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A New Type of Topological Vector space Via t^{ω} Approach Structure

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Abstract: The major aim of this work is to introduce the connotation of t^{ω}- approach topological vector space, which is an protraction of t^{ω}- topological vector space. In this paper, we clarify the relation between metric space and an approach space, we define t^{ω}-contraction function and prove many new properties and exist some results, furthermore, we define t^{ω}- approach group and introduce some examples in the t^{ω}- approach group. We describe the t^{ω}- approach subgroup and find the relation between t^{ω}- topological space and t^{ω}- approach space. We give the important and adequate condition to get t^{ω}- approach topological space. Finally, we define t^{ω}- approach vector space, t^{ω}- approach subspace, and t^{ω}- topological vector space.

Keywords:t^{ω}-approach distance, t^{ω}- approach group, t^{ω}- approach vector space, t^{ω}-approach topological space.

1-INTRODUCTION AND PRELIMINARIES

In the functional analysis, the idea of a topological vector space performs an essential role, so, the theory of approach spaces is playing a central topic in quantitative domain theory, moreover there are many special types of an approach structure in a functional analysis, measure theory, probability space, and approximation theory. In fact, the fundamental variance between an approach and metric spaces is contained in an approach space , it is easy to diagnose all the point-set distances, so, a point-set distance does not equal the infimum over the considered set of all the point distances, like in the metric condition, if topological is originated by a topological space, then, it is became an approach space and this an approach space is said to be a metric if it is originated by a metric space. R. Lowen [12] studied and understood an approach space in 1987. A considerable inspection of an approach spaces can be found in the monographs of Lowen [8]. The theory of an approach space regarded a generalization of metric spaces and Topological spaces, and based upon point-to-set distances instead of a point-to-point distances. M. Barn and M. Qasim [1, 2] introduced a local distance-approach spaces, Approach spaces, and gauge approach spaces and compared them with usual , approach spaces. E. Colebuders, M. Sion, W. Van Den Haute [3] proved many sufficient consequences on real valued contractions.

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J. Martínez − Moreno[1], A. Roldán[2] and C. Roldán[2] [4] studied the concept of fuzzy approach spaces as spaces generalization of fuzzy metric spaces and defined some Properties of fuzzy approach. G. Gutierres, D. Hofmann [5] established the concept of completeness for approach spaces and proved some properties in completeness approach spaces. K. Van Opdenbosch [6] defined a new isomorphic characterizations of approach spaces, pre-approach spaces, convergence approach spaces, uniform gauge spaces, topological spaces, and convergence spaces, pre-topological spaces, metric spaces, and uniform spaces. In approach spaces, The measures of Lindelof and separability are studied by R. Baekeland and R. Lowen [7]. R. Lowen and S. Verwulgen [8] defined an approach vector spaces. Approach group spaces, semi-group spaces, and uniformly convergent are defined by R. Lowen and B. Windels [9]. R. Lowen and M. Sion [10, 12] gave and studied the definitions of some separation axioms in the approach spaces and defined the relationship of the –axiom, regular and completely regular and also studied of normed linear spaces and from a normed real vector space $(X, \| \|)$, R. Lowen, C. Van Olmen, and T. Vroegrijk [11] found the relationship between Functional Ideas and Topological Theories. Some notions and relations in an approach theory are discussed by R. Lowen and C. Van Olmen [13]. R. Lowen [14] discussed and studied the development of fundamental theory of approximation. W. Li, Dexue Zhang [15] introduced the Smyth complete. R. K. Abbas and B. Y. Hussein [16,17,18] introduced new results in approach space and contraction by use distance between set and point and found new results in app-normed space. In this research, we want to define an approach group space in the right perspective in the concept of an approach vector spaces and then we want to use the topological approach structure to establish a canonical counterpart of the classical topological vector space. This paper, we define t^{ω}-approach distance and introduce some basic definitions, so, explain the relationship between metric space and approach space, we discuss the notion of metric space is an approach space and show that the converse is not true, moreover, we define the contraction function and prove some properties related to contractions. Furthermore, we introduce the definition of approach group, approach semi-group, approach subgroup, and give some examples in an approach group and introduce the definition of an approach vector space, furthermore, we introduce some examples in approach vector space. Finally, we present the definition of topological vector space and an approach sub-space, so, we prove some theorems in this manner. At the end of this paper we discuss the important conclusions.

2- t^{ω}- APPROACH METRIC SPACE

We start by definition of central notion of this paper, namely t^{ω}-approach distance.

Definition1: Let χ be a non-empty set. A collection $(t^{\omega})_{\omega<\infty}$ of functions $t^{\omega} : 2^{\chi} \times 2^{\chi} \to [0,\infty]$ is known as t^{ω}-approach distance on χ if this function satisfies the following properties:

$(t_1)\forall \omega \in \mathbb{R}^+, \forall A,B \in 2^{\chi} : t^{\omega}(A,B) = 0 \Rightarrow A = B$,

$(t_2)\forall \omega \in \mathbb{R}^+, \forall A \in 2^{\chi}$, then $t^{\omega}(A,\emptyset) = \infty$,

$(t_3)\forall \omega \in \mathbb{R}^+, \forall A,B,C \in 2^{\chi}: t^{\omega}(A, B \cap C) = max\{t^{\omega}(A,B), t^{\omega}(A,C)\}$,

$(t_4)\ \forall \omega \in \mathbb{R}^+, \forall A,B \in 2^{\chi}, \forall \varepsilon < \omega: t^{\omega}\left(A^{\omega}_{(\varepsilon)}, B\right) \leq t^{\omega}(A,B) + \varepsilon$ where

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$A^{\omega}_{(\varepsilon)} = \{x \in \chi | t^{\omega}(\{x\}, B) \leq \omega - \varepsilon\}$.

