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Strong Co-Secure Domination in Graphs

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ABSTRACT

Let $G = (V, E)$ be a graph. A subset D of the vertex set $V(G)$ of a graph G is a strong co-secure dominating set if every vertex $v \in V - D$ there exists $u \in D$ such that $uv \in E(G)$ then $D\backslash\{u\} \cup \{v\}$ and $\deg(u) \geq \deg(v)$. The strong co-secure domination number is the minimum cardinality of a strong co-secure dominating set of G, and it is denoted by $\gamma_{scsd}(G)$. The strong co-secure dominating set of G is found for path, cycle, helm graph, closed helm graph, Petersen graph, gear graph, Tadpole graph, and Butterfly graph.

Keywords: *Domination, Strong Domination, Co-secure Domination, Strong Co-Secure Domination.*

1 Introduction

All the graphs considered here are finite, connected, and undirected. Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. In our literature survey, we can find many authors have introduced various new parameters by formulating conditions on the dominating set D. In the graph, domination vertices or edges or both assigned values have been subjected to the condition. A dominating set D is a subset of $V(G)$ such that every vertex in $V - D$ is adjacent to at least one vertex in D. The domination number is denoted by $\gamma(G)$ is the minimum cardinality of a dominating set of G (Cockayne E.J & Hedetniemi. S (1977) [4], T.W. Haynes, *et al.*, (1998) [5], Sampathkumar & Latha (1994), [10]). Further, D is a co-secure dominating set if every $u \in D$ there exist $v \in V - D$ such that $uv \in E(G)$ and $D\backslash\{u\} \cup \{v\}$ is a dominating set of G. The co-secure domination number is denoted by $\gamma_{cs}(G)$ is the minimum cardinality of the co-secure dominating set of G. The co-secure dominating set was studied and introduced (S. Arumugam *et al.*, (2014) [1], Aleena Joseph *et al.*, (2018) [2], D.Bhuvaneshwari *et al.*, (2019) [3]).

A subset $D \subseteq V(G)$ is a strong dominating set if every vertex $v \in V - D$, there exists $u \in D$ such that $uv \in E(G)$ then $\deg(u) \geq \deg(v)$. The minimum cardinality of a strong dominating set is called the strong domination number of G, and it is denoted by $\gamma_s(G)$. A strong dominating set of cardinality $\gamma_s(G)$ is γ_s-set of G. Strong domination was introduced (Hettingh M.A *et al.*, (1998) [6], D. Rautenbach (1999) [8], Sampathkumar & Latha (1996) [9]). Besides, the two-step perfect domination was introduced by Jasmine, S. S., & Ambika, S. M (2020) [7].

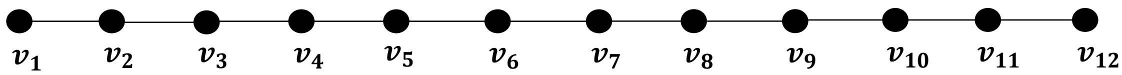
Now, we introduce a new domination parameter called Strong Co-secure domination. A dominating set is a strong co-secure dominating set if for every $u \in D$ there exists $v \in V - D$ such that $uv \in E(G)$, then $D\backslash\{u\} \cup \{v\}$ is a dominating set of G and degree of u greater than or equal to the degree of v.

The Strong Co-secure domination number of G is the minimum cardinality of a Strong dominating set, and it is denoted by $\gamma_{scsd}(G)$.

This paper investigates the Strong Co-secure domination number of some graphs.

2 Examples of strong co-secure domination

2.1 Consider the path P_{12}



D=$\{v_2, v_4, v_6, v_8, v_{10}, v_{12}\}$ is the strong co-secure dominating set of P_{12} ,

$$\gamma_{scsd}(G) = 6.$$

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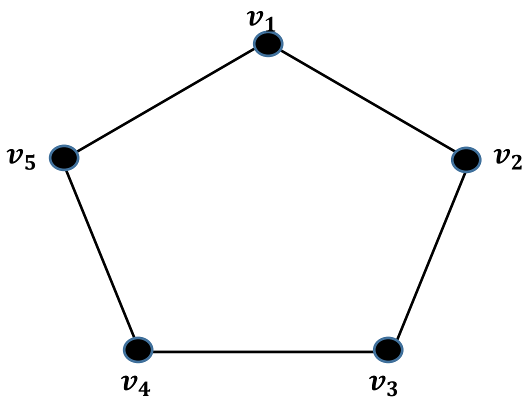
2.2 Consider the cycle C_5



$D = \{v_1, v_4\}$ Is the strong co-secure domination set of C_5 ,

$$\gamma_{scsd}(G) = 2$$

3 MAIN RESULTS

Definition 3.1

A dominating set for a graph $G = (V, E)$ is a subset of $V(G)$ such that every vertex in D is adjacent to at least one member of D.

