = (A \cup P) \cap (B \cup P) = (A \cap B) \cup P$, then $C^2 = [(A \cap B) \cup P]^2 \subseteq (A \cap B)^2 \cup P \subseteq AB \cup P \subseteq P$. But P is quasi-prime and so $C \subseteq P \subseteq (A \cup P) \cap (B \cup P) = C$. Thus $P = (A \cup P) \cap (B \cup P)$ and since P is irreducible, we have $P = A \cup P$ or $P = B \cup P$. Hence A is a subset of P or B is a subset of P and so P is prime. This proves the result.

Lemma 3.3 Every ideal of A of S is the intersection of totally irreducible ideals of S containing A.

Proof. When $A = S$, the result is obvious. Suppose A is proper, then there exists an element $x \notin A$. Let the collection of union of all ideals of S be M not containing x containing A and not containig x. We claim that M is totally irreducible. For, suppose $M = \cap K_\alpha$, where K_α are ideals of S. Then there is atleast one α such that $x \notin K_\alpha$. Hence $K_\alpha \subseteq M$. But $M \subseteq K_\alpha$, so that $M = K_\alpha$. Obviously $A \subseteq \cap T_i$, where T_i is a totally irreducible ideal containing A. If $A \subset \cap T_i$, then by the above argument, we can find a totally irreducible ideal containing A and not containing $\cap T_i$, which is a contradiction. Hence the result.

Corollary 3.1 Every ideal A of S is the intersection of irreducible ideals of S that contain A.

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Proof. Since a totally irreducible ideal is irreducible, the result follows from Lemma 3.3

Proposition 3.1 An ideal A of S is QP if and only if $A = qr(A)$.

The following lemma gives a characterisation of prime termed as quasi-prime ideals M-systems $(QM$-systems).

Lemma 3.4 An ideal P if S is [(i)]

1. P' a M-system if and only if S is Prime.

2. Quasi-prime if and only if P' is a QM-system.

Proof. [(i)]

1. Let P be a prime. Two ideals A, B in S. $P' \cap A \neq \varnothing$ and $P' \cap B \neq \varnothing$, then $A \nsubseteq P$ and $B \nsubseteq P$. Hence $AB \nsubseteq P$ so that $P' \cap AB \neq \varnothing$. Therefore P' is an M-system. The converse follows similarity.

2. we can prove from (i).

Proposition 3.2 Let A be an ideal of S and K an M-system disjoint with A. Then A is contained in an ideal P, maximal with respect to the property of not meeting K. Further P is prime.

Proof. The set P which is union of all ideals of S disjoint with K is clearly an ideal containing A and is maximal with respect to the property of not meeting K. Now we claim that prime P. For, let $B \nsubseteq P$ and $C \nsubseteq P$ $(B, C$ ideals of $S)$. Then $K \cap B \neq \varnothing$ and $K \cap C \neq \varnothing$ and K being an M-system, we have $K \cap BC \neq \varnothing$. Suppose $BC \subseteq P$. Then $K \cap BC \subseteq K \cap P = \varnothing$, which contradicts our assumption, which proves P is prime.

Proposition 3.3 For any ideal A of S, we have: [(i)]

1. $r(A) = \{x \in S \mid every\ M - system\ containing\ x\ meets\ A\}$

2. $qr(A) = \{x \in S \mid every\ QM - system\ containing\ x\ meets\ A\}$

Proof. [(i)]

1. If the set $A^{(M)} = \{x \in S \mid every\ M - system\ containing\ x\ meets\ A\}$, then clearly $A^{(M)} \subseteq r(A)$. Conversely, suppose that $x \notin A^{(M)}$. To prove the result, it is enough to find a prime ideal containing A and not containing x. Since $x \notin A^{(M)}$, there exists an M-system K such that K contains x and $K \cap A = \varnothing$. Hence by Proposition 3.1 in [10], there exists a prime ideal P containing A and not meeting K. Thus $x \notin P$ and this completes the proof.

2. we can prove from (i).

Proposition 3.4 An ideal is prime in a right duo seminearring if and only if it is totally prime.

Proposition 3.5 A of S an ideal, $r(A) \subseteq \sqrt{A}$. when S is right duo, then $r(A) = \sqrt{A}$.

Proof. Let $x \in r(A)$ and $K = \{x, x^2, \ldots\}$ be the subseminearring generated by x in S. Evidently K is an M-system containing x and so $K \cap A \neq \varnothing$ (by Proposition 3.3 (i)). Hence $x^n \in A$, for some integer $n \geq 1$ and so $x \in \sqrt{A}$. (i.e.) $r(A) \subseteq \sqrt{A}$. To prove the equality, assume now that S is a right duo seminearring.

$x \in \sqrt{A} \Rightarrow x^n \in A$, for some integer $n \geq 1$.

$\Rightarrow x^n \in P_i$, for every prime ideal $P_i \supseteq A$.

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$\Rightarrow x \in P_i$, since each P_i is also totally prime (by Proposition 3.4).

$\Rightarrow x \in \cap P_i = r(A)$.

Thus $\sqrt{A} \subseteq r(A)$ and so $r(A) = \sqrt{A}$, as required.

We next proceed to prove the equality of $r(A)$ and $qr(A)$ for any ideal A of S the proof of which requires the following results.

Lemma 3.5 If R is any QM-system in S then for each $a \in R$, we can find an M-system K in S such that $a \in K$ and $K \subseteq R$.