A pair (χ, t^{ω}) where the function t^{ω} is a distance and this pair is called t^{ω}- approach space and denoted by t^{ω}-app-spaces.

Now, we proved some properties of the t^{ω}-app- distance function. We start with the following definition:

Proposition1: Every metric space (χ, d) is t^{ω} –app- space.

Proof : We define $t^{\omega}{}_{d} : 2^{\chi} \times 2^{\chi} \to [0, \infty]$ by

$$t^{\omega}{}_{d}(A, B) = \begin{cases} \infty & B = \emptyset \\ inf_{x\in A, y\in B} d(x, y) & B \neq \emptyset \end{cases}$$

$t^{\omega}{}_{d}$ is t^{ω}-distance on χ is known as the t^{ω} – approach distance originated by d.The corresponding $(\chi, d) \mapsto (\chi, t^{\omega}{}_{d})$ defines, a space of the form $(\chi, t^{\omega}{}_{d})$ is said to be a t^{ω} – approach metric space.

(1) If $B \neq \emptyset, \forall \omega < \infty, \forall A, B \subseteq \chi: t^{\omega}{}_{d}(A, B) = inf_{x\in A, y\in B} d(x, y) = 0 \Rightarrow x = y$, then $A = B$.

If $B = \emptyset, t^{\omega}{}_{d}(A, \emptyset) = inf\ \emptyset = \infty$.

(2) If $B = \emptyset, \forall \omega < \infty: t^{\omega}{}_{d}(A, \emptyset) = inf d(A, \emptyset) = \infty$

(3) $\forall A, B, C \in 2^{\chi}$,

- if $C = \emptyset, B \neq \emptyset$, then $B \cap C = \emptyset$, so
  $t^{\omega}{}_{d}(A, B \cap C) = inf_{a\in A, b\in B\cap C} d(A, \emptyset) = \infty = \{\infty \vee inf_{b\in B\cap C} d(a, b)\}$
  $= \{inf_{\emptyset} d(a, \emptyset) \vee inf_{b\in B\cap C} d(a, b)\}$
  $= max\left(t^{\omega}{}_{d}(A, B), t^{\omega}{}_{d}(A, C)\right) \ldots\ldots\ldots..(1)$
- if $C \neq \emptyset, B = \emptyset$, then $B \cap C = \emptyset$ so, by the same way, the proof will be like (1)
- if $C = B = \emptyset$, then $B \cap C = \emptyset$, therefor
  $t^{\omega}{}_{d}(A, B \cap C) = inf_{a\in A, b\in B\cap C} d(a, b) = inf_{\emptyset} d(a, \emptyset) = \infty = \{\infty \vee \infty\}$
  $\{inf_{\emptyset} d(x, \emptyset) \vee inf_{\emptyset} d(x, \emptyset)\} = max(t^{\omega}{}_{d}(x, A), t^{\omega}{}_{d}(x, B))$
- if $C \neq \emptyset, B \neq \emptyset$, then $t^{\omega}{}_{d}(a, B \cap C) = inf_{a\in A, b\in B\cap C} d(a, b) =$
  $inf_{a\in A, b\in B\wedge b\in C} d(a, b) = inf_{a\in A, b\in B} d(a, b) \vee inf_{a\in A, b\in C} d(a, b)$
  $= max\{inf_{a\in A, b\in B} d(a, b), inf_{a\in A, b\in C} d(a, b)\} = max(t^{\omega}{}_{d}(A, B), t^{\omega}{}_{d}(A, C))$

(4) If $B = \emptyset$, then $\vee \varepsilon < \omega, t^{\omega}{}_{d}\left(A^{\omega}_{(\varepsilon)}, B\right) = t^{\omega}{}_{d}\left(A^{\omega}_{(\varepsilon)}, \emptyset\right) \leq t^{\omega}{}_{d}(A, \emptyset) + \varepsilon$.

Let $\omega < \infty, A, B \subseteq X$. In case that $A^{\omega}_{(\varepsilon\ )} \cap B^{c} \neq \emptyset$for all $\varepsilon < \omega, so, t^{\omega}{}_{d}(\{x\}, B) \leq \omega - \varepsilon$, for all $\varepsilon < \omega, x \in A^{\omega}_{(\varepsilon)}$, then by t_4

$t^{\omega}{}_{d}\left(A^{\omega}_{(\varepsilon)}, B\right) = inf_{a\in A_{(\varepsilon)}, b\in B} d(a, b) \leq inf_{a\in A, b\in B} d(a, b) + inf\{\varepsilon < \omega: \{x \in$
$\chi t\omega x, B \leq \omega - \varepsilon\} \leq t\omega A, B + \varepsilon$■

Definition2: An app- space of the type $(\chi, t^{\omega}{}_{d})$ for some metric d on χ is called t^{ω}- app- metric space and the distance of the type $t^{\omega}$is known as t^{ω}- app- metric distance.

Example1: For all $\omega \in \mathbb{R}^{+}$, the discrete t^{ω}- app- distance structure t^{ω} on χ is given as for all $x \in \chi$ and $A \subseteq \chi$,

$$t^{\omega}(A, B) = \begin{cases} 0 & x \in B \\ \infty & x \notin B. \end{cases}$$

Remark1:t^{ω}- app- metric space is not necessary metric space.

We explain this remark in the following example:

for all $A, B \subseteq [0, \infty]$, let

$$t^{\omega}{}_{p}(A, B) = \begin{cases} max\{sup_{A \in 2^X} A - \{ sup_{B \in 2^X} B , 0\}\}, & A \neq \emptyset, B \neq \emptyset \\ \infty & A \; or \; B = \emptyset \end{cases}$$

Clear $t^{\omega}{}_{p}$ is an t^{ω}-approach distance on [0,∞], then p = ([0, ∞], t^{ω}) is an app- space and plays an very essential purpose in the theory of approach spaces, p is not t^{ω}- app-metric space.