Definition 3.2

Let G be a graph $uv \in E(G)$. Then u strongly dominates V if $\deg(u) \geq \deg(v)$. If every vertex in $V - D$ is strongly dominated by some element of u in D then D is called a strong dominating set.

Definition 3.3

A set $D \subseteq V(G)$ of a graph $G = (V, E)$ is a co-secure dominating set if for every $u \in D$ there exists $v \in v - D$ such that $uv \in E(G)$ and $D\backslash\{u\} \cup \{v\}$ is a dominating set of G.

Definition 3.4

A subset D of the vertex set $V(G)$ of a graph G is a strong co-secure dominating set if for every $u \in D$ there exists $v \in v - D$ such that $uv \in E(G)$ and $D\backslash\{u\} \cup \{v\}$ is a dominating set of G then u strongly dominates v if $\deg(u) \geq \deg(v)$. The strong co-secure domination number of G is the minimum cardinality of a strong co-secure dominating set, and it's denoted by $\gamma_{scsd}(G)$.

Theorem 3.5

For any path graph $P_n (n \geq 4)$, then

$$\gamma_{scsd}(P_n) = \begin{cases} \frac{n}{2} & if\ n\ is\ even \\ \frac{n-1}{2} & if\ n\ is\ odd \end{cases}$$

Proof:

Let G be a path graph and let $V = \{v_1, v_2, v_3, \dots, v_n\}$ be the vertices of G.

Case (i): If n is even

Let $V = \{v_1, v_2, \dots, v_{n-2}\}$ be the vertices of $V(G)$, then $D = \{v_1, v_3, \dots, v_{n-2}\}$. Where D is a dominating set of $V(G)$ and $V - D = \{v_2, v_4, \dots, v_{n-2}\}$.

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Every vertex in $V-D$ is at least one vertex in D there exists $u \in D$ such that $uv \in E(G)$ satisfies the condition $D\backslash\{u\} \cup \{v\}$ and this set is a dominating set of $V(G)$.

Hence D is a co-secure dominating set and also, every vertex of $V-D$ has a degree of u greater than or equal to the degree of v.

Therefore, D is the strong co-secure dominating set of $V(G)$

$$\therefore \ \gamma_{scsd}(P_n) = \frac{n}{2}$$

Case (ii): If n is odd

Let $V(G) = \{v_1, v_2, \dots, v_{2n+1}\}$.

Now, $D = \{v_2, v_4, \dots, v_{2n+1}\}$ is a dominating set of $V(G)$ and $V-D = \{v_1, v_3, \dots, v_{2n+1}\}$. Every vertex in $V-D$ there exists $u \in D$ such that $uv \in E(G)$ satisfies for the condition degree of u greater than or equal to the degree of v and also each vertex $v \in V-D$ is adjacent to $u \in D$ such that $uv \in E(G)$ then we can choose one vertex in $u_1 \in D$ and remove it from the dominating set $D(D-\{u_1\})$ and also choose a vertex V from $V-D(v_1 \in V-D)$ then it satisfies $D\backslash\{u\} \cup \{v\}$ is a dominating set.

Hence D is a strong co-secure dominating set

$$\therefore \ \gamma_{scsd}(P_n) = \frac{n-1}{2}$$

Theorem 3.6

For any cycle $C_m (m \geq 4)$, then

$$\gamma_{scsd}(C_m) = \begin{cases} \frac{m}{2} & m = 2n, \ n \geq 2 \\ \frac{m-1}{2} & m = 2n+1, \ n \geq 2 \end{cases}$$

Proof:

Let G be a cycle graph and let $V = \{v_1, v_2, v_3, \dots, v_n\}$ be the vertices of G.

