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Canonical Transform for Pan Integrable Boehmians

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ABSTRACT

In this paper, pan-integral is used to address integrable Boehmians for canonical sine and cosine transforms, as well as their properties and convolution theorems.

Key Words: Canonical transform; Canonical sine transform; Canonical cosine transform; Convolution; Boehmians; pan-integral.

1 Introduction

The linear canonical transform (LCT), a generalisation of the Fourier transform (FT), has a wide range of applications in many fields, including signal processing and optics. We refer in [5], [6], [7]. A new convolution structure for the LCT and its property of translation invariance are introduced by [1], convolution theorems [2] and integrable Boehmians [4]. Yang et al. introduced the pan-integral and discussed its properties in [16]. Pan-integral research should be pursued further by Wang et al., Yang and Song [14], [15], [16]. Recent research has looked into the links between pan integrals and other integrals were in [3], [10], [12], [11], [13].
From [17], Let $\oplus$ be a binary operation on $\bar{R}_+$ (where $\bar{R}_+ = [0, \infty]$). The pair $(\bar{R}_+, \oplus)$ is called a commutative isotonic semigroup and $\oplus$ is called a pan-addition on $\bar{R}_+$ iff $\oplus$ satisfies the following requirements:

(1) $a \oplus b = b \oplus a$;
(2) $(a \oplus b) \oplus c = a \oplus (b \oplus c)$;
(3) $a \leq b \implies a \oplus c \leq b \oplus c$ for any c;

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(4) a $\oplus$ 0 = a;
(5) $\lim_n a_n$ and $\lim_n b_n$ exist $\implies$ $\lim_n(a_n \oplus b_n)$ exists, and
$\lim_n(a_n \oplus b_n) = lim_n a_n \oplus lim_n b_n$.

From (1)and (3), we have
a $\leq$ b and c $\leq$ d $\implies$ a $\oplus$ c $\leq$ b $\oplus$ d.

Let $\odot$ be a binary operaion on $\bar{R}_+$. The triple ($\bar{R}_+$, $\oplus$, $\odot$), where $\oplus$ is a pan-addition on $\bar{R}_+$, is called a commutative isotonic semiring with respect to $\oplus$ and $\odot$ iff $\odot$ satisfies the following requirements:

(1) a $\odot$ b = b $\odot$ a;
(2) (a $\odot$ b) $\odot$ c = a $\odot$ (b $\odot$ c);
(3) (a $\oplus$ b) $\odot$ c = (a $\odot$ c) $\oplus$ (b $\odot$ c);
(4) a $\leq$ b $\implies$ a $\odot$ c $\leq$ b $\odot$ c for any c;
(5) a $\neq$ 0 and b $\neq$ 0 $\Leftrightarrow$ a $\odot$ b $\neq$ 0;
(6) there exists I ϵ $\bar{R}_+$, such that I $\odot$ a = a, for any a ϵ $\bar{R}_+$;
(7) $\lim_n a_n$ and $\lim_n b_n$ exist and are finite $\implies$ $\lim_n(a_n \odot b_n) = lim_n a_n \odot lim_n b_n$.

The operation $\odot$ is called a pan-multiplication on $\bar{R}_+$, and the number I is called the unit element of ($\bar{R}_+$, $\oplus$, $\odot$)
From (1) and (4), we derive
a $\leq$ b and c $\leq$ d $\implies$ a $\odot$ c $\leq$ b $\odot$ d.
It is easy to see that (5) implies that a $\odot$ 0 = 0 and 0 $\odot$ a = 0 for any a ϵ $\bar{R}_+$.

If (X, $\mathscr{F}, \mu$) (where X$-$ Universe of discourse, $\mathscr{F}-$ σ-ring, $\mu-$ Set function) is a fuzzy measure space, and ($\bar{R}_+$, $\oplus$, $\odot$) is a commutative isotonic semiring, then (X, $\mathscr{F}$, μ, $\bar{R}_+$, $\oplus$, $\odot$) is called a pan-space.

Let $f_p \in \mathbf{F}$ (where **F** is the set of finite non-negative measurable functions on a measurable space (X, $\mathscr{F}$)) and A ϵ $\mathscr{F}$. The pan-integral of f_p on A with respect to μ, which is denoted by $(p)\int_X f_p d\mu$, is given by

$$(p)\int_A f_p d\mu = \sup_{0 \leq s \leq f_p, s \in S} P(s|A),$$

The set of all pan-simple measurable functions is denoted by S.

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Let μ be additive, When $\oplus$ is the common addition and $\odot$ is the common multiplication, we have

$$(\mathrm{p}) \int_A f_p d\mu = \int_A f_p d\mu$$

for any $f_p \in \mathbf{F}$ and A $\in \mathscr{F}$; that is, the pan-integral and Lebesgue's integral is coincide.
In this article, pan integral depends only on usual addition $(+)$ and usual multiplication $(.)$ instead of pan-addition $(\oplus)$ and pan-multiplication $(\odot)$.

Consider $\mathcal{L}^1_{pan}(\mathcal{R})$ space of complex absolutely pan integrable functions with $\|f_p\| = (p)\int_{\mathcal{R}} |f_p(t)|dt$ in usual manner. Linear Canonical transform for pan integrable is defined by

$$\mathcal{L}_{pan}(v) = \mathcal{L}_{pan}\{f_p(t)\}(v) = \begin{cases} \frac{1}{\sqrt{2\pi \mathrm{ib}}} e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2} (p) \int_{-\infty}^{\infty} e^{\frac{-i}{b}vt} e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f_p(t)dt, & \text{if } b\neq 0 \\ \sqrt{d} e^{\frac{i}{2}\mathrm{cdv}^2} f_p(dv), & \text{if } b=0 \end{cases}$$

where $\mathcal{L}_{pan}$ denote canonical pan operator with linearity. Using the above definition we can derive Canonical cosine pan and canonical sine pan transforms. It is defined as follows.