$d_{t^{\omega}} : \chi \times \chi \rightarrow [0, \infty] : d_{t^{\omega}}(x, y) = t^{\omega}{}_{p}(\{x\}, \{y\})$

Clear $t^{\omega}{}_{p}$ is t^{ω}-app–space but it is not metric space because the condition:

$d_{t^{\omega}}(x, y) \neq d_{t^{\omega}}(y, x)$.

Example2: Define $t^{\omega}{}_{E}: 2^{[0,\infty]} \times 2^{[0,\infty]} \rightarrow [0, \infty]$

$$t^{\omega}{}_{E}(A, B) := \begin{cases} 0 & a, b = \infty , B \text{ unbounded} \\ inf_{a \in A, b \in B}|a - b| & a, b < \infty \end{cases}$$

We verify that this function is t^{ω}-app- distance on $[0, \infty]$.

(1) Suppose $a, b < \infty$, $\forall \omega < \infty$ and $A \subseteq \chi$

then $t^{\omega}{}_{E}(A, B) = inf_{a \in A, b \in B}\big|a - b\big| = 0 \Rightarrow a = b$, then $A = B$

(2) Let $a, b < \infty$, then$t^{\omega}{}_{E}(A, \emptyset) = inf_{\emptyset}(A, \emptyset) = \infty$.

If $a, b = \infty$, we obtain$t^{\omega}{}_{E}(A, \emptyset) = \infty$.

(3) Suppose that $a, b < \infty$, let $A, B, C \in 2^{\chi}$,

- if $B = \emptyset \; and \; C \neq \emptyset$
  then we get $t^{\omega}{}_{E}(A, B \cap C) = t^{\omega}{}_{E}(A, \emptyset) = inf_{\emptyset}(A, \emptyset) = \infty = \{\infty \vee inf_{a \in B}|x - a|\} = max\, \{t^{\omega}{}_{E}(A, B), t^{\omega}{}_{E}(A, C)\} \ldots . .(2)$
- if $B \neq \emptyset$ and $C = \emptyset$, then $B \cap C = \emptyset$, so, by the same way, the proof is like (2)
- if $B = C = \emptyset$, then, $B \cap C = \emptyset$, therefor, $t^{\omega}{}_{E}(A, B \cap C) = t^{\omega}{}_{E}(x, \emptyset) = inf_{\emptyset}(x, \emptyset) = \infty = \{\infty \vee \infty\} = max\, \{t^{\omega}{}_{E}(A, B), t^{\omega}{}_{E}(A, C)\}$
- if $B \neq \emptyset, C \neq \emptyset$, then $t^{\omega}{}_{E}(A, B \cap C) = inf_{a \in A, b \in B \cap C}|a - b| = inf_{a \in A, b \in B \wedge b \in C}|a - b| = inf_{a \in A, b \in B}|a - b| \vee inf_{a \in A, b \in C}|a - b| = max\, \{t^{\omega}{}_{E}(A, B), t^{\omega}{}_{E}(A, C)\}$

(4) Let $B = \emptyset$, then $\varepsilon < \omega, t^{\omega}{}_{E}\big(A^{\omega}_{(\varepsilon)}, B\big) = t^{\omega}{}_{E}\big(A^{\omega}_{(\varepsilon)}, \emptyset\big) \leq t^{\omega}{}_{E}(A, \emptyset) + \varepsilon$.

Let $\omega < \infty, A, B \subseteq \chi$. In case that $A^{\omega}_{(\varepsilon)} \cap B^{c} \neq \emptyset$for all $\varepsilon < \omega, so, t^{\omega}{}_{E}(\{x\}, B) \leq \omega - \varepsilon$, for all $\varepsilon < \omega, x \in A^{\omega}_{(\varepsilon)}$, then by t_4

$$\begin{aligned} t^{\omega}{}_{E}\big(A^{\omega}_{(\varepsilon)}, B\big) &= inf_{a \in A_{(\varepsilon)}, b \in B}|a - b| \\ &\leq inf_{a \in A, b \in B}|a - b| + inf\{\varepsilon < \omega: \{x \in \chi | t^{\omega}(\{x\}, B) \leq \omega - \varepsilon\} \\ &\leq t^{\omega}(A, B) + \varepsilon. \end{aligned}$$

$B \subset 2^{[0,\infty]}$ is unbounded. We get that $t^{\omega}{}_{E}(A^{\omega}_{(\varepsilon)}, B) = \infty$,

and $t^{\omega}{}_{E}(A^{\omega}_{(\varepsilon)}, B) = \infty$, and thus $t^{\omega}{}_{E}\big(A^{\omega}_{(\varepsilon)}, B\big) \leq t^{\omega}{}_{E}(A, B) + \varepsilon$.

Hence, $(2^{[0,\infty]}, t^{\omega}{}_{E})$ is t^{ω}- app- space.

3-MAIN PROPERTIES OF CONTRACTIONS

Definition3: Let (χ, t^{ω}) and $(Y, t^{\omega'})$ are app- spaces. A function $\varrho: \chi \rightarrow Y$ is known as $t^{\omega}-$ contraction if for all $\omega < \infty$, and for all $A, B \in 2^{\chi}, t^{\omega'}(\varrho((A), \varrho(B)) \leq k\, t^{\omega}(A, B)$, for some $k \in [0,1]$.