Case (i): Let $G = C_{2n}, n \geq 2$

Let $\{v_1, v_2, \dots, v_{2n}\}$ be the vertices of $V(G)$ then $D = \{v_1, v_2, v_4 \dots, v_{2n}\}$ is a dominating set, and $V - D = \{v_3, v_5, \dots, v_{2n-1}, v_{2n}\}$.

Every vertex in $V-D$ there exists $u \in D$ such that $uv \in E(G)$ satisfies the condition $D\backslash\{u\} \cup \{v\}$ is adjacent to at least one vertex in D. Hence D is a co-secure dominating set and also, every vertex in $V-D$ has $\deg(u) \geq \deg(v)$.Hence D is a c strong co-secure dominating set of $V(G)$.

$$\therefore \ \gamma_{scsd}(C_{2n}) = \frac{m}{2}$$

Case (ii): Let $G = C_{2n+1}, n \geq 2$

Let $V(G) = \{v_1, v_2, \dots, v_{2n}, v_{2n+1}\}$ be the vertices of G.

Now, $D = \{v_1, v_3, \dots, v_{2n+1}\}$ is a strong dominating set and also D is a co-secure dominating set of C_{2n+1}.

Since $\left[\frac{m-1}{2}\right] = \gamma_{csd}(c_{2n+1}) = \gamma_{scsd}(c_{2n+1})$

Then, $\gamma_{csd}(c_{2n+1}) = \gamma_{scsd}(c_{2n+1})$

$$\therefore \ \gamma_{scsd}(c_{2n+1}) = \frac{m-1}{2}$$

Definition 3.7

The **helm H_n** is a graph obtained from a wheel graph by adjoining a pendant edge at each vertex of the cycle.

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Theorem 3.8

From the helm graph $\boldsymbol{H_n}$, then $\gamma_{scsd}(H_n) = n + 1 \; if \; n \geq 3$

Proof:

Let G be a helm graph and let $V = \{v_1, v_2, v_3, \dots, v_n\}$ be the vertices of G.

The helm graph $\boldsymbol{H_n}$ has $2n + 1$ vertices and it contains the wheel W_n and $n - 1$ pendant vertices. Let $\{v_1, v_2, \dots, v_{2n+1}\}$ be the vertices of a graph. Then $D = \{v_2, v_3, \dots, v_{2n+1}\}$ is a dominating set. Every $u \in D$ there exists every vertex $v \in V - D$ is adjacent with at least one vertex such that $D\setminus\{u\} \cup \{v\}$ is the dominating set and $\deg(u) \geq \deg(v)$.

Hence D becomes a strong co-secure dominating set.

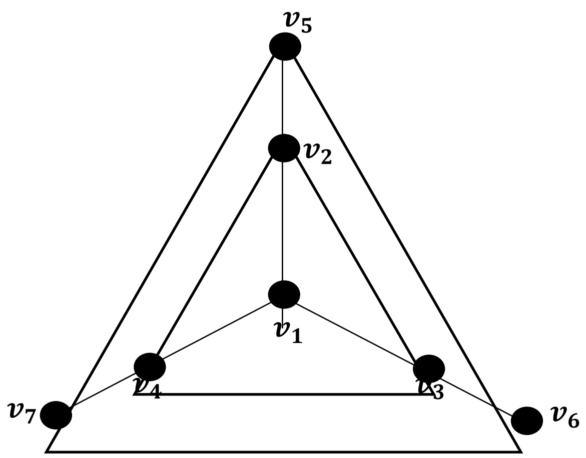
$$\therefore \; \gamma_{scsd}(H_n) = n + 1$$

Definition 3.9

The **closed helm $\boldsymbol{CH_n}$** is the graph obtained from a helm by adjoining each pendant vertex to form a cycle.

Illustration 3.10

The **closed helm graph $\boldsymbol{CH_3}$** shown in the figure**:**



First, we have $D = \{v_2, v_7\}$ is a dominating set and $V - D = \{v_1, v_3, v_4, v_5, v_6\}$.

We can choose one vertex in $u \in D$ if $u = \{v_7\}$ and also, we can choose $v \in V - D$ if we choose one vertex $v = \{v_6\}$ then $uv \in E(G)$ and remove it from the dominating set of u we get $D - \{u\} = \{v_2\}$.