Definition 1.1 The canonical cosine pan transform (CCPT) of $f_p \in \mathcal{L}^1_{pan}(\mathcal{R})$ for b $\neq$ 0, is defined by $\mathcal{C}_{pan}(f_p(t))(v) = \frac{1}{\sqrt{2\pi \mathrm{ib}}} e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2} (p) \int_{-\infty}^{\infty} cos(\frac{v}{b}t) e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f_p(t)dt$ where $\mathcal{C}_{pan}$ is CCPT operator.

Definition 1.2 The canonical sine pan transform (CSPT) of $f_p \in \mathcal{L}^1_{pan}(\mathcal{R})$ for b $\neq$ 0, is defined by $\mathcal{S}_{pan}(f_p(t))(v) = \frac{1}{\sqrt{2\pi \mathrm{ib}}} e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2} (p) \int_{-\infty}^{\infty} sin(\frac{v}{b}t) e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} f_p(t)dt$ where $\mathcal{S}_{pan}$ is CSPT operator.

Lemma 1.1 If $f_p \in \mathcal{L}^1_{pan}(\mathcal{R})$, then $\mathcal{C}_{pan}(f_p(t))$ and $\mathcal{S}_{pan}(f_p(t))$ are in $\mathcal{L}^1_{pan}(\mathcal{R})$.
Proof. Consider,

$$\|\mathcal{C}_{pan}(f_p(t))\|_1 = |(p) \int_{\mathcal{R}} \mathcal{C}_{pan}(f_p(t))dt|$$

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$$\leq (p)\int_{\mathcal{R}} |\frac{1}{2\pi \mathrm{ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}(p)\int_{-\infty}^{\infty} cos(\frac{v}{b}t)e^{\frac{i}{2}\left(\frac{\mathrm{a}}{\mathrm{b}}\right)t^2} f_p(t)dt|dt$$
$$\leq (p)\int_{\mathcal{R}} |f_p(t)|dt$$
$$= \|f_p\|_1$$

$$\|\mathcal{S}_{pan}(f_p(t))\|_1 = |(p)\int_{\mathcal{R}} \mathcal{S}_{pan}(f_p(t))dt|$$
$$\leq (p)\int_{\mathcal{R}} |\frac{1}{2\pi \mathrm{ib}} e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}(p)\int_{-\infty}^{\infty} sin(\frac{v}{b}t)e^{\frac{i}{2}\left(\frac{\mathrm{a}}{\mathrm{b}}\right)t^2} f_p(t)dt|dt$$
$$\leq (p)\int_{\mathcal{R}} |f_p(t)|dt$$
$$= \|f_p\|_1$$

Definition 1.3 A sequence (ψ_{p_n}) in $\mathcal{L}^1_{pan}(\mathcal{R})$ is called pan-delta sequence if

(1) $(p)\int_{\mathcal{R}} e^{\left\{\frac{\mathrm{ia}}{\mathrm{b}}\right\}tx}\psi_{p_n}(t)dt = 1, \ \forall n \in \mathcal{N}$ and $x \in \mathcal{R}$

(2) $\|\psi_{p_n}\| \leq \mathcal{M}$,

(3) $\lim_{n\to\infty}(p)\int_{|t|>\varepsilon} |\psi_{p_n}(t)|dt.$ for every $\varepsilon > 0$

The collection pan-delta sequences is indicated by $\blacktriangle_0$.

Example 1.1 For $x \in \mathcal{R}$, show that

$$\delta_{p_n}(t) = \begin{cases} e^{\left\{\frac{-\mathrm{ia}}{\mathrm{b}}\right\}tx} n^2 t, & \text{if } 0 \leq t \leq \frac{1}{n} \\ e^{\left\{\frac{-\mathrm{ia}}{\mathrm{b}}\right\}tx} n^2(\frac{2}{n} - t), & \text{if } \frac{1}{n} \leq t \leq \frac{2}{n} \\ 0, & \text{otherwise} \end{cases}$$

is a member of $\blacktriangle_0$

Solution: For $x \in \mathcal{R}$,

1 2

$$(1)\ \ (p)\int_{\mathcal{R}} \delta_{p_n}(t)dt = (p)\int_0^{\frac{1}{n}} e^{\left\{\frac{\mathrm{ia}}{\mathrm{b}}\right\}tx} e^{\left\{\frac{-\mathrm{ia}}{\mathrm{b}}\right\}tx} n^2 t dt + (p)\int_{\frac{1}{n}}^{\frac{2}{n}} e^{\left\{\frac{\mathrm{ia}}{\mathrm{b}}\right\}tx} e^{\left\{\frac{-\mathrm{ia}}{\mathrm{b}}\right\}tx} n^2\left(\frac{2}{n} - t\right)dt = 1$$

$$(2)\ \ \|\delta_{p_n}\| = (p)\int_0^{\frac{1}{n}} |n^2 t| dt + (p)\int_{\frac{1}{n}}^{\frac{2}{n}} \left|n^2\left(\frac{2}{n} - t\right)\right| dt \le \mathcal{M} \text{ for some } \mathcal{M} \in \mathcal{R}\text{ , and } \forall n \in \mathcal{N}$$

$$(3)\ \ \lim_{n\to\infty}(p)\int_{|x|>\varepsilon} |\delta_{p_n}(t)| dt = (p)\int_0^{\frac{1}{n}} |n^2 t| dt + (p)\int_{\frac{1}{n}}^{\frac{2}{n}} \left|n^2\left(\frac{2}{n} - t\right)\right| dt = 0, \text{ for } \varepsilon > \tfrac{2}{n}.$$