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Proposition2: Let (χ, t^{ω}) be $t^{\omega}-$ app- space and $\varrho: (\chi, t^{\omega}) \rightarrow (\chi, t^{\omega})$, Then for all $\omega < \infty$ and for any $A, B \in 2^{\chi}$.
1. (Identity map)$Ix: (\chi, t^{\omega}) \rightarrow (\chi, t^{\omega})$ is a $t^{\omega}-$ contraction
2. The constant map is $t^{\omega}-$contraction.
Proof : (1) Let $\omega < \infty$ and $A, B \in 2^{\chi}$,
$t^{\omega'}(\varrho((A), \varrho(B)) = t^{\omega'}(Ix((A), Ix(B)) = t^{\omega}(A, B) \leq k\, t^{\omega}(A, B)$.
Thus, Ix is a $t^{\omega}-$contraction for some $k \in [0,1]$.
(2) Let $\varrho: (2^{\chi}, t^{\omega}) \rightarrow (2^{\chi}, t^{\omega})$ defined as follows: $\varrho(A) = \{a\}, \varrho(B) = \{b\} \forall a, b \in \chi, \forall \omega < \infty$.Then
$t^{\omega'}\left(\varrho(A), \varrho(B)\right) = t^{\omega'}(\{a\}, \{b\}) = 0 \leq t^{\omega}(A, B) \leq k\, t^{\omega}(A, B)$for some $k \in [0,1]$.
■
Proposition3: Let $(\chi_i, t^{\omega}{}_i)_{i \in I}$ be a family of t^{ω}-app- space that any $i \in I$. Then $\forall \omega < \infty$,the projections $p_{rs}: \prod_{i \in I} \chi_i \rightarrow \chi_i$ are $t^{\omega}-$contractions.
Proof: Let $\omega < \infty, A, B \in 2^{\chi}, p_{rs}: \prod \chi_i \rightarrow \chi_i$ projection function,

$$t_i^{\omega'}(p_{rs}(A_i), p_{rs}(B)) = t_i^{\omega'}(p_{rs}(A_1, \ldots, A_k), p_{rs}((\prod_{i \in I} B_i)), \quad i \in I$$

$$(t_i^{\omega'}((A_i), (B_i)) \leq (k\, t_1^{\omega}(A_1, B_1) \times (k\, t_2^{\omega}(A_2, B_2) \times \ldots \times (k\, t_i^{\omega}(A_i, B_i), \ldots)$$

$$= k^n \prod_{i \in I} t^{\omega}{}_i(A_i, B_i) = k^n t^{\omega}(\prod_{i \in I} A_i, \prod_{i \in I} B_{i \in I}), \quad \text{for } i \in I, n \geq 1$$

Hence $t_i^{\omega'}\left(p_{rs}(A_i), p_{rs}(B)\right) \leq k^n t^{\omega}(\prod_{i \in I} A_i, B)\ \forall n \geq 1$, let $m = k^n$, then
$t_i^{\omega'}\left(p_{rs}(A_i), p_{rs}(B)\right) \leq m\, t^{\omega}(\prod_{i \in I} A_i, B)$, for some $m \in [0,1]$.
Thus, p_{rs} (A) is $t^{\omega}-$contraction. ■
Definition4: A triple $(\chi, t^{\omega}, *)$ is known as t^{ω}- app- semi-group if and only if
1. (χ, t^{ω}) is t^{ω}- app- space.
2. $(\chi, *)$ is a semi-group.
3. $*: 2^{\chi} \otimes 2^{\chi} \rightarrow 2^{\chi}: (A, B) \mapsto A * B$ is a $t^{\omega}-$contraction such that $A * B = \{x * y : x \in A, y \in B\}$.
Definition5: The triple $(\chi, t^{\omega}, *)$ is known as t^{ω}- app- group if satisfy the following:
a) (χ, t^{ω}) is t^{ω}- app- space.
b) $(\chi, *)$ is a group.
c) $*: 2^{\chi} \otimes 2^{\chi} \rightarrow 2^{\chi}: (A, B) \mapsto A + B$ is the t^{ω} –contraction such that $A + B = \{x + y : x \in A, y \in B\}$.
d) $-: 2^{\chi} \rightarrow 2^{\chi}: A \rightarrow -A$ is $t^{\omega}-$contraction such that $x \in A, -x \in -A$.
Example3: Let $\mathbb{R}$ is a set of real numbers and + is an addition binary operation, then $\forall \omega < \infty, (\mathbb{R}^n, t^{\omega}, +)$ with a usual distance t^{ω} and a usual addition is t^{ω} -app- group.
Proof: We must prove that $(\mathbb{R}^n, t^{\omega})$ is t^{ω}- app- space with usual distance t^{ω}:
Let$\omega < \infty$,for$i = 1, \ldots, n$,for all $A, B, C \in 2^{\mathbb{R}^n}$.Then$t^{\omega}: 2^{\mathbb{R}^n} \times 2^{\mathbb{R}^n} \rightarrow [0, \infty]$, such that $A = A_i = (A_1, A_2, \ldots, A_n)$ is defined as follows:

$$t^{\omega}(A_i, B) = \begin{cases} inf_{x_i \in A_i, a \in B} d(x_i, a) & A \neq \emptyset, B \neq \emptyset \\ \infty & A \text{ or } B = \emptyset \end{cases}$$

(1) If $A, B \neq \emptyset$,

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Let $\omega < \infty$,for all $x = (x_1, x_2, \dots x_n) \in \mathbb{R}^n$: $t^{\omega}(A_i, B) = inf_{x_i \in A_i, a \in B} d(x_i, a) = inf_{x_i \in A_i, x_i \in \{x_i\}} d(x_i, x_i) = 0 \Longrightarrow x_i = a$, then$A_i = B$ for all $i = 1, \dots, n$.
If $B = \emptyset, t^{\omega}(A, \emptyset) = inf_{\emptyset} d(x_i, \emptyset) = \infty, if\ A = \emptyset, t^{\omega}(\emptyset, B) = inf_{\emptyset} d(\emptyset, a) = \infty$

(2)

$t^{\omega}(A_i, \emptyset) = inf_{\emptyset} d(x_i, \emptyset) = inf_{\emptyset}\big(d(x_1, \emptyset), d(x_2, \emptyset), \dots, d(x_n, \emptyset)\big) = \infty$. Then$t^{\omega}(A_i, \emptyset) = \infty$, for all $x = xi \in \mathbb{R}^n$, for all$A_i \in 2^{\mathbb{R}^n}$, $\forall i = 1, \cdots n$.