Proof. Since $a_1 = a \in R$, $R \cap (a_1) \neq \varnothing$ and hence, R is being a QM-system, we have $R \cap (a_1)^2 \neq \varnothing$. Choose $a_2 \in R \cap (a_1)^2$. Then $R \cap (a_2)^2 \neq \varnothing$. Thus proceeding, we can find $a_{n+1} \in R \cap (a_n)^2, n \geq 2$. We claim that $K = \{a_1, a_2, \ldots\}$ is the required M-system. To see this, let $K \cap A \neq \varnothing$ and $K \cap B \neq \varnothing$ $(A, B$ ideals of $R)$.

Suppose $a_m \in A$ and $a_n \in B$. Then for $k = max\{m, n\}$, we have $a_{k+1} \in R \cap (a_k)^2 \subseteq (a_k)^2 \subseteq (a_m)(a_n) \subseteq AB$.

Hence $K \cap AB \neq \varnothing$, proving that K is an M-system. Since $K \subseteq R$, the proof is complete.

Theorem 3.1 For any ideal A of S, $r(A) = qr(A)$.

Proof. Clearly, $qr(A) \subseteq r(A)$. Let $a \in r(A)$ and consider any QM-system R containing a. Then by Lemma 3.5 we can find an M-system K such that $a \in K$ and $K \subseteq R$. Since $a \in r(A)$, K meets A (by Proposition 3.3(i)). Hence R meets A and so $a \in qr(A)$ (by Proposition 3.3(ii)). Thus $r(A) \subseteq qr(A)$ and this completes the proof.

Corollary 3.2 *S said to be a seminearring of quasi-prime ideal iff $A = r(A)$ for every A in S.*

Proof. This followed by the Theorem 3.1 and Proposition 3.1

The above Corollary, combined with Proposition 3.5 gives the following corollary.

Corollary 3.3 A right duo-seminearring S be a seminearring ideal of quasi-prime iff $A = \sqrt{A}$, for every A of S. For any ideal A, we call $r(A) = qr(A)$, the prime radical of A. When the seminearring S contains 0, we write $rad\ S = r(0)$ and call it the prime radical of S. We call a seminearring S with 0 as P-semi-simple if $rad\ S = 0$.

In the rest of this section S will denote a seminearring with 0.

Proposition 3.6 For any seminearring S, $rad\ S$ is a nil ideal. If S is right duo, then $rad\ S$ is precisely the set of all the nilpotent elements of S.

Proof. By Proposition 3.5, $rad\ S = r(0) \subseteq \sqrt{(0)}$ $(=$ the set of the nilpotent elements of $S)$ and $rad\ S = \sqrt{(0)}$, if S is right duo.

Proposition 3.7 A P-semi-simple seminearring S has no nilpotent ideals with a non-zero value. Conversely if S has no nilpotent ideals with a non-zero value and further S is right duo, then S is P-semi-simple.

Proof. Suppose A is a nilpotent of S, so that $A^m = 0$, for some integer $m \geq 1$. Then $A^m \subseteq r(0)$ and since $r(0)$ is quasi-prime, we get $0 \neq A \subseteq r(0) = rad\ S = 0$ (as S is P-semi-simple).

This is a contradiction and so S has no nilpotent ideals with a non-zero value.

Suppose now that S is right duo and has no nilpotent ideals with a non-zero value. If possible, let $a(\neq 0) \in rad\ S$ then $a^n = 0$, for some integer $n \geq 1$. Hence $(a)^n = 0$ (S being right duo) and so (a) is nilpotent ideals with a non-zero value of S. This contradiction shows that $rad\ S = 0$. Hence S is P-semi-simple, as required.

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Corollary 3.4 A semi-simple right duo seminearring S is P-semi-simple.

Proof. Since S is semi-simple, every ideals A of S is idempotent and so S has no non-zero nilpotent ideal. Hence S is P-semi-simple, by Proposition 3.7.

Theorem 3.2 The prime radical $rad\ S$ of S is precisely the collection of the nilpotent strong elements in S.

Proof. Let $a_0 \in S$ such that $a_0 \notin rad\ S$. Then a prime ideal P of S will exists such as $a_0 \notin P$. Hence P' is an M-system (by lemma 3.4(i)). Since $a_0 \in P'$, $P' \cap (a_0) \neq \varnothing$ and so $P' \cap (a_0)^2 \neq \varnothing$. Choose $a_1 \in P' \cap (a_0)^2$. Then $P' \cap (a_1)^2 \neq \varnothing$. Thus proceeding, we can find $a_{n+1} \in P' \cap (a_n)^2, n \geq 1$. We hence obtain a sequence $\{a_0, a_1, \dots\}$ with $a_{n+1} \in (a_n)^2$. Since $0 \in P'$, it follows that no a_i in this sequence can be 0. Thus a_0 is not quite nilpotent.

Conversely, let $a \in S$ is not strongly nilpotent. Then there exists a sequence $L = \{a_0, a_1, \dots\}$ with $a_0 = a$ and $a_{n+1} \in (a_n)^2$ such that $0 \notin L$. Hence $0 \cap L = \varnothing$. Let P be the union of every ideals of S disjoint with L. Evidently $a \notin P$. We now claim that P is prime. Let A, B be two ideals of S such that $A, B \nsubseteq P$. Then $A \cap L \neq \varnothing$ and $B \cap L \neq \varnothing$. Let $a_m \in A$ and $a_n \in B$. If $k = max\{m, n\}$ then $a_{k+1} \in (a_k)^2 \subseteq (a_m)(a_n) \subseteq AB$ and $a_{k+1} \notin P$, as $P \cap L = \varnothing$. Hence $AB \nsubseteq P$ and so P is prime. But $a \notin P$, so that $a \notin rad\ S$. This proves the theorem.

Conclusion

Even in the hypothesis of the nearring, a right ideal is not the same as a left ideal. This heightens the need to learn more about a seminearring of this calibre. In this paper, certain fruitful outcomes of seminearring of the right pair that do not follow the aforementioned rule are observed.

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