2 Canonical Cosine Pan Transform

Consider $f_p,\ g_p \in \mathcal{L}^1_{pan}(\mathcal{R})$ and define $\widehat{f_p}(t) = e^{\frac{i}{2}\left(\frac{\mathrm{a}}{\mathrm{b}}\right)t^2} f_p(t)$, $\widehat{g_p}(|x-t|) = e^{\frac{i}{2}\left(\frac{\mathrm{a}}{\mathrm{b}}\right)(x-t)^2} g_p(x-t)$, $\widehat{g_p}(x+t) = e^{\frac{i}{2}\left(\frac{a}{b}\right)(x+t)^2} g_p(x+t)$ and convolution $(f_p * g_p)$ as

$$(f_p * g_p)(t) = (p)\int_{-\infty}^{\infty} \frac{e^{\frac{-i}{2}\left(\frac{\mathrm{a}}{\mathrm{b}}\right)t^2}}{2} \widehat{f_p}(x)\left[\widehat{g_p}(x+t) + \widehat{g_p}(|x-t|)\right]dx.$$

Lemma 2.1 If $f_p,\ g_p \in \mathcal{L}^1_{pan}(\mathcal{R})$, then $\widehat{f_p}$ and $\widehat{g_p}$ are in $\mathcal{L}^1_{pan}(\mathcal{R})$.

Lemma 2.2 If $f_p,\ g_p \in \mathcal{L}^1_{pan}(\mathcal{R})$, then $f_p * g_p \in \mathcal{L}^1_{pan}(\mathcal{R})$.

Lemma 2.3 If $f_p,\ g_p \in \mathcal{L}^1_{pan}(\mathcal{R})$, then $f_p * g_p = g_p * f_p$.

Lemma 2.4 If $f_p,\ g_p,\ h_p \in \mathcal{L}^1_{pan}(\mathcal{R})$, then $f_p * (g_p * h_p) = (f_p * g_p) * h_p$.

Theorem 2.1 (Canonical cosine pan convolution) If $f_p,\ g_p \in \mathcal{L}^1_{pan}(\mathcal{R})$, $F_{p_c}(v)$ and $G_{p_c}(v)$ are the canonical cosine pan transform of $f_p(t)$ and $g_p(t)$, then

$$\mathcal{C}_{pan}[(f_p * g_p)(t)](v) = \sqrt{2\pi\mathrm{ib}}\ e^{\frac{-i}{2}\left(\frac{\mathrm{d}}{\mathrm{b}}\right)v^2} F_{p_c}(v).\ G_{p_c}(v).$$

Proof. For $f_p,\ g_p \in \mathcal{L}^1_{pan}(\mathcal{R})$, we have

$$\mathcal{C}_{pan}[(f_p * g_p)(t)](v) = \frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{\sqrt{2\pi\mathrm{ib}}}(p)\int_{-\infty}^{\infty} cos\left(\frac{v}{b}t\right) e^{\frac{i}{2}\left(\frac{\mathrm{a}}{\mathrm{b}}\right)t^2}$$

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$$\times\left\{(p)\int_{-\infty}^{\infty}\frac{e^{\frac{-i}{2}\left(\frac{a}{b}\right)t^2}}{2}\widehat{f_p}(x)[\widehat{g_p}(x+t)+\widehat{g_p}(|x-t|)]dx\right\}dt$$

$$=\frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{2\sqrt{2\pi ib}}(p)\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}f_p(x)\left\{(p)\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)(x-t)^2}[g_p(x-t)\right.$$

$$\left.+e^{\frac{i}{2}\left(\frac{a}{b}\right)(x+t)^2}g_p(x+t)]cos(\frac{v}{b}t)dt\right\}dx$$

$$=\frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{2\sqrt{2\pi ib}}(p)\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}f_p(x)\ (p)\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)(x-t)^2}g_p(x-t)cos(\frac{v}{b}t)dtdx$$

$$+\frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{2\sqrt{2\pi ib}}(p)\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}f_p(x)\ (p)\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)(x+t)^2}g_p(x+t)cos(\frac{v}{b}t)dtdx=\mathscr{I}_1+\mathscr{I}_2$$

put $x-t=y$ in $\mathscr{I}_1$ and $x+t=y$ in $\mathscr{I}_2$

$$=\frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{2\sqrt{2\pi ib}}(p)\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}f_p(x)\ (p)\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)(y)^2}g_p(y)cos(\frac{v}{b}(y+x))dydx$$

$$+\frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{2\sqrt{2\pi ib}}(p)\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}f_p(x)\ (p)\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)(y)^2}g_p(y)cos(\frac{v}{b}(y-x))dydx$$

$$=\frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{2\sqrt{2\pi ib}}(p)\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}f_p(x)\ (p)\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)(y)^2}[cos(\frac{v}{b}(y+x))+cos(\frac{v}{b}(y-x))]g_p(y)dydx$$

$$=(p)\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}f_p(x)\left\{\frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{2\sqrt{2\pi ib}}\ (p)\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)(y)^2}[2cos(\frac{v}{b}y)cos(\frac{v}{b}x)]g_p(y)dy\right\}dx$$

$$=\frac{\sqrt{2\pi ib}}{e^{\frac{i}{2}\left(\frac{d}{b}\right)(v)^2}}\frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)(v)^2}}{\sqrt{2\pi ib}}(p)\int_{-\infty}^{\infty}e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2}cos(\frac{v}{b}x)f_p(x)G_{p_c}(v)dx$$

$$=\sqrt{2\pi ib}e^{\frac{-i}{2}\left(\frac{d}{b}\right)(v)^2}F_{p_c}(v)G_{p_c}(v).$$