(3)

$t^{\omega}(A, B \cap C) = inf_{x_i \in A_i, a \in B \cap C} d(x_i, a) = inf_{x_i \in A_i, a \in B} d(x_i, a) \vee inf_{x_i \in A_i, a \in C} d(x_i, a), \forall i = 1, \cdots, n$

$\leq max\{inf_{x_i \in A_i, a \in B} d(x_i, a), inf_{x_i \in A_i, a \in C} d(x_i, a)\}$,

$= max(inf_{x_i \in A_i, a \in B}\{(d(x_i, a)), inf_{x_i \in A_i, a \in C}((d(x_i, a))$

$= max\big(t^{\omega}(A_i, B), t^{\omega}(A_i, C)\big)$, for all $x = xi \in \mathbb{R}^n$, for all$A_i \in 2^{\mathbb{R}^n}$ and for all $i = 1, \cdots, n$

(4) (a) Let $\omega < \infty, A, B \subseteq \mathbb{R}^n$. In case that $A^{\omega}_{i_{(\varepsilon)}} \cap B^c \neq \emptyset$for all $\varepsilon < \omega, so, t^{\omega}(\{x_i\}, B) \leq \omega - \varepsilon$, for all $\varepsilon < \omega, x_i \in A^{\omega}_{i_{(\varepsilon)}}$, then by t_4

$$t^{\omega}\left(A^{\omega}_{i_{(\varepsilon)}}, B\right) = inf_{x_{i_s} \in A_{i_{(\varepsilon)}}, a \in B} d(x_{i_s}, a)$$
$$\leq inf_{x_i \in A_i, a \in B} d(x_i, a) + inf\{\varepsilon < \omega : \{x \in X | t^{\omega}(\{x_i\}, B) \leq \omega - \varepsilon\}$$
$$\leq t^{\omega}(A_i, B) + \varepsilon. \quad , \forall x \in \mathbb{R}^n, \forall i, s = 1, \cdots, n$$

Then $(\mathbb{R}^n, t^{\omega})$ is t^{ω}- app- space.

(b) It is obvious that $(\mathbb{R}^n, +)$ is a group.

(c) $\forall \omega < \infty$ and $A, B, C \in 2^{\mathbb{R}^n}$

$$t^{\omega'}\big(\varrho(A_i), \varrho(B + C)\big) = t^{\omega'}(\varrho(A_i), \varrho(B) + \varrho(C))$$
$$= inf_{x_i + y_i \in A_i, a \in B, b \in C} d(x_i + y_i, a + b)$$
$$\leq inf_{x_i + y_i \in A_i, a \in B} d(x_i + y_i, a) + inf_{x_i + y_i \in A_i, b \in C} d(x_i + y_i, b) = t^{\omega}(A_i, B) + t^{\omega}(A_i, C)$$
$$\leq k t^{\omega}(A_i, B) + k t^{\omega}(A_i, C) = k(t^{\omega}(A_i, B) + t^{\omega}(A_i, C)), \text{ for some } k \in [0,1]$$

(d)$t^{\omega'}(\varrho(A_i), \varrho(B)) = t^{\omega'}\big(\varrho(-A_i), \varrho(-B)\big) = t^{\omega'}\big(-\varrho(A_i), -\varrho(B)\big) = inf_{-x_i \in A_i, -a \in B} d(-x_i, -a)$

$= inf_{x_i \in A_i, a \in B} d(x_i, a) = t^{\omega}(A_i, B) \leq k t^{\omega}(A_i, B)$, for all$x_i \in \mathbb{R}^n$and for some $k \in [0,1]$

So, the inverse function is t^{ω} −contraction,

Hence, $(\mathbb{R}^n, t^{\omega}, +)$ is $t^{\omega}-$ approach group. ■

Example4: Let $\mathbb{Z}$ is a set of all integer numbers and + is an addition binary operation, then $\forall \omega < \infty, (\mathbb{Z}, t^{\omega}, +)$is t^{ω} – app- group with the usual distance $t^{\omega}$and usual addition.

Proof: We will prove that $(\mathbb{Z}, t^{\omega})$ is t^{ω} − app- space with usual distance t^{ω}:

$$t^{\omega}(A, B) = \begin{cases} inf_{x \in A, a \in B} d(x, a) & A \neq \emptyset, B \neq \emptyset \\ \infty & A\ or\ B = \emptyset \end{cases}$$

such that $t^{\omega}: 2^{\mathbb{Z}} \times 2^{\mathbb{Z}} \to [0, \infty], \forall x \in \mathbb{Z}, \forall \omega < \infty, \forall A, B \in 2^{\mathbb{Z}}$

(a) We prove $(\mathbb{Z}, t^{\omega})$ is $t^{\omega}-$ approach spaceby $((t_1), (t_2), (t_3), (t_4)$ of definition1)

(b) It is very clear that $(\mathbb{Z}, +)$ is a group.