$\therefore D - \{u\} \cup \{v\} = \{v_2, v_6\}$ it is also a dominating set. Hence it is called co-secure domination.

Also, every vertex of $V - D$ is strongly dominated by some element of u in D.

Therefore, $\deg(u) \geq \deg(v)$ it satisfies the above figure. It has called the strong dominating set.

Hence there is a strong co-secure dominating set.

$$\therefore \; \gamma_{scsd}(CH_n) = 2$$

Theorem 3.11

For the closed helm graph CH_n , then

$$\gamma_{scsd}(\,CH_n) = \begin{cases} \frac{n+2}{2} & if \; n \; is \; even \\ \frac{n+1}{2} & if \; n \; is \; odd \end{cases}$$

Proof:

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Let G be a closed helm graph and let $V = \{v_1, v_2, v_3, \dots, v_n\}$ be the vertices of G.

Case (i): If n is even

Let $V(G) = \{v_1, v_2, \dots, v_{2n+2}\}$ be the vertices of $V(G)$ then $D = \{v_1, v_2, v_7, \dots, v_{2n+2}\}$ be the dominating set and $V - D = \{v_3, v_4, \dots, v_{2n+2}\}$. Every vertex in $V - D$ is dominated by at least one vertex adjacent to D and it has a co-secure dominating set.

Also, for every $u \in D$, there exists $v \in V - D$ such that $uv \in E(G)$ then the degree of u greater than or equal to the degree of v. So D is called the strong dominating set.

Hence D is the strong co-secure dominating set of $V(G)$.

In general, $n = 2k,\ k = 1,2, \dots, n$

$$\therefore\ \gamma_{scsd}(CH_n) = \frac{n+2}{2}$$

Case (ii): If n is odd

Let $\{v_1, v_2, \dots, v_n\}$ be vertices of $V(G)$.Then $D = \{v_2, v_7, \dots, v_{2n+1}\}$ is a dominating set and $V - D = \{v_1, v_3, \dots, v_{2n+1}\}$. Every $u \in D$ there exists $v \in V - D$ such that $uv \in E(G)$ then satisfies both condition $D\backslash\{u\} \cup \{v\}$ and $\deg(u) \geq \deg(v)$.

Then D is called a strong co-secure dominating set.

$$\therefore\ \gamma_{scsd}(CH_n) = \frac{n+1}{2}$$

Theorem 3.12

For a graph G be Petersen graph, then $\gamma_{scsd}(G) = n - 6$

Proof:

Let G be a Petersen graph and let $V = \{v_1, v_2, v_3, \dots, v_{10}\}$ be the vertices of G.

The Petersen graph is an undirected graph with ten vertices and 15 edges.

Denote the vertices, $V(G) = \{v_1, v_2, \dots, v_{10}\}$ of G.Then, $D = \{v_1, v_2, v_6, v_9\}$ is a dominating set of $V(G)$ and $V - D = \{v_3, v_4, v_5, v_7, v_8, v_{10}\}$.

Each vertex in $V - D$ has adjacent to at least one vertex in D and also, every $u \in D$ there exists $v \in V - D$, such that $uv \in E(G)$ then $D\backslash\{u\} \cup \{v\}$ and degree of u greater than or equal to the degree of v.

Hence D satisfies the strong co-secure domination of $V(G)$.

For the Petersen graph $V(G)$,

$$\therefore\ \gamma_{scsd}(G) = n - 6$$

Definition 3.13

The **gear graph** is a wheel graph with a graph vertex added between each pair of adjacent graph vertices of the outer cycle.

Theorem 3.14

For the gear graph $G_n (n \geq 3)$, $\gamma_{scsd}(G_n) = \left[\frac{n}{2}\right] + 3$

Proof:

Let G be a gear graph and let $V = \{v_1, v_2, v_3, \dots, v_n\}$ be the vertices of G.

The gear graph G_n has $\{G_1, G_2, \dots, G_{n-2}\}$.

Now, $D = \{v_1, v_4, \dots, v_{n-2}\}$ is a dominating set and $V - D = \{v_2, v_3, \dots, v_{n-2}\}$.

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Every $u \in D$ there exist $v \in V - D$ such that $uv \in E(G)$ in D then $D\backslash\{u\} \cup \{v\}$ is a dominating set. Therefore, it is called a co-secure dominating set.