3 Canonical cosine Pan Boehmians

Definition 3.1 [9] A pair of sequences $(f_{p_\ell},\phi_{p_\ell})$ is known as quotient of sequences, denoted by f_{p_ℓ}/ϕ_{p_ℓ}, $f_{p_\ell}\in\mathscr{L}^1_{pan}(\mathscr{R})$, $\phi_{p_\ell}\in\blacktriangle_0$ and $f_{p_m}*\phi_{p_\ell}=f_{p_\ell}*\phi_{p_m}$ for every $\ell,m\in\mathscr{N}$.

Definition 3.2 [9] Two sequences $\{f_{p_\ell}/\zeta_{p_\ell}\}$ and $\{g_{p_\ell}/\phi_{p_\ell}\}$ are *equivalent* if $f_{p_m} * \phi_{p_\ell} = g_{p_\ell} * \zeta_{p_m}$ for all $l, m \in \mathcal{N}$ and $f_{p_\ell}, g_{p_\ell} \in \mathcal{L}^1_{pan}(\mathcal{R})$, $\zeta_{p_\ell}, \phi_{p_\ell} \in \blacktriangle_0$. The equivalence class of quotients described above is known as pan integrable Boehmians. All pan integrable Boehmians space denoted by $\mathscr{P}^*_{L^1} = \mathscr{P}_{L^1}(\mathcal{L}^1_{pan}(\mathcal{R}), \blacktriangle_0, *)$. A function $f_p \in \mathcal{L}^1_{pan}(\mathcal{R})$ can be diagnosed with the Boehmian is $[f_{p_\ell} * \delta_{p_\ell}/\delta_{p_\ell}]$, where $(\delta_{p_\ell}) \in \blacktriangle_0$ and members of $\mathscr{P}^*_{L^1}$ denoted by $[f_{p_\ell}/\delta_{p_\ell}]$. If $F_p = [f_{p_\ell}/\delta_{p_\ell}]$, then $F_p * \delta_{p_\ell} = f_{p_\ell} \in \mathcal{L}^1_{pan}(\mathcal{R})$.

Construct $\mathscr{P}_{L^1} = (\mathcal{L}^1_{pan}(\mathcal{R}), \mathcal{C}_0(\mathcal{R}) \cap \mathcal{L}^1_{pan}(\mathcal{R}), \blacktriangle_0, \circ)$ where $\blacktriangle_0 = \{\mathcal{C}_{pan}(\delta_{p_n}) : (\delta_{p_n}) \in \blacktriangle_0\}$, $\mathcal{C}_0(\mathcal{R})$−complex valued infinitely vanishing continuous functions and the operator $\circ$ is pointwise multiplication(usual). Denote $\mathcal{C}_{pan}(f_{p_n})/\mathcal{C}_{pan}(\delta_{p_n})$ an element of $\mathscr{P}_{L^1}$ and also which is the range of CCPT on $\mathscr{P}^*_{L^1}$.

Lemma 3.1 If $(\delta_{p_n}) \in \blacktriangle_0$, then $\mathcal{C}_{pan}(\delta_{p_n}) \to 1$ as $n \to \infty$ uniformly on each compact set in $\mathcal{L}^1_{pan}(\mathcal{R})$.
Proof. Let $(\delta_{p_n}) \in \blacktriangle_0$. Then for any compact subset $\mathcal{K}$ of $\mathcal{R}$ such that $\lim_{n\to\infty} (p) \int_{\mathcal{K}} |\delta_{p_n}(t)dt| \to 0$ and $\exists\, \mathcal{M} > 0$ such that $\left|cos(\frac{v}{b}t)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} - \sqrt{2\pi ib}\; e^{\frac{-i}{2}\left(\frac{d}{b}\right)v^2} e^{\left(\frac{ia}{b}\right)xt}\right| < \mathcal{M}$. Choose $|t| > \varepsilon$ and $\varepsilon > 0$. Then

$$\|\mathcal{C}_{pan}(\delta_{p_n}) - 1\| = (p)\int_{\mathcal{R}} \left|\frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{\sqrt{2\pi ib}}(p)\int_{-\infty}^{\infty} cos(\frac{v}{b}t)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\delta_{p_n}(t)dt - (p)\int_{-\infty}^{\infty} e^{\left(\frac{ia}{b}\right)xt}\delta_{p_n}(t)dt\right|dv$$

$$= (p)\int_{\mathcal{R}} \left|\frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{\sqrt{2\pi ib}}(p)\int_{-\infty}^{\infty} \left\{cos(\frac{v}{b}t)e^{\frac{i}{2}\left(ab\right)t^2} - \sqrt{2\pi ib}e^{\frac{-i}{2}\left(\frac{d}{b}\right)v^2}e^{\left(\frac{ia}{b}\right)xt}\right\}\delta_{p_n}(t)dt\right|dv$$

$$\leq \frac{\mathcal{M}}{\sqrt{2\pi ib}}(p)\int_{\mathcal{R}}\left\{(p)\int_{\mathcal{K}} |\delta_{p_n}(t)|dt\right\}dv \to 0 \; as \; n \to \infty.$$