(c) Now, we prove$+: 2^{\mathbb{Z}} \oplus 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}: (A,B) \to A + B\ is\ t^{\omega}$ – contraction for all $x + y \in A + B \in 2^{\mathbb{Z}} : x \in A, y \in B$, for all $\omega < \infty\ and$ for all $B, C, D \subset \mathbb{Z}$, such that $a \in D, b \in C$

$t^{\omega'}\big(\varrho(A), \varrho(D + C)\big) = t^{\omega'}\big(A + B, \varrho(D) + \varrho(C))\big) = t^{\omega'}(x + y, a + b)$
$= inf_{x+y \in A+B, a\in D, b\in C} d((x + y), (a + b)) = inf_{x+y\in A+B,a\in D,b\in C}|(x + y) - (a + b)|$
$\leq inf_{x+y\in A+B,a\in D,b\in C}(|(x + y) - a| + |b - (x + y)|)$
$= inf_{x+y\in A+B,a\in D}(|(x + y) - a|) + inf_{x+y\in A+B,b\in C}(|(x + y) - b|)$
$= inf_{x+y\in A+B,a\in D} d((x + y) - a) + inf_{x+y\in A+B,b\in C} d((x + y) - b)$
$= t^{\omega}(A, D) + t^{\omega}(A, C) \leq kt^{\omega}(A, D) + k\,t^{\omega}(A, C) = k(t^{\omega}(A, D) + t^{\omega}(A, C)),$
for some $k \in [0,1]$.

(d)
$t^{\omega'}\big(\varrho(A), \varrho(B)\big) = t^{\omega'}\big(\varrho(-A), \varrho(-B)\big) = t^{\omega'}\big(-\varrho(A), -\varrho(B)\big) =$
$inf_{-x\in A,-a\in B} d(-x, -a) =$
$= inf_{-x\in A,-a\in B}|-x - (-a)| = inf_{-x\in A,-a\in B}|x - a| = inf_{x\in A,a\in B} d(x, a) = t^{\omega}(A, B) \leq$
$kt^{\omega}(A, B)$, for some $k \in [0,1]$.■

Then, the set of all integer numbers with the usual distance and addition operation is t^{ω}-app- space.

Definition6: (t^{ω}-app- sub-group): Let $(\chi, t^{\omega}, *)$ is t^{ω}- app- group and $Y \subseteq \chi$, then $\forall \omega \in \mathbb{R}^{+}$, $(Y, t^{\omega}, *)$ is called t^{ω}- app- sub-group if satisfy the following:

(a) (Y, t_{Y}^{ω}) is t^{ω}- app- space.
(b) $(Y, *)$ is a sub-group.
(c) ϱ: $2^{Y} \times 2^{Y} \to 2^{Y}$ with $\varrho(A, B) = A * B^{-1}$ is t^{ω} –contraction such that $A * B^{-1} = \{x * y^{-1}: x \in A, y \in B\}$.

Example5: Let $\mathbb{Z}$ is the set of all integer elements and subset of $\mathbb{R}$ the set of all real numbers with usual distance t^{ω}, such that

$$t^{\omega}(A, B) = \begin{cases} \infty & if\ A\ orB = \emptyset \\ inf_{y\in B}|x - y| & if\ A \neq \emptyset, B \neq \emptyset \end{cases},$$

then $(\mathbb{Z}, t^{\omega}, +)$ is t^{ω}- app- sub-group of $(\mathbb{R}, t^{\omega}, +)$

Proof: It is very obvious that $(\mathbb{Z},, t^{\omega})$ is t^{ω}- app- space and $(\mathbb{Z},, +)$ is sub-group of $(\mathbb{R}, +)$

(a) $(\mathbb{Z}, t^{\omega})$ is t^{ω}- app- space by example (5).
(b) $(\mathbb{Z}, +)$ is a sub-group because :

- $\forall a, b \in \mathbb{Z}, \forall A, B \in 2^{\mathbb{Z}} \to A + B \in 2^{\mathbb{Z}}$ (closed).
- $\emptyset \in 2^{\mathbb{R}} \to \emptyset \in 2^{\mathbb{Z}}$.
- if$A \in 2^{\mathbb{Z}} \to -A \in 2^{\mathbb{R}} \to -A \in 2^{\mathbb{Z}}$, (inverse).

Now, we will prove that $*: 2^{\mathbb{Z}} \otimes 2^{\mathbb{Z}} \to 2^{\mathbb{Z}}: (A, B^{-1}) \mapsto A - B is t^{\omega}$ –contraction when $A - B = \{x - y: x \in A, y \in B\}$.

- Let $B = \emptyset$ and $C = \emptyset$, then
$t^{\omega'}(\varrho(A, B), \varrho(B, C)) = t^{\omega'}(A - B, B - C)) = t^{\omega'}(A - B, \emptyset) \leq t^{\omega}(A - B, \emptyset) = \infty$
$= k\infty = kt^{\omega}(A - B, \emptyset)$, for all $a, b, \in \mathbb{Z}$, for all $A, B, C \in 2^{\mathbb{Z}}$, for some $k \in [0,1]$ ... (3).
- Now, by the same way in(3)we can prove the $t^{\omega} -$ onractionIf $B = \emptyset$ and $C \neq \emptyset$.