Moreover, every vertex $v \in V - D$ there exists $u \in D$ such that $\deg(u) \geq \deg(v)$.

Hence D is a strong dominating set of G_n.

$\therefore$ D become both condition $D\backslash\{u\} \cup \{v\}$ and degree of u greater than or equal to the degree of v.

Hence, D is a strong co-secure dominating set of G_n.

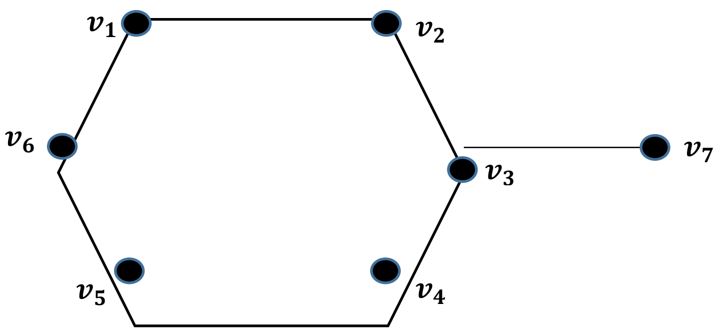
$$\therefore \ \gamma_{scsd}(G_n) = \left[\frac{n}{2}\right] + 3$$

Definition 3.15

The (m, n)- **tadpole graph** is a special type of graph consisting of a cycle graph on m vertices and a path graph on n vertices connected with a bridge.

Example 3.16

The **Tadpole graph $T_{(6,1)}$** shown in the figure**:**



First, we have $D = \{v_1, v_3, v_5\}$ is a dominating set and $V - D = \{v_2, v_4, v_6, v_7\}$.

We can choose one vertex in $u \in D$ if $u = \{v_1\}$ and also, we can choose $v \in V - D$ if we choose one vertex $v = \{v_2\}$ then remove it from the dominating set of u we get $D - \{u\} = \{v_3, v_5\}$.

$\therefore$ $D - \{u\} \cup \{v\} = \{v_2, v_3, v_5\}$ it is also a dominating set. Hence it is called co-secure domination.

Also, every vertex of $V - D$ is strongly dominated by some element of u in D.

Therefore, $\deg(u) \geq \deg(v)$ it satisfies the above figure. It has called the strong dominating set.

Hence there is a strong co-secure dominating set.

$$\therefore \ \gamma_{scsd}\left(T_{(m,n)}\right) = 3$$

Theorem 3.17

For the tadpole graph, $(n \geq 4)$

$$\gamma_{scsd}\left(T_{(m,n)}\right) = \begin{cases} \frac{n}{2} & if\ n\ is\ even \\ \frac{n-1}{2} & if\ n\ is\ odd \end{cases}$$

Proof:

Let G be a tadpole graph and let $V(G) = \{v_1, v_2, v_3, \ldots, v_n\}$ be the vertices of G.

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Case (i): If n is even

Let $V(G) = \{v_1, v_2, \dots, v_{n-2}\}$ then $D = \{v_1, v_3, \dots, v_{n-2}\}$ is a dominating set of $V(G)$.

Every $u \in D$ there exists $v \in V - D$ such that $uv \in E(G)$, then we can choose one vertex in $u_1 \in D$ and remove it from the dominating set D $(D - \{u_1\})$ and also chose one vertex v from $V - D$ $(v_1 \in V - D)$ then satisfies the condition $D \backslash \{u\} \cup \{v\}$ is a dominating set of $V(G)$.

Therefore, it is called a co-secure dominating set.

Moreover, for every vertex of $V - D$ there exists $u \in D$ such that $uv \in E(G)$ then the degree of u greater than or equal to the degree of v.

Then, it has a strong dominating set.

Hence satisfies both condition $D \backslash \{u\} \cup \{v\}$ and $\deg(u) \geq \deg(v)$ it has been called the strong co-secure dominating set of $V(G)$.

$$\therefore \gamma_{scsd}\left(T_{(m,n)}\right) = \frac{n}{2}$$

Case (ii): If n is odd

Let $V(G) = \{v_1, v_2, \dots, v_{2n+1}\}$ the $D = \{v_1, v_3, \dots, v_{2n+1}\}$, where D is a dominating set of $V(G)$. Every $u \in D$ there exists $v \in V - D$, such that $uv \in E(G)$ then $D \backslash \{u\} \cup \{v\}$ and degree of u greater than or equal to the degree of v.