Lemma 3.2 If $f_{p_n} \to f_p$ as $n \to \infty$ in $\mathcal{L}^1_{pan}(\mathcal{R})$ and $\mathcal{C}_{pan}(\psi_{p_n}) \in \blacktriangle_0$. Then $f_{p_n} \circ \mathcal{C}_{pan}(\psi_{p_n}) \to f_p$ as $n \to \infty$ in $\mathcal{L}^1_{pan}(\mathcal{R})$.
Proof. Since $\mathcal{C}_{pan}(\psi_{p_n}) \in \blacktriangle_0$ and lemma 3.1, we have,

$$\begin{aligned}\|f_{p_n} \circ \mathcal{C}_{pan}(\psi_{p_n}) - f_p\| \\ &= \|f_{p_n} \circ \mathcal{C}_{pan}(\psi_{p_n}) - f_p \circ \mathcal{C}_{pan}(\psi_{p_n}) + f_p \circ \mathcal{C}_{pan}(\psi_{p_n}) - f_p\| \\ &\leq \|f_{p_n} - f_p\| \; \|\mathcal{C}_{pan}(\psi_{p_n})\| + \leq \|f_p\| \; \|\mathcal{C}_{pan}(\psi_{p_n}) - 1\| \to 0 \; as \; n \to \infty.\end{aligned}$$

We define CCPT $\mathcal{C}_{pan} : \mathscr{P}^*_{L^1} \to \mathscr{P}_{L^1}$ as $\mathcal{C}_{pan}[f_{p_n}/\delta_{p_n}] = \mathcal{C}_{pan}(f_{p_n})/\mathcal{C}_{pan}(\delta_{p_n})$ where $[f_{p_n}/\delta_{p_n}] \in \mathscr{P}^*_{L^1}$. Suppose $[f_{p_n}/\delta_{p_n}] \in \mathscr{P}^*_{L^1}$ then $f_{p_n} * \delta_{p_m} = f_{p_m} * \delta_{p_n} \forall\; m, n \in \mathcal{N}$ which imply $\mathcal{C}_{pan}[(f_{p_n} * \delta_{p_m})(t)](v) = \mathcal{C}_{pan}[(f_{p_m} * \delta_{p_n})(t)](v)$. Applying both side convolution, we get $\mathcal{C}_{pan}(f_{p_n})/\mathcal{C}_{pan}(\delta_{p_n}) = \mathcal{C}_{pan}(f_{p_m})/\mathcal{C}_{pan}(\delta_{p_m})$ $\forall m, n \in \mathcal{N}$. Hence CCPT is well-defined on $\mathscr{P}_{L^1}$.

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Lemma 3.3 If $(f_{p_n}) \in \mathcal{L}^1_{pan}(\mathcal{R}), n \in \mathcal{N}$ then,

$$\mathcal{C}_{pan}[f_{p_n}](v) = \sqrt{\frac{1}{2\pi \mathrm{ib}}} e^{\frac{i}{2}\left(\frac{\mathrm{d}}{\mathrm{b}}\right)v^2} \ (p)\int_{-\infty}^{\infty} cos(\frac{v}{b}t) e^{\frac{i}{2}\left(\frac{\mathrm{a}}{\mathrm{b}}\right)t^2} f_{p_n}(t)dt$$

converges uniformly on each compact subset of $\mathcal{R}$.
Proof. If δ_{p_n} is a pan-delta sequence, then $\mathcal{C}_{pan}[\delta_{p_n}]$ converges uniformly on each compact set to the constant function 1. Hence for each compact set $\mathcal{K}$, $\mathcal{C}_{pan}[\delta_{p_m}] > 0$ on $\mathcal{K}$ for almost all m $\in \mathcal{N}$ and

$$\begin{aligned}\mathcal{C}_{pan}(f_{p_n}) = \mathcal{C}_{pan}(f_{p_n})\frac{\mathcal{C}_{pan}(\delta_{p_m})}{\mathcal{C}_{pan}(\delta_{p_m})} &= \frac{e^{\frac{i}{2}\left(\frac{\mathrm{d}}{\mathrm{b}}\right)v^2}}{\sqrt{2\pi \mathrm{ib}}}\frac{\mathcal{C}_{pan}(f_{p_n} * \delta_{p_m})}{\mathcal{C}_{pan}(\delta_{p_m})} \\ &= \frac{e^{\frac{i}{2}\left(\frac{\mathrm{d}}{\mathrm{b}}\right)v^2}}{\sqrt{2\pi \mathrm{ib}}}\frac{\mathcal{C}_{pan}(f_{p_m} * \delta_{p_n})}{\mathcal{C}_{pan}(\delta_{p_m})} = \frac{\mathcal{C}_{pan}(f_{p_m})}{\mathcal{C}_{pan}(\delta_{p_m})}\mathcal{C}_{pan}(\delta_{p_n}) \ on \ \mathcal{K}\end{aligned}$$

Applying $n \to \infty$, we get $\mathcal{C}_{pan}[f_{p_n}] \to \frac{\mathcal{C}_{pan}(f_{p_m})}{\mathcal{C}_{pan}(\delta_{p_m})}$ on $\mathcal{K}$.