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- Let $B \neq \emptyset$ and $C = \emptyset$, then

$$t^{\omega'}(\varrho(A,B),\varrho(B,C)) = t^{\omega'}(A - B, B - C)) = t^{\omega'}(A - B, B \cap C^c)$$
$$= t^{\omega'}(A - B,(B \cap 2^{\mathbb{Z}})) = t^{\omega'}(A - B, B) \leq t^{\omega}(A - B, B)$$
$$\leq k\, t^{\omega}(A - B, B) \text{ for all } a,b \in \mathbb{Z}, A,B,C \in 2^{\mathbb{Z}}, \text{for some } k \in [0,1]$$

Let $B \neq \emptyset$ and $C \neq \emptyset$, then

$$t^{\omega'}(\varrho(A,B),\varrho(B,C)) = t^{\omega'}(A - B, B - C) = t^{\omega'}(A - B, B \cap C^c)$$
$$\leq \{ t^{\omega}(A - B, B) \vee t^{\omega}(A - B, C^c)\}$$
$$\leq max\{ t^{\omega}(A - B, B), t^{\omega}(A - B, C^c)\}$$
$$\leq max\{k\, t^{\omega}(A - B, B), k\, t^{\omega}((A - B, C^c)\}, \text{for all } a,b, \in \mathbb{Z}, A,B,C$$
$$\in 2^{\mathbb{Z}}, \text{for some } k \in [0,1].$$

If $B \neq \emptyset$,

$$t^{\omega'}\big(\varrho(A,B),\varrho(B,C)\big) = t^{\omega'}(A - B, B - C) \leq t^{\omega}(A - B, B - C)$$
$$= inf_{a\in A,b\in B,c\in C}|a - b, b - c| \leq inf_{a\in A,b\in B,c\in C}\{|a - b| + |b - c|\}$$
$$\leq inf_{a\in A,b\in B}|a - b| + inf_{b\in B,c\in C}|b - c|$$
$$= t^{\omega}(A,B) + t^{\omega}(B,C) \leq k t^{\omega}(A,B) + k\, t^{\omega}(B,C) = k\, t^{\omega}((A,B),(B,C)).$$

Then $(Z, t^{\omega}, +)$ is t^{ω}-app- sub-group. ■

3-t^{ω}- APPROACH VECTOR SPACES

Definition7: Let F be a field and let χ be a non-empty power set with two binary operations :an addition and a scalar multiplication, $\forall \omega < \infty, t^{\omega}$ is an app- distance on 2^{χ}, then, a quadruple $(\chi, t^{\omega}, +, .)$ is said to be t^{ω}- approach vector space if satisfy the following:

1) $(\chi, t^{\omega}, +)$ is t^{ω}- app-group.
2) $(\chi, t^{\omega}, .)$is t^{ω}- app-semi group.
3) $\mu.(B + C) = \mu.B + \mu.C$, for all $\mu \in F$, for all $B, C \in 2^{\chi}$.
4) $(B + C).\mu = B.\mu + C.\mu$, for all $\mu \in F$, for all $B, C \in 2^{\chi}$.
5) $(\mu.\vartheta).C = \mu(\vartheta C)$, for all $C \in 2^{\chi}$, for all $\mu, \vartheta \in F$.
6) $I.A = A$, for all $A \in 2^{\chi}$.

Example6: Let $\mathbb{R}$ be a set of all real numbers, then $\forall \omega < \infty$, a quadruple $(\mathbb{R}, t^{\omega}, +, .)$ with a usual distance t^{ω}, addition $(+)$ and scalar multiplication $(.)$ is t^{ω}-app−vector space.

1) $(\mathbb{R}, t^{\omega}, +)$ is t^{ω}- app- group.
2) It is obvious $(\mathbb{R}, t^{\omega}, .)$ is t^{ω}- app- semi group.
3) Since $\mathbb{R}$ is a vector space: $\mu(B + C) = \mu.B + \mu.C$, $\forall \mu \in \mathbb{R}, \forall B, C \in 2^{\mathbb{R}}$
4) $(B + C).\mu = B.\mu + C.\mu$, $\forall \mu \in \mathbb{R}, \forall B, C \in 2^{\mathbb{R}}$.
5) $(\mu.\vartheta).C = \mu.(\vartheta.C)$, for all $C \in 2^{\mathbb{R}}$, for all $\mu, \vartheta \in \mathbb{R}$.
6) $I.A = A$ for all $A \in 2^{\mathbb{R}}$. Then $(\mathbb{R}, t^{\omega}, +, \cdot)$ is t^{ω}- approach vector space.

Example7: Let $(\mathbb{R}^n, +, .)$be an Euclidean space with an usual addition and scalar multiplication binary operations, and a usual distance t^{ω}, then it is t^{ω}- app-vector space such that

$$t^{\omega}(A_i, B) = \begin{cases} \infty & A\ or B = \emptyset \\ inf_{x_i \in A_i, a\in B}|x_i - a| & A \neq \emptyset, B \neq \emptyset \end{cases}$$

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For all $i = 1,2, \dots \dots ., n$, for all $x_i \in A_i$, for all $A_i, B \in 2^{\mathbb{R}^n}, and\ \forall\omega < \infty$ such that $t^\omega: 2^{\mathbb{R}^n} \times 2^{\mathbb{R}^n} \to [0,\infty]$, $(\mathbb{R}^n, t^\omega, +)$ is t^ω- app-group from example (7)

(1) By example (7) , $(\mathbb{R}^n, t^\omega, +)$ is t^ω- app-group, so, $(\mathbb{R}^n, t^\omega)$ is t^ω- app-space.

(2) It is clear that $\mu. A_i \in 2^{\mathbb{R}^n}$, for all $A_i \in 2^{\mathbb{R}^n}$, for all $\mu \in F$.