Hence, D is called the strong co-secure dominating set of $V(G)$.

$$\therefore \gamma_{scsd}\left(T_{(m,n)}\right) = \frac{n-1}{2}$$

Theorem 3.18

For a butterfly graph, $(n \geq 2)$

$$\gamma_{scsd}(BF_{\ n}) = \begin{cases} \frac{n+2}{2} & if\ n\ is\ even \\ \frac{n+3}{2} & if n\ is\ odd \end{cases}$$

Proof:

Let G be a butterfly graph and let $\{v_1, v_2, v_3, \dots, v_n\}$ be the vertices of G.

Case (i): If n is even

Let $V(G) = \{v_1, v_2, \dots, v_{2n+2}\}$ then $D = \{v_1, v_3, \dots, v_{2n+2}\}$, where D is a dominating set of $V(G)$.

Every $u \in D$ there exists $v \in V - D$ such that $uv \in E(G)$ satisfies the condition $D \backslash \{u\} \cup \{v\}$ is a dominating set of $V(G)$.

Hence D is a co-secure dominating set and also, every vertex in $V - D$ has a degree of u greater than or equal to the degree of v then D is called a strong dominating set.

Then D is the strong co-secure domination of $V(G)$.

$$\therefore \gamma_{scsd}(BF_{\ n}) = \frac{n+2}{2}$$

Case (ii): If n is odd

Let $V(G) = \{v_1, v_2, \dots, v_{2n+3}\}$ then $D = \{v_1, v_3, \dots, v_{2n+3}\}$ is a dominating set of $V(G)$ and $V - D = \{v_2, v_4, \dots, v_{2n+3}\}$.

Every $u \in D$ there exists $v \in V - D$ such that $uv \in E(G)$ satisfies the condition $D \backslash \{u\} \cup \{v\}$ is a dominating set of $V(G)$ it is called a co-secure dominating set of $V(G)$ and also, every vertex in $V - D$ has a degree of u greater than or equal to a degree of v. Therefore, D satisfies both strong and co-secure dominating sets.

Hence D becomes a strong co-secure dominating set of $V(G)$.

$$\therefore \gamma_{scsd}(BF_{\ n}) = \frac{n+3}{2}$$

CONCLUSION:

This paper describes two new variants of domination parameters such as strong, co-secure domination of various graphs. The theorems and examples can be easily understood with the given concepts. Also, we derived the general results on the strong co-secure domination parameters. Several new graph families that recognize strong co-secure domination can be investigated in the future.

REFERENCES:

[1] Arumugam S, Karan Embadi, Martin Manrique, Co-secure and secure domination in graphs, Utilitas Mathematica, 94, (2014), PP.167-182.

[2] Aleena Joseph, Sangeetha V, Bounds on Co-secure domination in graphs, International Journal on Mathematics Trends and Technology (IJMTT), vol.5, (2018), PP, 158-164.

[3] Bhuvaneshwari D, Meenakshi S, Co-secure set domination in graphs, International Journal of Recent Technology and Engineering (IJRTE), (2019).

[4] Cockayne E.J and Hedetniemi S, towards a theory of domination in graphs, Networks fall (1977), 247-261.

[5] Haynes T.W, Hedetniemi S.T, Slater P.J, Domination in Graphs-Advanced Topics (1998), Marcel Decker, New York.

[6] Hettingh J.H and Henning M.A, on strong domination in graphs, J.Combin Math. comb.Comput. 26, (1998), 73-82.

[7] Jasmine, S. S., & Ambika, S. M, 2-step perfect domination on graphs. International Journal of Engineering and Technology, 7(5), (2020), 7678-7686.

[8] Rautenbach D, The influence of special vertices on strong domination, Discrete Math. 197/198, (1999), 683-690.

[9] Sampathkumar E & Pushpalatha L, Strong Weak domination and domination balance in a graph, Discrete Math, 161, (1996), 235-242.

[10] Sampathkumar E and Pushpalatha L, Set domination in graphs, Journal of Graph Theory, vol.18, (1994).