Theorem 3.1 The CCPT $\mathcal{C}_{pan} : \mathscr{P}^*_{L^1} \to \mathscr{P}_{L^1}$ is consistent with $\mathcal{C}_{pan} : \mathcal{L}^1_{pan}(\mathcal{R}) \to \mathcal{L}^1_{pan}(\mathcal{R})$.
Proof. Let $f_p \in \mathcal{L}^1_{pan}(\mathcal{R})$. Then $f_p \in \mathscr{P}_{L^1}$ is $[(f_p * \delta_{p_n})/\delta_{p_n}]$ where δ_{p_n} is pan-delta sequence. By definition, $\mathcal{C}_{pan}((f_p * \delta_{p_n})/\delta_{p_n}) = \frac{\sqrt{2\pi \mathrm{ib}}}{e^{\frac{i}{2}\left(\frac{\mathrm{d}}{\mathrm{b}}\right)v^2}} \ \mathcal{C}_{pan}(f_p)\mathcal{C}_{pan}(\delta_{p_n})/\mathcal{C}_{pan}(\delta_{p_n})$ which represents Boehmian $\frac{\sqrt{2\pi \mathrm{ib}}}{e^{\frac{i}{2}\left(\frac{\mathrm{d}}{\mathrm{b}}\right)v^2}} \ \mathcal{C}_{pan}(f_p)$ in $\mathscr{P}_{L^1}$.

Theorem 3.2 The CCPT $\mathcal{C}_{pan} : \mathscr{P}^*_{L^1} \to \mathscr{P}_{L^1}$ is bijection.
Proof. Let $[f_{p_n}/\phi_{p_n}]$, $[g_{p_m}/\psi_{p_m}] \in \mathscr{P}^*_{L^1}$ with $\mathcal{C}_{pan}([f_{p_n}/\phi_{p_n}]) = \mathcal{C}_{pan}([g_{p_m}/\psi_{p_m}])$. Since $\mathcal{C}_{pan}[f_{p_n}/\phi_{p_n}] = \mathcal{C}_{pan}(f_{p_n})/\mathcal{C}_{pan}(\phi_{p_n})$ and $\mathcal{C}_{pan}[g_{p_m}/\psi_{p_m}] = \mathcal{C}_{pan}(g_{p_m})/\mathcal{C}_{pan}(\psi_{p_m})$. From which we get, $\mathcal{C}_{pan}(f_{p_n} * \psi_{p_m}) = \mathcal{C}_{pan}(g_{p_m} * \phi_{p_n}) \forall m, n \in \mathcal{N}$. Since CCPT $\mathcal{C}_{pan}$ on $\mathcal{L}^1_{pan}(\mathcal{R})$ is 1-1, $(f_{p_n} * \psi_{p_m}) = (g_{p_m} * \phi_{p_n}) \forall m, n \in \mathcal{N}$. It follows $[f_{p_n}/\phi_{p_n}] = [g_{p_m}/\psi_{p_m}]$ and hence $\mathcal{C}_{pan}$ is 1-1. Since CCPT $\mathcal{C}_{pan}$ on $\mathcal{L}^1_{pan}(\mathcal{R})$ is onto, it follows that $\mathcal{C}_{pan}([f_{p_n}/\phi_{p_n}]) = [g_{p_m}/\psi_{p_m}]$. Hence $\mathcal{C}_{pan} : \mathscr{P}^*_{L^1} \to \mathscr{P}_{L^1}$ is bijective map.

4 Canonical Sine Pan Transform

Consider $f_p, g_p \in \mathcal{L}^1_{pan}(\mathcal{R})$ and define $\widehat{f_p}(t) = e^{\frac{i}{2}\left(\frac{\mathrm{a}}{\mathrm{b}}\right)t^2} f_p(t)$, $\widehat{g_p}(|x-t|) = e^{\frac{i}{2}\left(\frac{\mathrm{a}}{\mathrm{b}}\right)(x-t)^2} g_p(x-t)$, $\widehat{g_p}(x+t) = e^{\frac{i}{2}\left(\frac{\mathrm{a}}{\mathrm{b}}\right)(x+t)^2} g_p(x+t)$ and convolution $(f_p \circledast g_p)$ as $(f_p \circledast g_p)(t) = (p)\int_{-\infty}^{\infty} \frac{e^{\frac{-i}{2}\left(\frac{\mathrm{a}}{\mathrm{b}}\right)t^2}}{2} \widehat{f_p}(x)[\widehat{g_p}(|x-t|) - \widehat{g_p}(x+t)]dx$ and $f_p \circledast g_p = f_p * g_p - f_p \odot g_p$, where

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$(f_p \odot g_p)(t) = \int_{\mathcal{R}} e^{\frac{-i}{2}(\frac{a}{b})t^2} \widehat{f_p}(x)\widehat{g_p}(x+t)dx, \forall x \in \mathcal{R}.$

Lemma 4.1 If f_p, $g_p \in \mathcal{L}^1_{pan}(\mathcal{R})$, then $f_p \odot g_p \in \mathcal{L}^1_{pan}(\mathcal{R})$.

Lemma 4.2 If f_p, $g_p \in \mathcal{L}^1_{pan}(\mathcal{R})$, then $f_p \odot g_p = g_p \odot f_p$.

Lemma 4.3 If f_p, g_p, $h_p \in \mathcal{L}^1_{pan}(\mathcal{R})$, then $f_p \odot (g_p \odot h_p) = (f_p \odot g_p) \odot h_p$.

Lemma 4.4 If f_p, $g_p \in \mathcal{L}^1_{pan}(\mathcal{R})$, then $f_p \circledast g_p = g_p \circledast f_p$.