(3) For all $A_i, B_i, C, D \in 2^{\mathbb{R}^n}$

$$t^{\omega'}(A_i + B_i, C + D) = inf_{x_i \in A_i, y_i \in B_i, a \in C, b \in D}|(x_i + y_i) - (a + b)|$$
$$= inf_{x_i \in A_i, y_i \in B_i, a \in C, b \in D}|x_i - a + y_i - b|$$
$$\leq inf_{x_i \in A_i, a \in C}|x_i - a| + inf_{,y_i \in B_i, b \in D}|y_i - b| = t^\omega(A_i, C) + t^\omega(B_i, D)$$
$$= kt^\omega(A_i, C) + kt^\omega(B_i, D) \leq k\{t^\omega(A_i, C) + t^\omega(B_i, D)\} \text{for some } k \in [0,1]$$

(4) $(A_i + B_i).\mu = A_i.\mu + B_i.\mu$, for all $\mu \in F. \forall A_i, B_i \in 2^{\mathbb{R}^n}$

(5) $(\mu.\vartheta).A_i = \mu(\vartheta A_i)$, for all $A_i \in 2^{\mathbb{R}^n}$, for all $\mu, \vartheta \in F$.

(6) $I.A_i = A_i.$, for all $A_i \in 2^{\mathbb{R}^n}$.

Thus, $(\mathbb{R}^n, t^\omega, +, .)$ is t^ω- app-vector space. ■

Proposition4: Let $(V, t^\omega, +, .)$ is t^ω- app-vector space, then $(V, t^\omega, +, .)$ is a vector space .

Proof: The proof is direct according to definition of t^ω- app − vector space ,$(V, t^\omega, +, .)$ satisfy the condition of vector space.

Remark2: The convers of proposition (7) is not true, we show that by the following example.

Definition8 (t^ω-app- subspace): let $\omega \in \mathbb{R}^+$,a subset Y of t^ω- app-vector space χ over the field F is called t^ω- app- subspace if satisfy the following:

(1) Y is a sub-space of a vector space $(\chi, +, .)$.

(2) (Y, t_Y^ω) is t^ω- app- space.

4-MAIN RESULT

(χ, τ) is known as a topological space and the collection of all open sets announces by τ, it structures derivate from τ, like the accompanied closure operator, will be announced by c_τ if no distraction can arise, we can also drop the reference to τ.

Definition8: Let (χ, t^ω) be t^ω app-space. For $A, B \in 2^\chi$, $\{x\} \subseteq A$, the set with center $\{x\}$ and radius $r > 0$ will be denoted by $B_r(\{x\}) = \{t \in \chi : t^\omega(\{t\}, \{x\}) < r, \forall\omega < \infty\}$, $B_r(\{x\})$ is called t^ω-open-ball.

Definition9: Let $(V, t^\omega, +, .)$ be t^ω- app- vector space on the field F, then t^ω- topological approach vector space V together with an inducing topology τ_V will be satisfy the two axioms:

(1) The map $+: 2^V \times 2^V \to 2^V, (A, B) \to A + B$ when $A + B = \{x + y : x \in A, y \in B\}$ is t^ω −contraction.

(2) The map $\cdot : 2^F \times 2^V \to 2^V$ is also t^ω−contraction.

It is denoted by a pair (V, τ, t_τ^ω).

Definition10: t_τ^ω- app-topological space is a topological space (χ, τ) that is associated with a natural app- space in the following way: we defined the function $t_\tau^\omega: 2^\chi \times 2^\chi \to [0,\infty]$ by

$$t_\tau^\omega(A,B) = \begin{cases} 0 & \text{if } x \in c_\tau(A) \cap c_\tau(B) \\ \infty & \text{if } x \notin c_\tau(A) \cap c_\tau(B) \end{cases}$$

for any $x \in \chi, \omega < \infty$ and $A, B \in 2^\chi$. t^ω- app-space of type $(\chi, \tau, t_\tau^\omega)$ for some topology τ on χ is called (t_τ^ω- topological app- space).

t^ω- distance of type t_τ^ω is called (a topological $t^\omega -$ distance).

We must introduce some results with their proofs in a topological app- space;

Now, we start with the following important proposition:

Proposition5: Let (χ, τ) be a topological space, then the function

$t_\tau^\omega : 2^\chi \times 2^\chi \to [0, \infty]$ which is defined as follows:

$$t_\tau^\omega(A,B) = \begin{cases} 0 & \text{if } x \in c_\tau(A) \cap c_\tau(B) \\ \infty & \text{if } x \notin c_\tau(A) \cap c_\tau(B) \end{cases}$$

is t^ω- distance on χ.

Proof: We prove that t_τ^ω is a distance.

(1) Let $A, B \in 2^\chi, \omega < \infty$ and $t_\tau^\omega(A,B) = 0$, then by definition of (t^ω-open-ball), $A = B$.

(2) We know that $c_\tau(\emptyset) = \emptyset$, thus $t_\tau^\omega(A, \emptyset) = \infty$.

(3) for any $A, B, C \in 2^\chi$, we obtien $c_\tau(B \cap C) = c_\tau(B) \cap c_\tau(C)$, then

$t_\tau^\omega(A, B \cap C) = max(t_\tau^\omega(A,B), t_\tau^\omega(A,C))$.

(4)for all $\epsilon < \infty$, we have $A_{(\epsilon)}^\omega = c_\tau(A)$, $B_{(\epsilon)}^\omega = c_\tau(B)$ and $A(\infty) = \chi$, so we get $t_\tau^\omega(A_{(\epsilon)}^\omega, B) \le t_\tau^\omega(A,B) + \epsilon$.■

5-CONCLUSION

We establish some problems in the theory of app- spaces: a generalization of t^ω- metric spaces and t^ω-topological spaces that is called t^ω- topological approach structure, therefore, we introduce some notions in t^ω-app- spaces as t^ω- app- vector space, t^ω-app- subspace, and t^ω-app- topological vector space, moreover, we give some examples in t^ω-app- space, t^ω- app- group, and t^ω- app- vector space. We explain every t^ω-app- space is metric space but the converse is not true and show that by give an example. We construct several new properties of contractions.

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