Lemma 4.5 If f_p, g_p, $h_p \in \mathcal{L}^1_{pan}(\mathcal{R})$, then $f_p \circledast (g_p * h_p) = (f_p \circledast g_p) * h_p$.

Theorem 4.1 (Canonical sine pan convolution) If f_p, $g_p \in \mathcal{L}^1_{pan}(\mathcal{R})$, $F_{p_s}(v)$ and $G_{p_c}(v)$ are the canonical sine and cosine pan transform of $f_p(t)$ and $g_p(t)$, then $\mathcal{S}_{pan}[(f_p \circledast g_p)(t)](v) = \sqrt{2\pi i b}\ e^{\frac{-i}{2}\left(\frac{d}{b}\right)v^2} F_{p_s}(v).\ G_{p_c}(v)$.

Proof. For f_p, $g_p \in \mathcal{L}^1_{pan}(\mathcal{R})$, we have

$$\mathcal{S}_{pan}[(f_p \circledast g_p)(t)](v) = \frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{\sqrt{2\pi i b}}(p)\int_{-\infty}^{\infty} sin(\frac{v}{b}t)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}$$

$$\times\left\{ (p)\int_{-\infty}^{\infty} \frac{e^{\frac{-i}{2}\left(\frac{a}{b}\right)t^2}}{2}\widehat{f_p}(x)\left[\widehat{g_p}(|x-t|) - \widehat{g_p}(x+t)\right]dx\right\}dt$$

$$= \frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{2\sqrt{2\pi i b}}(p)\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} f_p(x)\left\{ (p)\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)(x-t)^2}[g_p(x-t)\right.$$

$$\left. - e^{\frac{i}{2}\left(\frac{a}{b}\right)(x+t)^2} g_p(x+t)]sin(\frac{v}{b}t)dt\right\}dx$$

$$= \frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{2\sqrt{2\pi i b}}(p)\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} f_p(x)\ (p)\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)(x-t)^2} g_p(x-t)sin(\frac{v}{b}t)dtdx$$

$$- \frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{2\sqrt{2\pi i b}}(p)\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} f_p(x)\ (p)\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)(x+t)^2} g_p(x+t)sin(\frac{v}{b}t)dtdx = \mathscr{I}_1 - \mathscr{I}_2$$

put $x - t = y$ in $\mathscr{I}_1$ and $x + t = y$ in $\mathscr{I}_2$

$$= \frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{2\sqrt{2\pi i b}}(p)\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} f_p(x)\ (p)\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)(y)^2} g_p(y)sin(\frac{v}{b}(y+x))dydx$$

$$- \frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{2\sqrt{2\pi i b}}(p)\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} f_p(x)\ (p)\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)(y)^2} g_p(y)sin(\frac{v}{b}(y-x))dydx$$

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$$= \frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{2\sqrt{2\pi ib}}(p)\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} f_p(x)\ (p)\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)(y)^2}[sin(\frac{v}{b}(y+x)) - sin(\frac{v}{b}(y-x))]g_p(y)dydx$$

$$= (p)\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} f_p(x)\left\{\frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{2\sqrt{2\pi ib}}\ (p)\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)(y)^2}[2cos(\frac{v}{b}y)sin(\frac{v}{b}x)]g_p(y)dy\right\}dx$$

$$= \frac{\sqrt{2\pi ib}}{e^{\frac{i}{2}\left(\frac{a}{b}\right)(v)^2}}\frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)(v)^2}}{\sqrt{2\pi ib}}(p)\int_{-\infty}^{\infty} e^{\frac{i}{2}\left(\frac{a}{b}\right)x^2} sin(\frac{v}{b}x)f_p(x)G_{p_c}(v)dx$$

$$= \sqrt{2\pi ib}e^{\frac{-i}{2}\left(\frac{d}{b}\right)(v)^2}F_{p_s}(v)G_{p_c}(v).$$

5 Canonical Sine Pan Boehmians

Definition 5.1 [9] Two sequences $\{f_{p_\ell}/\zeta_{p_\ell}\}$ and $\{g_{p_\ell}/\phi_{p_\ell}\}$ are *equivalent* if $f_{p_m} \circledast \phi_{p_\ell} = g_{p_\ell} \circledast \zeta_{p_m}$ for all $l, m \in \mathcal{N}$ and $f_{p_\ell}, g_{p_\ell} \in \mathcal{L}^1_{pan}(\mathcal{R})$, $\zeta_{p_\ell}, \phi_{p_\ell} \in \blacktriangle_0$. The equivalence class of quotients described above is known as pan integrable Boehmians. All pan integrable Boehmians space denoted by $\mathscr{P}^{\circledast}_{L^1} = \mathscr{P}_{L^1}(\mathcal{L}^1_{pan}(\mathcal{R}), \blacktriangle_0, \circledast)$. A function $f_p \in \mathcal{L}^1_{pan}(\mathcal{R})$ can be diagnosed with the Boehmian is $[f_{p_\ell} \circledast \delta_{p_\ell}/\delta_{p_\ell}]$, where $(\delta_{p_\ell}) \in \blacktriangle_0$ and members of $\mathscr{P}^{\circledast}_{L_1}$ denoted by $[f_{p_\ell}/\phi_{p_\ell}]$. Suppose F = $[f_{p_\ell}/\phi_{p_\ell}]$ then $F \circledast \delta_{p_\ell} = f_{p_\ell} \in \mathcal{L}^1_{pan}(\mathcal{R})$.

We define $\mathcal{S}_{pan} : \mathscr{P}^{\circledast}_{L_1} \to \mathscr{P}_{L_1}$ as $\mathcal{S}_{pan}[f_{p_n}/\delta_{p_n}] = \mathcal{S}_{pan}(f_{p_n})/\mathcal{C}_{pan}(\delta_{p_n})$, where $[f_{p_n}/\delta_{p_n}] \in \mathscr{P}^{\circledast}_{L_1}$. Suppose $[f_{p_n}/\delta_{p_n}] \in \mathscr{P}^{\circledast}_{L_1}$ then $f_{p_n} \circledast \delta_{p_m} = f_{p_m} \circledast \delta_{p_n}\ \forall\ m, n \in \mathcal{N}$ which imply $\mathcal{S}_{pan}[(f_{p_n} \circledast \delta_{p_m})(t)](v) = \mathcal{S}_{pan}[(f_{p_m} \circledast \delta_{p_n})(t)](v)$ which imply $\mathcal{S}_{pan}(f_{p_n})/\mathcal{C}_{pan}(\delta_{p_n}) = \mathcal{S}_{pan}(f_{p_m})/\mathcal{C}_{pan}(\delta_{p_m})\ \forall m, n \in \mathcal{N}$. Hence CSPT is well-defined on $\mathscr{P}_{L_1}$.

Lemma 5.1 If $(\delta_{p_n}) \in \blacktriangle_0$, then $\mathcal{S}_{pan}(\delta_{p_n}) \to 1$ as $n \to \infty$ uniformly on each compact set in $\mathcal{L}^1_{pan}(\mathcal{R})$.
Proof. Let $(\delta_{p_n}) \in \blacktriangle_0$. Then for any compact subset $\mathcal{K}$ of $\mathcal{R}$ such that $\lim_{n\to\infty} (p)\int_{\mathcal{K}} |\delta_{p_n}(t)dt| \to 0$ and $\exists\ \mathcal{M} > 0$ such that $\left|sin(\frac{v}{b}t)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2} - \sqrt{2\pi ib}\ e^{\frac{-i}{2}\left(\frac{d}{b}\right)v^2}e^{\left(\frac{ia}{b}\right)xt}\right| < \mathcal{M}$. Choose $|t| > \varepsilon$ and $\varepsilon > 0$. Then

$\|\mathcal{S}_{pan}(\delta_{p_n}) - 1\|$

$$= (p)\int_{\mathcal{R}} \left|\frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{\sqrt{2\pi ib}}(p)\int_{-\infty}^{\infty} sin(\frac{v}{b}t)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}\delta_{p_n}(t)dt - (p)\int_{-\infty}^{\infty} e^{\left(\frac{ia}{b}\right)xt}\delta_{p_n}(t)dt\right|dv$$

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$$= (p)\int_{\mathcal{R}}\left|\frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{\sqrt{2\pi ib}}(p)\int_{-\infty}^{\infty}\left\{sin(\frac{v}{b}t)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}-\sqrt{2\pi ib}e^{\frac{-i}{2}\left(\frac{d}{b}\right)v^2}e^{\left(\frac{ia}{b}\right)xt}\right\}\delta_{p_n}(t)dt\right|dv$$
$$\leq \frac{\mathcal{M}}{\sqrt{2\pi ib}}(p)\int_{\mathcal{R}}\left\{(p)\int_{\mathcal{K}}|\delta_{p_n}(t)|dt\right\}dv \to 0 \ as \ n \to \infty.$$

Lemma 5.2 If $[f_{p_n}/\delta_{p_n}] \in \mathscr{P}^{\circledast}_{L^1}, n \in \mathcal{N}$, then

$$\mathcal{S}_{pan}[f_{p_n}](v) = \sqrt{\frac{1}{2\pi ib}}e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}\ (p)\int_{-\infty}^{\infty}cos(\frac{v}{b}t)e^{\frac{i}{2}\left(\frac{a}{b}\right)t^2}f_{p_n}(t)dt$$

converges uniformly on each compact subset of $\mathcal{R}$.

Proof. If δ_{p_n} is a pan-delta sequence, then $\mathcal{S}_{pan}[\delta_{p_n}]$ converges uniformly on each compact set to the constant function 1. Hence for each compact set $\mathcal{K}$, $\mathcal{S}_{pan}[\delta_{p_m}] > 0$ on $\mathcal{K}$ for almost all m $\in \mathcal{N}$ and

$$\mathcal{S}_{pan}(f_{p_n}) = \mathcal{S}_{pan}(f_{p_n})\frac{\mathcal{C}_{pan}(\delta_{p_m})}{\mathcal{C}_{pan}(\delta_{p_m})} = \frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{\sqrt{2\pi ib}}\frac{\mathcal{S}_{pan}(f_{p_n}\circledast\delta_{p_m})}{\mathcal{C}_{pan}(\delta_{p_m})}$$
$$= \frac{e^{\frac{i}{2}\left(\frac{d}{b}\right)v^2}}{\sqrt{2\pi ib}}\frac{\mathcal{S}_{pan}(f_{p_m}\circledast\delta_{p_n})}{\mathcal{C}_{pan}(\delta_{p_m})} = \frac{\mathcal{S}_{pan}(f_{p_m})}{\mathcal{C}_{pan}(\delta_{p_m})}\mathcal{C}_{pan}(\delta_{p_n}) \ on \ \mathcal{K}$$

Applying $n \to \infty$, we get $\mathcal{S}_{pan}[f_{p_n}] \to \frac{\mathcal{S}_{pan}(f_{p_m})}{\mathcal{C}_{pan}(\delta_{p_m})}$ on $\mathcal{K}$.

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